Filomat 32:12 (2018), 4351–4364
https://doi.org/10.2298/FIL1812351R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence of Iterates of q-Bernstein and (p, q)-Bernstein Operators and the Kelisky-Rivlin Type Theorem

Shagufta Rahman[a], **M. Mursaleen**[a], **Ali H. Alkhaldi**[b]

[a]*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*
[b]*Department of Mathematics, College of Science, King Khalid university, P.O. Box 9004, Abha, Saudi Arabia*

Abstract. Recently, Radu [Note on the iterates of q and (p, q)-Bernstein operators, Scientific Studies and Research, Series Mathematics and Informatics, 26(2) (2016) 83-94] has investigated the convergence of iterates of q-Bernstein polynomial and (p, q)-Bernstein polynomial with the aids of weakly Picard operators theory. In this article, we establish Kelisky-Rivlin type theorem on the power of the q-Bernstein operators for two dimensional case, (p, q)-Bernstein operators and bivariate (p, q)-Bernstein operators by using contraction principle.

1. Introduction and Preliminaries

In 1912, Bernstein [11] introduced a sequence of operators $B_n : C[0, 1] \to C[0, 1]$ defined by

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \ x \in [0, 1], \tag{1}$$

for $n \in \mathbb{N}$ and $f \in C[0, 1]$. These sequence of polynomials possess remarkable properties i.e., positivity, linearity, end point interpolation property and many more, due to this fact these polynomial and their generalization have been intensively studied.

In 1967, Kelisky and Rivlin [18] were the first to investigate the power of the well known Bernstein polynomials, which are defined recursively as $B_n^k(f; x) = B_n(B_n^{k-1}(f; x); x)$ for $k > 1$. They studied the convergence of $B_n^k(f; x)$ as $k \to \infty$, both in the case that k is independent of n and when k is a function of n.

Theorem 1.1. *(Kelisky and Rivlin [18]) If $n \in \mathbb{N}^*$ is fixed, then for all $f \in C[0, 1]$,*

$$\lim_{k\to\infty} B_n^k(f; x) = f(0) + [f(1) - f(0)]x, \quad x \in [0, 1].$$

2010 *Mathematics Subject Classification*. 41A10, 41A36, 47H10
Keywords. Iterates; Contraction principles; Picard iteration; q-Bernstein operator; (p, q)-Bernstein operator
Received: 15 December 2017; Accepted: 15 August 2018
Communicated by Dragan Djordjević
Email addresses: rahmanshagufta14@gmail.com (Shagufta Rahman), mursaleenm@gmail.com (M. Mursaleen), ahalkhaldi@kku.edu.sa (Ali H. Alkhaldi)

Later on several researcher have been investigated iterates properties of Bernstein operator from different point of view [13, 15, 34].
In 1987, Lupas [21] introduced a q-analogue of the Bernstein operator and in 1997 another generalization of these operators based on q-integer was introduced by Phillips [32]. For $q = 1$ these polynomials are the classical ones. In 2002, Oruc and Tuncer [30] and in 2007, Xiang *et al.* [36] studied the convergence properties for iterates of q-Bernstein polynomial. The convergence of $B_n(f, q; x)$ as $q \to \infty$ and the convergence of the iterates $B_n^{j_n}(f, q; x)$ as both $n \to \infty$ and $j_n \to \infty$ have been investigated by S. Ostrovska in 2003 (see [31]). In 2013, Wang and Zhou [35] studied the iterates properties for q-Bernstein Stancu operators.

Fixed point theory is a very wide topic of mathematical research and it has extensive applications in various fields within mathematics as well as outside it. In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the Banach Contraction Principle (BCP) which is one of the most important results of analysis and considered as the main source of metric fixed point theory. Today we have many generalization of this result.
In 2014, Jleli and Samet [16] introduced the class of JS-contraction mapping and generalized the Banach Contraction Principle.
Recently, Altun, Arifi, Jleli, Lashin and Samet generalized the work of Jleli and Samet and gave a fixed point theorem for a new class of JS- contraction mapping (see [9]). They also investigated the iterates property of q-Bernstein-Stancu operators and q-Bernstein-Stancu operators of nonlinear type as an application of Theorem 1.2.

Theorem 1.2. *([9]) Let E be a group with respect to a certain operation $+$. Let X be a subset of E endowed with a certain metric d such that (X, d) is complete. Let $X_0 \subset X$ be a closed subset of X such that X_0 is a subgroup of E. Let $T : X \to X$ be a given mapping satisfying*

$$(x, y) \in X \times X, x - y \in X_0 \Rightarrow d(Tx, Ty) \leq kd(x, y),$$

where $k \in (0, 1)$ is a constant. Suppose that the operation mapping $\pm : X \to X$ defined by

$$\pm(x, y) = x \pm y, \quad (x, y) \in X \times X$$

is continuous with respect to the metric d. Moreover, suppose that

$$x - Tx \in X_0, \quad x \in X.$$

Then we have

1. *For every $x \in X$, the picard sequence $\{T^n x\}$ converges to a fixed point of T.*
2. *For every $x \in X$,*

$$(x + X_0) \cap Fix(T) = \{\lim_{n\to\infty} T^n x\},$$

where $Fix(T)$ is the set of all fixed point of T, that is,

$$Fix(T) = \{x \in X : Tx = x\}.$$

Now, in this paper as an application of Theorem 1.2, we want to extend the study of the iterates of q-Bernstein operators in two dimensional case. Also, our aim is to study the convergence for iterates of (p, q)-Bernstein operators and bivariate (p, q)-Bernstein operators.

