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Application of Double Natural Decomposition Method for Solving Singular One Dimensional Boussinesq Equation

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Abstract. In this work, a combined form of the double Natural transform method with the Adomian decomposition method is developed for analytic treatment of the linear and nonlinear singular one dimensional Boussinesq equations. Two examples are provided to illustrate the simplicity and reliability of this method. Moreover, the results show that the proposed method is effective and is easy to implement for certain problems in science and engineering.

1. introduction

The nonlinear one dimensional Boussinesq equation is used to model flows of water in unconfined aquifers. Recently many researchers are interested in seeking analytical solutions of nonlinear ordinary and partial differential equations. Many research works have adopted the method of integral transform in order to solve certain of partial differential equations, for example in [16, 18, 19], double Sumudu transform is used to an evolution equation of population dynamic. The powerful mathematical methods such as Adomian decomposition method (ADM) [1–4], the modified double Laplace decomposition method [6–8] have been suggested for obtaining the exact and approximate analytic solutions of nonlinear problems. Construction of soliton solutions and periodic solution of Boussinesq equation by modified decomposition method are given in [9, 10] and a solitary wave solution of the Boussinesq equation with power law nonlinearity has been derived in [11]. Natural transform was first introduced by [5] and its properties were studied by [13]. Natural Decomposition Method was used to solve coupled system of nonlinear pde's [12]. In [15] the Natural Homotopy Perturbation Method has been successfully applied to linear and nonlinear Schrodinger equation. Recently double Natural transform has been applied to solve telegraph, wave and partial integro-differential equations see [14]. The aim of this paper is to propose an analytic solution of the singular one dimensional Boussinesq equation by using a modified double Natural decomposition method (MDNDM). This analytical technique basically confirms how the double Natural transform can be used to approximate the solutions of the nonlinear differential equations with the linearization of non-linear terms by using Adomian polynomials. We now recall the following definition of the Natural transform given by [5]. Over the set of functions

$$A = \left\{ \begin{array}{c} f(t) : \exists M, \tau_1, \tau_2 > 0, \text{ such that} \\ |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ , if } t \in (-1)^j \times [0, \infty), \; j = 1, 2. \end{array} \right\}$$

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the Natural transform is defined by

$$\mathbf{N}^{+}\left[f\left(t\right)\right]=R\left(s;u\right)=\frac{1}{u}\int_{0}^{\infty}e^{-\frac{s}{u}t}f(t)dt\quad Re\left(s\right),Re\left(u\right)>0, \tag{1}$$

where the variables u and s are complex variables of the natural transform.

2. Properties of Double Natural Transform

In this section, we discuss the basic concepts and properties of double Natural transform used in the sequel.

$$\begin{aligned}\mathbf{N}^{+}_{x,t}\left[f\left(x,t\right)\right] &= \frac{1}{uv}\int_{0}^{\infty}\int_{0}^{\infty}e^{-\frac{p}{u}x-\frac{s}{v}t}f(x,t)dt\,dx,\\ Re\left(s\right),Re\left(p\right) &> 0,\ Re\left(u\right),Re\left(v\right)>0.\end{aligned} \tag{2}$$

We can rewrite Eq.(2) in other form as

$$\mathbf{N}^{+}_{x,t}\left[f\left(x,t\right)\right]=\int_{0}^{\infty}\int_{0}^{\infty}e^{-px-st}f\left(ux,vt\right)dt\,dx. \tag{3}$$

where $\mathbf{N}^{+}_{x,t}$ indicates double Natural transform and. Double inverse Natural transform is defined by

$$\mathbf{N}^{-1}_{p,s;u,v}\left(\mathbf{N}^{+}_{x,t}\left[f\left(x,t\right)\right]\right)=f\left(x,t\right)=\frac{1}{(2\pi i)^{2}}\int_{\alpha-i\infty}^{\alpha-i\infty}\int_{\beta-i\infty}^{\beta-i\infty}e^{\frac{p}{u}x+\frac{s}{v}t}\mathbf{N}^{+}_{x,t}\left[f\left(x,t\right)\right]ds\,dp,$$

where $\mathbf{N}^{-1}_{p,s;u,v}$ indicates double inverse Natural transform. The reader can read more about the Double Natural transform in [14].