2. Iterates of q-Bernstein Operators in Two Dimension

In this section we generalize the work of Kelisky and Rivlin on the power of the q-Bernstein operator to the two dimensional case as an application of Theorem 1.2.

q-calculus plays an important role in approximation theory. We recall some definitions of q-calculus (see [35]).
Let $k \in \mathbb{N}$ and $q \in (0,1)$ then q-integer $[k]_q$ is defined as

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q} & \text{if } q \neq 1, \\ \\ k & \text{if } q = 1. \end{cases}$$

Set $[0]_q = 0$. The q-factorial $[k]_q!$ is defined as

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q & \text{if } k \in \mathbb{N}, \\ \\ 1 & \text{if } k = 0. \end{cases}$$

and for $k \in \mathbb{N}$, q-binomial coefficient $\begin{bmatrix} k \\ r \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} k \\ r \end{bmatrix}_q = \begin{cases} \frac{[k]_q!}{[r]_q![k-r]_q!} & \text{if } 1 \leq r \leq k, \\ \\ 1 & \text{if } r = 0, \\ 0 & \text{if } r > k. \end{cases}$$

The q-analogue of $(1+x)_q^n$ is the polynomial

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)...(1+q^{n-1}x) & \text{if } n = 1,2,..., \\ 1 & \text{if } n = 0. \end{cases}$$

Iterates of q-Bernstein polynomials are defined as

$$B_{n,q}^{M+1}(f;x) = B_{n,q}(B_{n,q}^M(f;x);x) \quad \text{for } M = 1,2,3...$$

and

$$B_{n,q}^1(f;x) = B_{n,q}(f;x).$$

It is obvious that q-calculus reduce to ordinary when $q = 1$.
Bivariate Bernstein polynomial over a triangle was introduced by Lorentz [20]. Let $\triangle$ be the standard triangle in $\mathbb{R}^2$,i.e.,

$$\triangle := \{(x,y) : x, y \geq 0 \text{ and } x + y \leq 1\}.$$

For $n \in \mathbb{N}$, $0 < q < 1$, $f \in C(\triangle)$ and $(x,y) \in \triangle$, define the bivariate q-Bernstein polynomial as

$$B_{n,q,\triangle}(f;x,y) := \sum_{i} \sum_{i+j \leq n} \sum_{j} b_{ij}^q(x,y) f\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big), \tag{2}$$

where

$$b_{ij}^q(x,y) = \frac{[n]_q!}{[i]_q![j]_q![n-i-j]_q!} x^i y^j (1-x-y)_q^{n-i-j}. \tag{3}$$

Lemma 2.1. *Let $e_{i,j}(x,y) = x^i y^j$, where $i, j \in \{0,1\}$ and $i + j < 2$ then for $0 < q < 1$, $(x,y) \in \triangle$, we have*

1. $B_{n,q,\triangle}(e_{0,0};x,y) = 1$,
2. $B_{n,q,\triangle}(e_{1,0};x,y) = x$,
3. $B_{n,q,\triangle}(e_{0,1};x,y) = y$.

Lemma 2.2. *The bivariate q-Bernstein operator over a triangle defined by* (2) *interpolates the function f at each vertex of the triangle, i.e.,*

$$B_{n,q,\triangle}(f;0,0) = f(0,0), \quad B_{n,q,\triangle}(f;1,0) = f(1,0), \quad B_{n,q,\triangle}(f;0,1) = f(0,1).$$

Theorem 2.3. *Let $B_{n,q,\triangle}(f;x,y)$ be the q-Bernstein operator defined in* (2)*. Then for all $f \in C(\triangle;\mathbb{R})$ and all $(x,y) \in \triangle$*

$$\lim_{M\to\infty} B^M_{n,q,\triangle}(f;x,y) = f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y.$$

Proof. Let $X = E = C(\triangle;R)$. We endow X with the metric d defined by

$$d(f,g) = \max\{|f(x,y) - g(x,y)| : (x,y) \in \triangle\}, \quad (f,g) \in X \times X.$$

Then (X,d) is a complete metric space.
Let $X_0 = \{f \in X : f(0,0) = f(0,1) = f(1,0) = 0\}$, then X_0 is a closed linear subspace of X. Let $(f,g) \in X \times X$ such that $f - g \in X_0$, that is,

$$(f,g) \in X \times X \text{ and } f(0,0) = g(0,0), \ f(1,0) = g(1,0), \ f(0,1) = g(0,1).$$

Let $(x,y) \in \triangle$ be fixed. Then, we have

$$\begin{aligned}
&|B_{n,q,\triangle}(f;x,y) - B_{n,q,\triangle}(g;x,y)| \\
&= \Big| \sum_{i+j\le n} b^q_{ij}(x,y) f\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big) - \sum_{i+j\le n} b^q_{ij}(x,y) g\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big)\Big| \\
&= \Big| \sum_{i+j\le n} b^q_{ij}(x,y)\Big(f\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big) - g\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big)\Big)\Big| \\
&\le \sum_{i+j\le n} b^q_{ij}(x,y)\Big|f\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big) - g\Big(\frac{[i]_q}{[n]_q}, \frac{[j]_q}{[n]_q}\Big)\Big| \\
&\le (1 - (b^q_{00}(x,y) + b^q_{10}(x,y) + b^q_{01}(x,y)))d(f,g) \\
&\le (1 - u_n)d(f,g),
\end{aligned}$$

where $u_n = \min\limits_{(x,y)\in\triangle} (b^q_{00}(x,y) + b^q_{10}(x,y) + b^q_{01}(x,y))$. Therefore, we have

$$(f,g) \in X \times X, f - g \in X_0 \Rightarrow d(B_{n,q,\triangle}(f;x,y), B_{n,q,\triangle}(g;x,y)) \le kd(f,g),$$

where $k = 1 - u_n \in (0,1)$.
Next, let $\alpha(x,y) := f(x,y) - B_{n,q,\triangle}(f;x,y), \quad (x,y) \in \triangle$.
We can easily check that

$$\alpha(0,0) = \alpha(1,0) = \alpha(0,1) = 0,$$

which yields

$$f(x,y) - B_{n,q,\triangle}(f;x,y) \in X_0, \quad f \in X.$$

Applying Theorem 1.2, we deduce that

$$(f + X_0) \cap Fix(B_{n,q,\triangle}(f;x,y)) = \{\lim_{M\to\infty} B^M_{n,q,\triangle}(f;x,y)\}.$$

Let $f \in X$, it is not difficult to observe that the function $\lambda : \triangle \to \mathbb{R}$ defined by

$$\lambda(x,y) := f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y, \quad (x,y) \in \triangle,$$

belongs to $Fix(B_{n,q,\triangle}(f;x,y))$. Moreover, for all $(x,y) \in \triangle$,

$$\lambda'(x,y) := \lambda(x,y) - f(x,y) = f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y - f(x,y).$$

Observe that

$$\lambda'(0,0) = \lambda'(1,0) = \lambda'(0,1) = 0.$$

Therefore, $\lambda' \in X_0 \Rightarrow \lambda \in f + X_0$. As consequence, we get

$$\lim_{M\to\infty} B^M_{n,q,\triangle}(f;x,y) = f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y,$$

which yield the desired result. □

Now, we consider bivariate q-Bernstein operator over a square $I = [0,1] \times [0,1]$ (see [10]). For $n, m \in \mathbb{N}$, $0 < q < 1$, $f \in C([0,1]^2)$ and $x, y \in [0,1]$, the bivariate q-Bernstein polynomial is denoted by $B_{n,m,q}(f;x,y)$ and is defined as

$$B_{n,m,q}(f;x,y) := \sum_{k=0}^{n}\sum_{j=0}^{m} b^q_{n,k}(x) b^q_{m,j}(y) f\left(\frac{[k]_q}{[n]_q}, \frac{[j]_q}{[m]_q}\right), \tag{4}$$

where

$$b^q_{n,k}(x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)^{n-k}_q$$

and

$$b^q_{m,j}(y) := \begin{bmatrix} m \\ j \end{bmatrix}_q y^j (1-y)^{m-j}_q.$$

Lemma 2.4. *([10]) Let $e_{i,j}(x,y) = x^i y^j$, $0 \le i + j \le 2$. For $x, y \in [0,1]$, $0 < q < 1$, we have*

1. $B_{n,m,q}(e_{0,0};x,y) = 1$,
2. $B_{n,m,q}(e_{1,0};x,y) = x$,
3. $B_{n,m,q}(e_{0,1};x,y) = y$,
4. $B_{n,m,q}(e_{1,1};x,y) = xy$,
5. $B_{n,m,q}(e_{2,0};x,y) = x^2 + \frac{x(1-x)}{[n]_q}$,
6. $B_{n,m,q}(e_{0,2};x,y) = y^2 + \frac{y(1-y)}{[m]_q}$.

Lemma 2.5. *([10]) The bivariate q-Bernstein operator defined by* (4) *interpolates the function f in the four corners of the unit square, i.e.,*

$$B_{n,m,q}(f;0,0) = f(0,0), \quad B_{n,m,q}(f;0,1) = f(0,1),$$

$$B_{n,m,q}(f;1,0) = f(1,0), \; B_{n,m,q}(f;1,1) = f(1,1).$$

Theorem 2.6. *Let $n, m \in \mathbb{N}^*$, $f \in C([0,1]^2;\mathbb{R})$ and $x, y \in [0,1]$. Then we have*

$$\begin{aligned}\lim_{M\to\infty} B^M_{n,m,q}(f;x,y) &= f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y \\ &\quad + [f(0,0) - f(1,0) - f(0,1) + f(1,1)]xy.\end{aligned}$$