Example 1: [14] Double Natural transform of the function $f(x,t)=e^{i(ax+bt)}$ is given by

$$\begin{aligned}\mathbf{N}^{+}_{x,t}\left[e^{i(ax+bt)}\right] &= \frac{1}{(p-aui)(s-bvi)}\\ &= \frac{ps-abuv+(aus+pbv)\,i}{(p^{2}+a^{2}u^{2})(s^{2}+b^{2}v^{2})}.\end{aligned}$$

Consequently,

$$\mathbf{N}^{+}_{x,t}\left[\cos(ax+bt)\right]=\frac{ps-abuv}{(p^{2}+a^{2}u^{2})(s^{2}+b^{2}v^{2})},$$

and

$$\mathbf{N}^{+}_{x,t}\left[\sin(ax+bt)\right]=\frac{aus+pbv}{(p^{2}+a^{2}u^{2})(s^{2}+b^{2}v^{2})}.$$

Example 2: [14] The double Natural transform of $f(x,t)=(xt)^{n}$ is given by

$$\mathbf{N}^{+}_{x,t}\left[(xt)^{n}\right]=\frac{(n!)^{2}\,u^{n}v^{n}}{p^{n+1}s^{n+1}},$$

where n is a positive integer. If $a(>-1)$ and $b(>-1)$ are real numbers, then

$$\mathbf{N}^{+}_{x,t}\left[x^{a}t^{b}\right]=\frac{u^{a}v^{b}\Gamma\left(a+1\right)\Gamma\left(b+1\right)}{p^{a+1}s^{b+1}},$$

then it follows from the definition of double natural transform that

$$\begin{aligned}\mathbf{N}^{+}_{x,t}\left[x^a t^b\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\frac{p}{u}x-\frac{s}{v}t}x^a t^b dt\,dx\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{p}{u}x}x^a\left(\frac{1}{v}\int_0^\infty e^{-\frac{s}{v}t}t^b dt\right)dx,\end{aligned}$$

by substituting $\frac{p}{u}x = r$, and $\frac{s}{v}t = q$, one gets

$$\begin{aligned}\mathbf{N}^{+}_{x,t}\left[x^a t^b\right] &= \frac{1}{u}\int_0^\infty\left(\frac{u}{p}r\right)^a e^{-r}\frac{u}{p}dr\frac{1}{v}\int_0^\infty e^{-\frac{s}{v}t}\left(\frac{v}{s}q\right)^b e^{-q}\frac{v}{s}dq\\ &= \frac{u^a}{p^{a+1}}\frac{v^b}{s^{b+1}}\int_0^\infty\int_0^\infty r^a q^b e^{-r}e^{-q}drdq\\ &= \frac{u^a v^b\Gamma(a+1)\Gamma(b+1)}{p^{a+1}s^{b+1}},\end{aligned}$$

where, gamma functions of a and b are defined by the uniformly convergent integral as follows.

$$\Gamma(a)\Gamma(b) = \int_0^\infty e^{-r}r^{a-1}dx\int_0^\infty e^{-q}q^{b-1}dt,\ ,a>0,\ b>0.$$

Example 3: The double natural transform for the following function

$$f(x,t) = H(x)\otimes H(t)\ln x\ln t,$$

is given by

$$\mathbf{N}^{+}_{x,t}\left[H(x)\otimes H(t)\ln x\ln t\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\frac{p}{u}x-\frac{s}{v}t}\ln x\ln t dt\,dx,$$

where the $H(x,t) = H(t)\otimes H(x)$ is a Heaviside function and $\otimes$ is a tensor product see [17] . let $\zeta = \frac{p}{u}x$ and $\eta = \frac{p}{u}t$ the integral becomes

$$\begin{aligned}\mathbf{N}^{+}_{x,t}\left[H(x)\otimes H(t)\ln x\ln t\right] &= \frac{1}{ps}\int_0^\infty e^{-\eta}\ln\left(\frac{v}{s}\eta\right)\left(\int_0^\infty e^{-\zeta}\ln\left(\frac{u}{p}\zeta\right)d\zeta\right)d\eta\\ &= \frac{1}{ps}(\gamma-\ln u+\ln p)(\gamma-\ln v+\ln s),\end{aligned}\tag{4}$$

where Euler constant defined by $\gamma = \int_0^\infty e^{-\eta}\ln(\eta)\,d\eta = 0.5772\cdot$ Also the Euler Mascheroni constant γ is defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j}-\log n\right).$$

More detail about Euler constant see [20]. We can use the result of the above example to evaluate the natural transform of the singular function $Pf[H(x)\otimes H(t)/xt]$. Where

$$Pf[H(x)\otimes H(t)/xt] = \frac{\partial^2}{\partial x\partial t}(H(x)\otimes H(t)\ln x\ln t)$$

we obtain, from Eq.(4)

$$\begin{aligned}\mathbf{N}^{+}_{x,t}\left(Pf[H(x)\otimes H(t)/xt]\right) &= \mathbf{N}^{+}_{x,t}\left(\frac{\partial^2}{\partial x\partial t}(H(x)\otimes H(t)\ln x\ln t)\right)\\ &= \frac{1}{uv}(\gamma-\ln u+\ln p)(\gamma-\ln v+\ln s)\end{aligned}$$

Existence Condition for the double Natural transform:

If $f(x,t)$ is an exponential order a and b as $x \to \infty$, $t \to \infty$, if there exists a positive constant K such that for all $x > X$ and $t > T$

$$\left|f(x,t)\right| \leq Ke^{ax+bt}, \tag{5}$$

it is easy to get,

$$f(x,t) = O(e^{ax+bt}) \quad \text{as} \quad x \to \infty, t \to \infty,$$

Or, equivalently,

$$\lim_{\substack{x\to\infty \\ t\to\infty}} e^{-\frac{\alpha}{u}x-\frac{\beta}{v}t}\left|f(x,t)\right| = K \lim_{\substack{x\to\infty \\ t\to\infty}} e^{-\left(\frac{\alpha}{u}-a\right)x-\left(\frac{\beta}{v}-b\right)t} = 0,$$

where $\frac{\alpha}{u} > a$ and $\frac{\beta}{v} > a$. The function $f(x,t)$ is called an exponential order as $x \to \infty, t \to \infty$, and clearly, it does not grow faster than $K\, e^{ax+bt}$ as $x \to \infty, t \to \infty$.

Theorem 2.1. *If a function $f(x,t)$ is a continuous function in every finite interval $(0,X)$ and $(0,T)$ and of exponential order e^{ax+bt}, then the double natural transform of $f(x,t)$ which is defined by $\mathbf{N}^+_{x,t}\left[f(x,t)\right]$ exists for all $p > \alpha, s > \beta$ and $u \neq 0, v \neq 0$.*

Proof. From the definition of the double natural transform of $f(x,t)$, we have

$$\begin{aligned}
\left|\mathbf{N}^+_{x,t}\left[f(x,t)\right]\right| &= \left|\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\frac{p}{u}x-\frac{s}{v}t} f(x,t)dt\,dx\right| \\
&\leq K\left|\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}-a\right)x-\left(\frac{s}{v}-b\right)t}dt\,dx\right| \\
&= \frac{k}{(p-au)(s-bv)}.
\end{aligned} \tag{6}$$

From Eq.(6) we have

$$\lim_{\substack{x\to\infty \\ t\to\infty}} \left|\mathbf{N}^+_{x,t}\left[f(x,t)\right]\right| = 0 \text{ or } \lim_{\substack{x\to\infty \\ t\to\infty}} R\left((p,s);(u,v)\right) = 0.$$

□

Theorem 2.2. *[14] Let $f(x,t)$, be a periodic function in A of the period $T > 0$ and $K > 0$. Then the double natural transform of $f(x,t)$ is given by,*

$$\mathbf{N}^+_{x,t}\left[f(x,t)\right] = \frac{\frac{1}{uv}\int_0^T\int_0^K e^{-px-st} f(ux,vt)dt\,dx}{\left(1-e^{-\frac{Kp}{u}}\right)\left(1-e^{-\frac{Ts}{v}}\right)}.$$

The natural transform of the convolution product:

Theorem 2.3. *Let $f(x)$ and $g(x)$ be integrable functions, if the convolution of $f(x)$ and $g(x)$ is given by*

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx. \tag{7}$$

then Natural transform of the convolution product is defined as follow

$$\mathbf{N}^+ [f(t) * g(t)] = uF(s;u)\, G(s;u). \tag{8}$$

Where $F(s;u)$ and $G(s;u)$ are Natural transform of the functions $f(t)$ and $g(t)$ respectively.