Proof. Let X=E=C($[0,1]^2;\mathbb{R}$). Define metric d on X

$$d(f,g)=\max\{|f(x,y)-g(x,y)|:x,y\in[0,1]\},\ (f,g)\in X\times X,$$

then (X,d) is a complete metric space.
Consider $X_0=\{f\in X: f(0,0)=f(0,1)=f(1,0)=f(1,1)=0\}$, then X_0 is a closed linear subspace of X. Let $(f,g)\in X\times X$ such that $f-g\in X_0$, that is,

$$(f,g)\in X\times X \text{ and } f(0,0)=g(0,0),\ \ f(1,0)=g(1,0),\ \ f(0,1)=g(0,1),\ \ f(1,1)=g(1,1).$$

Let $x,y\in[0,1]$ be fixed. Then we have

$$\begin{aligned}
&|B_{n,m,q}(f;x,y)-B_{n,m,q}(g;x,y)|\\
&=\Big|\sum_{k=0}^{n}\sum_{j=0}^{m}b^q_{n,k}(x)b^q_{m,j}(y)f\Big(\frac{[k]_q}{[n]_q},\frac{[j]_q}{[m]_q}\Big)-\sum_{k=0}^{n}\sum_{j=0}^{m}b^q_{n,k}(x)b^q_{m,j}(y)g\Big(\frac{[k]_q}{[n]_q},\frac{[j]_q}{[m]_q}\Big)\Big|\\
&=\Big|\sum_{k=0}^{n}\sum_{j=0}^{m}b^q_{n,k}(x)b^q_{m,j}(y)\Big\{f\Big(\frac{[k]_q}{[n]_q},\frac{[j]_q}{[m]_q}\Big)-g\Big(\frac{[k]_q}{[n]_q},\frac{[j]_q}{[m]_q}\Big)\Big\}\Big|\\
&\le\sum_{k=0}^{n}\sum_{j=0}^{m}b^q_{n,k}(x)b^q_{m,j}(y)\Big|f\Big(\frac{[k]_q}{[n]_q},\frac{[j]_q}{[m]_q}\Big)-g\Big(\frac{[k]_q}{[n]_q},\frac{[j]_q}{[m]_q}\Big)\Big|\\
&\le(1-(b^q_{n,0}(x)b^q_{m,0}(y)+b^q_{n,0}(x)b^q_{m,m}(y)+b^q_{n,n}(x)b^q_{m,0}(y)+b^q_{n,n}(x)b^q_{m,m}(y)))d(f,g)\\
&\le(1-u_{n,m})d(f,g),
\end{aligned}$$

where

$$u_{n,m}=\min_{x,y\in[0,1]}(b^q_{n,0}(x)b^q_{m,0}(y)+b^q_{n,0}(x)b^q_{m,m}(y)+b^q_{n,n}(x)b^q_{m,0}(y)+b^q_{n,n}(x)b^q_{m,m}(y))>0.$$

Therefore, we have

$$(f,g)\in X\times X, f-g\in X_0\Rightarrow d(B_{n,m,q}(f;x,y),B_{n,m,q}(g;x,y))\le k'd(f,g),$$

where $k'=(1-u_{n,m})\in(0,1)$.
Next, let $\beta(x,y):=f(x,y)-B_{n,m,q}(f;x,y),\ x,y\in[0,1]$.
We can easily check that

$$\beta(0,0)=\beta(0,1)=\beta(1,0)=\beta(1,1)=0,$$

which yields

$$f(x)-B_{n,m,q}(f;x,y)\in X_0,\quad f\in X.$$

By Theorem 1.2, we deduce that

$$(f+X_0)\cap Fix(B_{n,m,q}(f;x,y))=\{\lim_{M\to\infty}B^M_{n,m,q}(f;x,y)\}.$$

Let $f\in X,\ x,y\in[0,1]$ it is not difficult to observe that the function $\mu:[0,1]\times[0,1]\to\mathbb{R}$ defined by

$$\begin{aligned}
\mu(x,y)\ &=\ f(0,0)+[f(1,0)-f(0,0)]x+[f(0,1)-f(0,0)]y\\
&\quad+[f(0,0)-f(1,0)-f(0,1)+f(1,1)]xy,
\end{aligned}$$

belongs to $Fix(B_{n,m,q}(f;x,y))$. Moreover, for all $x,y\in[0,1]$,

$$\begin{aligned}
\mu'(x,y)\ &:=\ \mu(x,y)-f(x,y)\\
&=\ f(0,0)+[f(1,0)-f(0,0)]x+[f(0,1)-f(0,0)]y\\
&\quad+[f(0,0)-f(1,0)-f(0,1)+f(1,1)]xy-f(x,y).
\end{aligned}$$