Theorem 2.4. *[14] Let $f(x,t)$ and $g(x,t)$ having double Natural transform. Then double Natural transform of the double convolution of the $f(x,t)$ and $g(x,t)$,*

$$(f * * g)(x,t) = \int_0^x \int_0^t f(\zeta,\eta)\, g(x-\zeta, t-\eta)\, d\zeta d\eta, \tag{9}$$

denoted by

$$\mathbf{N}^+_{x,t} [f(x,t) * * g(x,t)] = uvF((p,s);(u,v))\, G((p,s);(u,v)). \tag{10}$$

Where $F((p,s);(u,v))$ and $G((p,s);(u,v))$ are double Natural transform of the functions $f(x,t)$ and $g(x,t)$ respectively. See [14].

The fundamental properties of double Natural transform of partial derivatives are given by the authors in [14], as follows: If double Natural transform of the function $f(x,t)$ is given by $\mathbf{N}^+_{x,t}[f(x,t)] = R((p,s);(u,v))$ then, the double Natural transform of $\frac{\partial f(x,t)}{\partial x}$, $\frac{\partial^2 f(x,t)}{\partial x^2}$, $\frac{\partial f(x,t)}{\partial t}$ and $\frac{\partial^2 f(x,t)}{\partial t^2}$ are given by

$$\mathbf{N}^+_{x,t}\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{p}{u} R(p,s;u,v) - \frac{1}{u} f(0,s;0,v), \tag{11}$$

$$\mathbf{N}^+_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{p^2}{u^2} R(p,s;u,v) - \frac{p}{u^2} f(0,s;0,v) - \frac{1}{u}\frac{\partial f(0,s;0,v)}{\partial x}, \tag{12}$$

and

$$\mathbf{N}^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{s}{v} R(p,s;u,v) - \frac{1}{v} f(p,0;u,0), \tag{13}$$

$$\mathbf{N}^+_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{s^2}{v^2} R(p,s;u,v) - \frac{s}{v^2} f(p,0;u,0) - \frac{1}{v}\frac{\partial f(p,0;u,0)}{\partial t}, \tag{14}$$

In the following theorem, we study double Natural transform of the functions $x^n g(x,t)$, $x^n \frac{\partial f(x,t)}{\partial t}$ and $x^n \frac{\partial^2 f(x,t)}{\partial t^2}$ as follows:

Theorem 2.5. *If double Natural transform of the partial derivatives $\frac{\partial f(x,t)}{\partial t}$ and $\frac{\partial^2 f(x,t)}{\partial t^2}$ are given by Eqs.(13) and Eq.(14), then double Natural transform of $x^n \frac{\partial f(x,t)}{\partial t}$, $x^n \frac{\partial^2 f(x,t)}{\partial t^2}$ and $x^n g(x,t)$ are given by*

$$(-u)^n \frac{d}{dp}\left(\mathbf{N}^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right]\right) = \mathbf{N}^+_{x,t}\left[x^n \frac{\partial f(x,t)}{\partial t}\right] \tag{15}$$

$$(-u)^n \frac{d}{dp}\left(\mathbf{N}^+_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]\right) = \mathbf{N}^+_{x,t}\left[x^n \frac{\partial^2 f(x,t)}{\partial t^2}\right], \tag{16}$$

and

$$(-u)^n \frac{d}{dp}\left(\mathbf{N}^+_{x,t}\left[g(x,t)\right]\right) = \mathbf{N}^+_{x,t}\left[x^n g(x,t)\right], \tag{17}$$

where $n = 1, 2, 3, ...$

Proof. Using the definition of double Natural transform of the first order partial derivatives one gets