Observe that

$$\mu'(0,0) = \mu'(1,0) = \mu'(0,1) = \mu'(1,1) = 0.$$

Therefore, $\mu' \in X_0 \Rightarrow \mu \in f + X_0$, As consequence, we get

$$\lim_{M\to\infty} d(B^M_{n,m,q}(f;x,y),\mu) = 0,$$

which completes the proof. □

3. (p,q)-Bernstein Operator

During the last two decades, the application of q-calculus have emerged as a new area in the field of approximation theory, and further (p,q)-calculus is a new generalization of the q-calculus. Recently, Mursaleen *et al.* [24, 25] applied (p,q)-calculus in approximation theory. For further developments of (p,q)-approximation one can refer to [1–6, 12, 14, 19, 22, 23, 26, 27, 29]. Let us recall some basic notations of (p,q)-calculus:
The (p,q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p-q},\ n = 0,1,2,...,\ 0 < q < p \leq 1.$$

The (p,q)-factorial $[n]_{p,q}!$ and (p,q)-binomial coefficients $\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}$ are defined as follows

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q} & \text{if } n \in \mathbb{N}, \\ 1 & \text{if } n = 0. \end{cases}$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \begin{cases} \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} & \text{if } 1 \leq k \leq n, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k > n. \end{cases}$$

Also, the (p,q)-binomial expansion is

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k,$$

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)...(p^{n-1}x+q^{n-1}y).$$

For $p = 1$, all the notions of (p,q)-calculus are reduced to q-calculus.
Mursaleen *et al.* [24, 25] introduced the (p,q)-analogue of Bernstein operators as follows:

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} b^{p,q}_{n,k}(x) f\Big(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\Big),\quad x \in [0,1], \tag{5}$$

where

$$b^{p,q}_{n,k}(x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{j=0}^{n-k-1}(p^j - q^j x).$$

Note that for $p = q = 1$, the (p,q)-Bernstein operators given by (5) turn out to be Bernstein operator given by (1).
We have the following basic result:

Lemma 3.1. *For* $x \in [0,1],\ 0 < q < p \leq 1$, *we have*

1. $B_{n,p,q}(e_0; x) = 1$,
2. $B_{n,p,q}(e_1; x) = x$,
3. $B_{n,p,q}(e_2; x) = \frac{p^{n-1}}{[n]_{p,q}}x + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2$,

where $e_i = t^i$ *for i=0,1,2.*

4. Iterates of (p, q)-Bernstein Operators

The iterates of positive and linear operators in various classes were intensively investigated in the last decades. The convergence of linear operators using the fixed point theory was introduced by Agratini and Rus [7, 8, 34]. Recently in 2016, Radu [33] investigate the convergence of iterates of q-Bernstein polynomial and (p,q)-Bernstein polynomial by using contraction principle.
Iterates of (p,q)-Bernstein polynomial are defined as

$$B^{M+1}_{n,p,q}(f; x) = B_{n,p,q}(B^{M}_{n,p,q}(f; x); x),\ \text{for}\ M = 1, 2, 3...$$

and

$$B^{1}_{n,p,q}(f; x) = B_{n,p,q}(f; x).$$

Lemma 4.1. *Let* $n \in \mathbb{N}^*,\ 0 < q < p \leq 1$. *Then,* $\sum\limits_{k=0}^{n} b^{p,q}_{n,k}(x) = 1$.

Lemma 4.2. *Let* $n \in \mathbb{N}^*$ *and* $0 < q < p \leq 1$, *then* $\min\limits_{x\in[0,1]}(b^{p,q}_{n,0}(x), b^{p,q}_{n,1}(x)) > 0$.

Lemma 4.3. (p,q)*-Bernstein polynomial defined by* (5) *possess the end point interpolation property, i.e.,*

$$B_{n,p,q}(f; 0) = f(0),\ B_{n,p,q}(f; 1) = f(1).$$

4.1. Kelisky-Rivlin Type Result for Linear (p,q)*-Bernstein Operator:*

Theorem 4.4. *Let* $n \in \mathbb{N}^*,\ 0 < q < p \leq 1$. *Then for every* $f \in C([0,1];\mathbb{R})$

$$\lim_{M\to\infty} B^{M}_{n,p,q}(f; x) = f(0) + [f(1) - f(0)]x, \quad x \in [0,1].$$

Proof. Let $X = E = C([0,1], \mathbb{R})$. Define

$$d(f, g) = \max\{|f(x) - g(x)| : x \in [0,1]\}, \quad (f, g) \in X \times X.$$

Then X endowed with metric d is a complete metric space.
Consider $X_0 = \{f \in X : f(0) = f(1) = 0\}$, then X_0 is a closed linear subspace of X. Let $(f, g) \in X \times X$ such that $f - g \in X_0$, that is,

$$(f, g) \in X \times X \text{ and } f(0) = g(0),\ f(1) = g(1).$$

Let $x \in [0,1]$ be fixed. Then we have

$$\begin{aligned}
& |B_{n,p,q}(f;x) - B_{n,p,q}(g;x)| \\
= & \left| \sum_{k=0}^{n} b_{n,k}^{p,q}(x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) - \sum_{k=0}^{n} b_{n,k}^{p,q}(x) g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \right| \\
= & \left| \sum_{k=0}^{n} b_{n,k}^{p,q}(x) \left(f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) - g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \right) \right| \\
= & \left| \sum_{k=1}^{n-1} b_{n,k}^{p,q}(x) \left(f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) - g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \right) \right| \\
\leq & \sum_{k=1}^{n-1} b_{n,k}^{p,q}(x) \left| f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) - g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \right| \\
\leq & \left(\sum_{k=1}^{n-1} b_{n,k}^{p,q}(x) \right) d(f,g).
\end{aligned}$$

Let

$$v_n = \min_{x \in [0,1]} (b_{n,0}^{p,q}(x) + b_{n,1}^{p,q}(x)), \tag{6}$$

then by using Lemma 4.1, we get $v_n > 0$ and

$$\begin{aligned}
\sum_{k=1}^{n-1} b_{n,k}^{p,q}(x) &= 1 - (b_{n,0}^{p,q}(x) + b_{n,n}^{p,q}(x)) \\
&\leq 1 - v_n.
\end{aligned}$$

Therefore, we have

$$(f,g) \in X \times X, f - g \in X_0 \Rightarrow d(B_{n,p,q}(f;x), B_{n,p,q}(g;x)) \leq t d(f,g),$$

where $t = 1 - v_n \in (0,1)$.
Next, let $\gamma(x) := f(x) - B_{n,p,q}(f;x),\ x \in [0,1]$.
We can easily check that

$$\gamma(0) = \gamma(1) = 0,$$

which yields

$$f(x) - B_{n,p,q}(f;x) \in X_0,\ f \in X.$$

Applying Theorem 1.2, we find that

$$(f + X_0) \cap Fix(B_{n,p,q}(f;x)) = \{\lim_{M \to \infty} B_{n,p,q}^{M}(f;x)\}$$

Let $f \in X$, it is not difficult to observe that the function $\omega : [0,1] \to \mathbb{R}$ defined by

$$\omega(x) = f(0) + [f(1) - f(0)]x,\ x \in [0,1],$$

belongs to $Fix(B_{n,p,q}(f;x))$. Moreover, for all $x \in [0,1]$

$$\omega'(x) := \omega(x) - f(x) = f(0)(1-x) + f(1)x - f(x)$$

Observe that

$$\omega'(0) = f(0) - f(0) = 0$$

and

$$\omega'(1) = f(1) - f(1) = 0.$$

Therefore, $\omega' \in X_0 \Rightarrow \omega \in f + X_0$. As consequence, we get

$$\lim_{M\to\infty} d(B^M_{n,p,q}(f;x), \omega) = 0,$$

which is the required result. □

4.2. Kelisky-Rivlin Type Result for Non-linear (p,q)-Bernstein Operator:

For $f \in C[0,1]$, $0 < q < p \leq 1$, and $n \in \mathbb{N}$, we define the non-linear (p,q)-Bernstein operators of degree n by

$$B^*_{n,p,q}(f;x) = \sum_{k=0}^{n} b^{p,q}_{n,k}(x)\left|f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right|, \quad x \in [0,1], \tag{7}$$

where

$$b^{p,q}_{n,k}(x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{j=0}^{n-k-1}(p^j - q^j x).$$

Theorem 4.5. *Let $n \in \mathbb{N}^*$, $0 < q < p \leq 1$. Then for every $f \in C([0,1];\mathbb{R})$*

$$\lim_{M\to\infty} (B^*_{n,p,q})^M(f;x) = f(0) + [f(1) - f(0)]x, \quad x \in [0,1].$$

Proof. Let $E = C([0,1],\mathbb{R})$ and X be defined by

$$X = \{f \in E : f(0) \geq 0, f(1) \geq 0\} \subseteq E.$$

We endow X with the metric d defined by

$$d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}, \quad (f,g) \in X \times X.$$

Then (X,d) is a complete metric space.
Define $X_0 = \{f \in X : f(0) = f(1) = 0\}$, then X_0 is a closed subgroup of X. Let $(f,g) \in X \times X$ such that $f - g \in X_0$. Fix $x \in [0,1]$, then we have

$$\begin{aligned}
& |B^*_{n,p,q}(f;x) - B^*_{n,p,q}(g;x)| \\
= & \left|\sum_{k=0}^{n} b^{p,q}_{n,k}(x)\left|f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right| - \sum_{k=0}^{n} b^{p,q}_{n,k}(x)\left|g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right|\right| \\
= & \left|\sum_{k=0}^{n} b^{p,q}_{n,k}(x)\left(\left|f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right| - \left|g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right|\right)\right| \\
= & \left|\sum_{k=1}^{n-1} b^{p,q}_{n,k}(x)\left(\left|f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right| - \left|g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right|\right)\right| \\
\leq & \sum_{k=1}^{n-1} b^{p,q}_{n,k}(x)\left|f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) - g\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)\right| \\
\leq & \left(\sum_{k=1}^{n-1} b^{p,q}_{n,k}(x)\right)d(f,g) \\
\leq & (1 - v_n)d(f,g),
\end{aligned}$$

where v_n is given by equation (6). Therefore, we have

$$(f,g) \in X \times X, f - g \in X_0 \Rightarrow d(B^*_{n,p,q}(f;x), B^*_{n,p,q}(g;x)) \leq td(f,g),$$

where $t = 1 - v_n \in (0,1)$.
For $x \in [0,1]$, let $\delta(x) := f(x) - B^*_{n,p,q}(f;x)$, then we have

$$\delta(0) = f(0) - |f(0)| = 0,$$

$$\delta(1) = f(1) - |f(1)| = 0,$$

which yields

$$f(x) - B^*_{n,p,q}(f;x) \in X_0,\ f \in X.$$

Applying Theorem 1.2, we have

$$(f + X_0) \cap Fix(B^*_{n,p,q}(f;x)) = \{\lim_{M\to\infty} (B^*_{n,p,q}(f;x))^M\}.$$