$$\mathbf{N}^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\frac{p}{u}x-\frac{s}{v}t}\frac{\partial f(x,t)}{\partial t}dt\,dx, \tag{18}$$

by taking the ^{n}th derivative with respect to p for both sides of Eq.(34), we have

$$\begin{aligned}
\frac{d^n}{dp^n}\left(\mathbf{N}^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right]\right) &= \frac{1}{uv}\int_0^\infty \int_0^\infty \frac{d^n}{dp^n}\left(e^{-\frac{p}{u}x-\frac{s}{v}t}\frac{\partial f(x,t)}{\partial t}dt\,dx\right) \\
&= \frac{(-1)^n}{uv}\int_0^\infty \int_0^\infty \left(\frac{x}{u}\right)^n \left(e^{-\frac{p}{u}x-\frac{s}{v}t}\frac{\partial f(x,t)}{\partial t}dt\,dx\right) \\
&= \frac{(-1)^n}{u^n}\left(\frac{1}{uv}\int_0^\infty \int_0^\infty x^n \left(e^{-\frac{p}{u}x-\frac{s}{v}t}\frac{\partial f(x,t)}{\partial t}dt\,dx\right)\right) \\
&= \frac{(-1)^n}{u^n}\mathbf{N}^+_{x,t}\left[x^n\frac{\partial f(x,t)}{\partial t}\right],
\end{aligned}$$

we obtain

$$(-u)^n \frac{d^n}{dp^n}\left(\mathbf{N}^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right]\right) = \mathbf{N}^+_{x,t}\left[x^n\frac{\partial f(x,t)}{\partial t}\right].$$

Similarly, we can prove Eqs.(16) and Eq.(17). □

3. Modified Double Natural Decomposition Method Applied to Singular One-dimensional Boussinesq Equation

The main aim of this section is to discuss the use of modified double Natural decomposition method (MDNDM) for solving linear and nonlinear Singular one-dimensional Boussinesq equation. In this section we define double Natural transform of the function $\psi(x,t)$ by $R(p,s;u,v)$. We suggest here two important problems.

First problem: We consider the following general linear singular one-dimensional boussinesq equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + a(x)\frac{\partial^4\psi}{\partial x^4} + b(x)\frac{\partial^4\psi}{\partial x^2\partial t^2} = f(x,t), \tag{19}$$

subject to initial condition

$$\psi(x,0) = f_1(x), \quad \frac{\partial\psi(x,0)}{\partial t} = f_2(x), \tag{20}$$

where $f(x,t)$, $f_1(x)$, $a(x)$ and $b(x)$ are known functions. First, we multiply both sides of Eq.(19) by x, and using the property of partial derivative of the double Natural transform and single Natural transform for Eq.(19) and Eq.(20) respectively and theorem 5.

$$\frac{d}{dp}\left[\frac{s^2}{v^2}R(p,s;u,v) - \frac{s}{v^2}f_1(p,u) - \frac{1}{v}f_2(p,u)\right] = -\frac{1}{u}\mathbf{N}^+_{x,t}[\Phi] + \frac{d}{dp}F(p,s;u,v), \tag{21}$$

where

$$\Phi = \frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) - xa(x)\frac{\partial^4\psi}{\partial x^4} - xb\left(x\right)\frac{\partial^4\psi}{\partial x^2\partial t^2},$$

simplifying Eq.(21), we have

$$\frac{d}{dp}\left[R\left(p,s;u,v\right)\right] = \frac{1}{s}\frac{d}{dp}f_1\left(p,u\right) + \frac{v}{s^2}\frac{d}{dp}f_2\left(p,u\right) - \frac{v^2}{us^2}\mathbf{N}^+_{x,t}\left[\Phi\right] + \frac{v^2}{s^2}\frac{d}{dp}F\left(p,s;u,v\right), \tag{22}$$

by integrating both sides of Eq.(22), from 0 to p with respect to p, we have

$$\begin{aligned} R\left(p,s;u,v\right) &= \frac{f_1\left(p,u\right)}{s} + \frac{vf_2\left(p,u\right)}{s^2} - \frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\Phi\right]dp \\ &\quad + \frac{v^2}{s^2}F\left(p,s;u,v\right), \end{aligned} \tag{23}$$