For $f \in X$, define the function $\omega : [0,1] \to \mathbb{R}$ by

$$\omega(x) = f(0) + [f(1) - f(0)]x,\ x \in [0,1].$$

We can easily observe that $\omega \in Fix(B^*_{n,p,q}(f;x)) \cap (f + X_0)$. As consequence, we get

$$\lim_{M\to\infty} d((B^*_{n,p,q}(f;x))^M, \omega) = 0,$$

which yield the desired result. □

5. Iterates of Bivariate (p,q)-Bernstein Operator

In 2016, Karaisa [17] defined the bivariate Bernstein operators based on (p,q)-integer and investigated their approximation property. Let $I = [0,1] \times [0,1]$, $f : I \to \mathbb{R}$ and $0 < q_1, q_2 < p_1, p_2 \leq 1$, then the bivariate Bernstein operator is denoted by $B^{(p_1,q_1)(p_2,q_2)}_{n,m}$ and is defined as follows:

$$B^{(p_1,q_1)(p_2,q_2)}_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} b^{p_1,q_1}_{n,k}(x) b^{p_2,q_2}_{m,j}(y) f\Big(\frac{[k]_{p_1,q_1}}{p_1^{k-n}[n]_{p_1,q_1}}, \frac{[j]_{p_2,q_2}}{p_2^{j-m}[m]_{p_2,q_2}}\Big), \tag{8}$$

where

$$b^{p_1,q_1}_{n,k}(x) = \frac{1}{p_1^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_1,q_1} p_1^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p_1^s - q_1^s x),$$

$$b^{p_2,q_2}_{m,j}(y) = \frac{1}{p_2^{\frac{m(m-1)}{2}}} \begin{bmatrix} m \\ j \end{bmatrix}_{p_2,q_2} p_2^{\frac{j(j-1)}{2}} y^j \prod_{s=0}^{m-j-1} (p_2^s - q_2^s y).$$

Lemma 5.1. *([17]) Let $e_{i,j}(x,y) = x^i y^j$, $0 \leq i + j \leq 2$. For $x, y \in [0,1]$, $0 < q_1, q_2 < p_1, p_2 \leq 1$, we have*

1. $B^{(p_1,q_1)(p_2,q_2)}_{n,m}(e_{0,0}; x, y) = 1,$
2. $B^{(p_1,q_1)(p_2,q_2)}_{n,m}(e_{1,0}; x, y) = x,$
3. $B^{(p_1,q_1)(p_2,q_2)}_{n,m}(e_{0,1}; x, y) = y,$
4. $B^{(p_1,q_1)(p_2,q_2)}_{n,m}(e_{1,1}; x, y) = xy,$

5. $B_{n,m}^{(p_1,q_1)(p_2,q_2)}(e_{2,0};x,y) = \frac{p_1^{n-1}}{[n]_{p_1,q_1}}x + \frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}x^2,$
6. $B_{n,m}^{(p_1,q_1)(p_2,q_2)}(e_{0,2};x,y) = \frac{p_2^{m-1}}{[m]_{p_2,q_2}}y + \frac{q_2[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}y^2.$

Lemma 5.2. *The generalized bivariate (p,q)-Bernstein operator defined by* (8) *interpolates the function f in the four corners of the unit square, i.e.,*

$$B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f;0,0) = f(0,0), \quad B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f;0,1) = f(0,1),$$

$$B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f;1,0) = f(1,0), \quad B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f;1,1) = f(1,1).$$

Theorem 5.3. *Let $n,m \in \mathbb{N}^*$, $0 < q_1,q_2 < p_1,p_2 \le 1$ and $x,y \in [0,1]$. Then for every $f \in C([0,1]^2;\mathbb{R})$*

$$\begin{aligned}\lim_{M\to\infty}(B_{n,m}^{(p_1,q_1)(p_2,q_2)})^M(f;x,y) &= f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y \\ &\quad + [f(0,0) - f(1,0) - f(0,1) + f(1,1)]xy.\end{aligned}$$