where $F\left(p,s;u,v\right)$, $F_1\left(p,u\right)$ and $F_2\left(p,u\right)$ are Natural transform of the functions $f\left(x,t\right)$, $f_1\left(x\right)$ and $f_2\left(x\right)$ respectively and the double Natural transform with respect to x, t is defined by $\mathbf{N}^+_{x,t}$. Operating with the double Natural transform inverse on both sides of Eq.(23), we obtain

$$\psi\left(x,t\right) = f_1\left(x\right) + tf_2\left(x\right) + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{s^2}F\left(p,s;u,v\right)\right] - \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\Phi\right]dp\right]. \tag{24}$$

The modified double Natural decomposition method (MDNDM) defines the solutions $\psi(x,t)$ by the infinite series as follows:

$$\psi\left(x,t\right) = \sum_{n=0}^{\infty}\psi_n\left(x,t\right). \tag{25}$$

Substituting Eq.(25) into Eq.(24), we have

$$\begin{aligned} \sum_{n=0}^{\infty}\psi_n\left(x,t\right) &= f_1\left(x\right) + tf_2\left(x\right) + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{s^2}F\left(p,s;u,v\right)\right] \\ &\quad - \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n\left(x,t\right)\right)\right)\right]dp\right] \\ &\quad + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\left(xa(x)\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{\infty}\psi_n\left(x,t\right)\right)\right)\right]dp\right] \\ &\quad + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\left(xb(x)\frac{\partial^4}{\partial x^2\partial t^2}\left(\sum_{n=0}^{\infty}\psi_n\left(x,t\right)\right)\right)\right]dp\right]. \end{aligned} \tag{26}$$

By comparing both sides of the Eq.(24), we get

$$\begin{aligned} \psi_0\left(x,t\right) &= f_1\left(x\right) + tf_2\left(x\right) + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{s^2}F\left(p,s;u,v\right)\right] \\ \psi_1\left(x,t\right) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi_0}{\partial x}\right)\right]dp\right] \\ &\quad + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\left(xa(x)\frac{\partial^4\psi_0}{\partial x^4} + xb(x)\frac{\partial^4\psi_0}{\partial x^2\partial t^2}\right)\right]dp\right]. \end{aligned} \tag{27}$$

In general, the recursive relation is given by

$$\begin{aligned}\psi_{n+1}(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=1}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right]\\ &+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[\left(xa(x)\frac{\partial^4}{\partial x^4}\left(\sum_{n=1}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right]\\ &+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[\left(xb(x)\frac{\partial^4}{\partial x^2\partial t^2}\left(\sum_{n=1}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right],\end{aligned} \tag{28}$$

where the double inverse Natural transform is denoted by $\mathbf{N}^{-1}_{p,s;u,v}$. Here we assume that double inverse Natural transform exists for each term in the right hand side of Eqs. (27) and (28). To illustrate this method for solving the singular one dimensional boussinesq equation, we assume $a(x) = 1, b(x) = -1$ and $f(x,t) = -x^2\sin t - 2\sin t$, in Eq.(19), we have the following example:

Example 4: Consider a singular one dimensional boussinesq equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^4\psi}{\partial x^4} - \frac{\partial^4\psi}{\partial x^2\partial t^2} = -x^2\sin t - 2\sin t, \tag{29}$$

subject to initial condition

$$\psi(x,0) = 0,\ \ \frac{\partial\psi(x,0)}{\partial t} = x^2, \tag{30}$$

by multiplying Eq.(29), by x and using the definition of partial derivatives of the double Natural transform, single Natural transform for Eq.s (30) respectively, we obtain,

$$\begin{aligned}\frac{dR}{dp} &= -\frac{v^2}{us^2}\mathbf{N}^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) - x\frac{\partial^4\psi}{\partial x^4} + x\frac{\partial^4\psi}{\partial x^2\partial t^2}\right] + \frac{3!u^2v^3}{p^4s^2(s^2+v^2)}\\ &+\frac{2v^3}{p^2s^2(s^2+v^2)} - \frac{6u^2v}{p^4s^2},\end{aligned} \tag{31}$$

by integrating both sides of Eq.(31), from 0 to p with respect to p, we have

$$\begin{aligned}R(p,s;u,v) &= -\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) - x\frac{\partial^4\psi}{\partial x^4} + x\frac{\partial^4\psi}{\partial x^2\partial t^2}\right]dp - \frac{2u^2v^3}{p^3s^2(s^2+v^2)}\\ &-\frac{2v^3}{ps^2(s^2+v^2)} + \frac{2!u^2v}{p^3s^2}.\end{aligned} \tag{32}$$

On using the inverse double Natural transform to Eq.(32), we obtain

$$\begin{aligned}\psi(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) - x\frac{\partial^4\psi}{\partial x^4} + x\frac{\partial^4\psi}{\partial x^2\partial t^2}\right]dp\right]\\ &+x^2\sin t + 2\sin t - 2t,\end{aligned} \tag{33}$$

putting Eq.(25) into Eq.(33), we will get

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right) - x\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right]\\ &-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^{+}_{x,t}\left[x\frac{\partial^4}{\partial x^2\partial t^2}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right]\\ &+x^2\sin t + 2\sin t - 2t.\end{aligned} \tag{34}$$

By using modified Natural decomposition method, we have

$$\psi_0 = x^2 \sin t + 2\sin t - 2t,$$

and

$$\begin{aligned}\psi_{n+1}(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right) - x\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right] \\ &\quad -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[x\frac{\partial^4}{\partial x^2\partial t^2}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right],\end{aligned}$$

now the components of the series solution are

$$\begin{aligned}\psi_1 &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_0(x,t)\right) - x\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{\infty}\psi_0(x,t)\right)\right]dp\right] \\ &\quad -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[x\frac{\partial^4}{\partial x^2\partial t^2}\left(\sum_{n=0}^{\infty}\psi_0(x,t)\right)\right]dp\right] \\ \psi_1 &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[2x\sin t\right]dp\right] = -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{-2uv}{p^2s^2\left(s^2+v^2\right)}\right], \\ &= 2t - 2\sin t\end{aligned}$$

and

$$\psi_2 = -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[0\right]dp\right] = 0.$$

Eventually, the approximate solution of the unknown functions given by

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= \psi_0 + \psi_1 + \psi_2 + ... \\ &= x^2\sin t + 2\sin t - 2t + 2t - 2\sin t + 0.\end{aligned}$$

Hence, the exact solution is given by

$$\psi(x,t) = x^2 \sin t.$$

Second problem: Consider the following general form of the nonlinear singular one dimensional boussinesq equation:

$$\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\psi_x\right) + a(x)\psi_{xxxx} - b(x)\psi_{xxtt} + c(x)\psi_t\psi_{xx} + d(x)\psi_x\psi_{xt} = g(x,t), \tag{35}$$

subject to initial condition

$$\psi(x,0) = g_1(x), \quad \frac{\partial\psi(x,0)}{\partial t} = g_2(x), \tag{36}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are arbitrary function. The first step, the method consists of multiplying both sides of Eq.(35) by x and applying double Natural transform, we have

$$\begin{aligned}&\mathbf{N}^+_{x,t}\left[x\psi_{tt} - \frac{\partial}{\partial x}\left(x\psi_x\right) + xa(x)\psi_{xxxx} - xb(x)\psi_{xxtt} + xc(x)x\psi_t\psi_{xx} + xd(x)x\psi_x\psi_{xt}\right] \\ =\; &\mathbf{N}^+_{x,t}\left[xf(x,t)\right].\end{aligned} \tag{37}$$

Using the differentiation property of double Natural transform, theorem 5 and initial condition given in Eq.(36), we get

$$\frac{d}{dp}\left[R\left(p,s;u,v\right)\right]=\frac{1}{s}\frac{d}{dp}g_1\left(p,u\right)+\frac{v}{s^2}\frac{d}{dp}g_2\left(p,u\right)-\frac{v^2}{us^2}\mathbf{N}^{+}_{x,t}\left[\Psi\right]+\frac{v^2}{s^2}\frac{d}{dp}G\left(p,s;u,v\right), \tag{38}$$

where,

$$\Psi=\frac{\partial}{\partial x}\left(x\psi_x\right)-xa(x)\psi_{xxxx}+xb\left(x\right)\psi_{xxtt}-xc\left(x\right)x\psi_t\psi_{xx}-xd\left(x\right)x\psi_x\psi_{xt},$$