Proof. Let $X = E = C([0,1]^2,\mathbb{R})$. Define $d : X \times X \to \mathbb{R}$ by

$$d(f,g) = \max\{|f(x,y) - g(x,y)| : x,y \in [0,1]\}.$$

Then d is a metric on X and (X,d) is a complete metric space.
Let $X_0 = \{f \in X : f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0\}$, then X_0 is a closed linear subspace of X. Let $(f,g) \in X \times X$ such that $f - g \in X_0$, i.e.,

$$f(0,0) = g(0,0),\ f(1,0) = g(1,0),\ f(0,1) = g(0,1),\ f(1,1) = g(1,1).$$

Let $x,y \in [0,1]$ be fixed. Then we have

$$\begin{aligned} & |B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f;x,y) - B_{n,m}^{(p_1,q_1)(p_2,q_2)}(g;x,y)| \\ = & \Big|\sum_{k=0}^{n}\sum_{j=0}^{m} b_{n,k}^{p_1,q_1}(x) b_{m,j}^{p_2,q_2}(y)\Big\{f\Big(\frac{[k]_{p_1,q_1}}{p_1^{k-n}[n]_{p_1,q_1}}, \frac{[j]_{p_2,q_2}}{p_2^{j-m}[m]_{p_2,q_2}}\Big) \\ & -g\Big(\frac{[k]_{p_1,q_1}}{p_1^{k-n}[n]_{p_1,q_1}}, \frac{[j]_{p_2,q_2}}{p_2^{j-m}[m]_{p_2,q_2}}\Big)\Big\}\Big| \\ \le & \sum_{k=0}^{n}\sum_{j=0}^{m} b_{n,k}^{p_1,q_1}(x) b_{m,j}^{p_2,q_2}(y)\Big|f\Big(\frac{[k]_{p_1,q_1}}{p_1^{k-n}[n]_{p_1,q_1}}, \frac{[j]_{p_2,q_2}}{p_2^{j-m}[m]_{p_2,q_2}}\Big) \\ & -g\Big(\frac{[k]_{p_1,q_1}}{p_1^{k-n}[n]_{p_1,q_1}}, \frac{[j]_{p_2,q_2}}{p_2^{j-m}[m]_{p_2,q_2}}\Big)\Big| \\ \le & (1 - (b_{n,0}^{p_1,q_1}(x)b_{m,0}^{p_2,q_2}(y) + b_{n,0}^{p_1,q_1}(x)b_{m,m}^{p_2,q_2}(y) + b_{n,n}^{p_1,q_1}(x)b_{m,0}^{p_2,q_2}(y) + b_{n,n}^{p_1,q_1}(x)b_{m,m}^{p_2,q_2}(y))) \\ & \times d(f,g) \\ \le & (1 - v_{n,m})d(f,g), \end{aligned}$$

where

$$v_{n,m} = \min_{x,y\in[0,1]}(b_{n,0}^{p_1,q_1}(x)b_{m,0}^{p_2,q_2}(y) + b_{n,0}^{p_1,q_1}(x)b_{m,m}^{p_2,q_2}(y) + b_{n,n}^{p_1,q_1}(x)b_{m,0}^{p_2,q_2}(y) + b_{n,n}^{p_1,q_1}(x)b_{m,m}^{p_2,q_2}(y)).$$

Therefore, we have

$$(f,g) \in X \times X, f - g \in X_0 \Rightarrow d(B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f;x,y), B_{n,m}^{(p_1,q_1)(p_2,q_2)}(g;x,y)) \le t' d(f,g),$$

where $t' = (1 - v_{n,m}) \in (0, 1)$.
Let $\eta(x, y) := f(x, y) - B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f; x, y), \quad x, y \in [0, 1]$, then

$$\eta(0, 0) = \eta(0, 1) = \eta(1, 0) = \eta(1, 1) = 0,$$

which yields

$$f(x) - B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f; x, y) \in X_0.$$

By Theorem 1.2, we find that

$$(f + X_0) \cap Fix(B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f; x, y)) = \{\lim_{M\to\infty} (B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f; x, y))^M\}.$$

For $f \in X$ and all $x, y \in [0, 1]$, we can easily check that the function $\varphi : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$\begin{aligned} \varphi(x, y) &= f(0, 0) + [f(1, 0) - f(0, 0)]x + [f(0, 1) - f(0, 0)]y \\ &\quad +[f(0, 0) - f(1, 0) - f(0, 1) + f(1, 1)]xy, \end{aligned}$$

belongs to $Fix(B_{n,m}^{(p_1,q_1)(p_2,q_2)}(f; x, y))$. Moreover, for all $x, y \in [0, 1]$

$$\begin{aligned} \varphi'(x, y) &:= \varphi(x, y) - f(x, y) \\ &= f(0, 0) + [f(1, 0) - f(0, 0)]x + [f(0, 1) - f(0, 0)]y \\ &\quad +[f(0, 0) - f(1, 0) - f(0, 1) + f(1, 1)]xy - f(x, y). \end{aligned}$$

Observe that

$$\varphi'(0, 0) = \varphi'(1, 0) = \varphi'(0, 1) = \varphi'(1, 1) = 0.$$

Therefore, $\varphi' \in X_0 \Rightarrow \varphi \in f + X_0$. As consequence, we get

$$\begin{aligned} \lim_{M\to\infty} (B_{n,m}^{(p_1,q_1)(p_2,q_2)})^M(f; x, y) &= f(0, 0) + [f(1, 0) - f(0, 0)]x + [f(0, 1) - f(0, 0)]y \\ &\quad +[f(0, 0) - f(1, 0) - f(0, 1) + f(1, 1)]xy. \end{aligned}$$

□

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