By integrating both sides of Eq.(38), from 0 to p with respect to p, we have

$$R\left(p,s;u,v\right)=\frac{G_1\left(p,u\right)}{s}+\frac{vG_2\left(p,u\right)}{s^2}-\frac{v^2}{us^2}\int_0^p\mathbf{N}^{+}_{x,t}\left[\Psi\right]dp+\frac{v^2}{s^2}G\left(p,s;u,v\right), \tag{39}$$

The second step in the double Natural decomposition method is that we represent solution as an infinite series as in Eq.(25). Using double inverse Natural transform for Eq.(39) we obtain

$$\psi\left(x,t\right)=g_1\left(x\right)+tg_2\left(x\right)+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{s^2}\mathbf{g}\left(p,s;u,v\right)\right]-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p\mathbf{N}^{+}_{x,t}\left[\Psi\right]dp\right]. \tag{40}$$

Moreover, the nonlinear terms $\psi_t\psi_{xx}$ and $\psi_x\psi_{xt}$ can be defined as follows:

$$\psi_t\psi_{xx}=N_1=\sum_{n=0}^{\infty}A_n,\quad \psi_x\psi_{xt}=N_2=\sum_{n=0}^{\infty}B_n, \tag{41}$$

We have a few terms of the Adomian polynomials for A_n and B_n that are given by

$$A_n=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{i=0}^{\infty}\left(\lambda^n\psi_n\right)\right]\right)_{\lambda=0}, \tag{42}$$

and

$$B_n=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2\sum_{i=0}^{\infty}\left(\lambda^n\psi_n\right)\right]\right)_{\lambda=0}, \tag{43}$$

where $n=0,1,2,....$ By substituting Eq.(42), Eq.(43) and Eq.(41) into Eq.(40), we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n\left(x,t\right) &= f_1\left(x\right)+tf_2\left(x\right)+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{s^2}\mathbf{g}\left(p,s;u,v\right)\right]\\ &-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p\mathbf{N}^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n\right)\right)\right]dp\right]\\ &+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p\mathbf{N}^{+}_{x,t}\left[xa(x)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxxx}-xb\left(x\right)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxtt}\right]dp\right]\\ &+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p\mathbf{N}^{+}_{x,t}\left[xc\left(x\right)\sum_{n=0}^{\infty}A_n+xd\left(x\right)\sum_{n=0}^{\infty}B_n\right]dp\right],\end{aligned} \tag{44}$$

where, A_n and B_n are given by

$$\begin{aligned}A_0 &= \psi_{0t}\psi_{0xx}\\ A_1 &= \psi_{0t}\psi_{1xx}+\psi_{1t}\psi_{0xx}\\ A_2 &= \psi_{0t}\psi_{2xx}+\psi_{1t}\psi_{1xx}+\psi_{2t}\psi_{0xx}\\ A_3 &= \psi_{0t}\psi_{3xx}+\psi_{1t}\psi_{2xx}+\psi_{2t}\psi_{1xx}+\psi_{3t}\psi_{0xx},\end{aligned} \tag{45}$$

and

$$\begin{aligned} B_0 &= \psi_{0x}\psi_{0xt} \\ B_1 &= \psi_{0x}\psi_{1xt} + \psi_{1x}\psi_{0xt} \\ B_2 &= \psi_{0x}\psi_{2xt} + \psi_{1x}\psi_{1xt} + \psi_{2x}\psi_{0xt}. \\ B_3 &= \psi_{0x}\psi_{3xt} + \psi_{1x}\psi_{2xt} + \psi_{2x}\psi_{1xt} + \psi_{3x}\psi_{0xt}. \end{aligned} \tag{46}$$

Therefore, from Eq.(44) above, we have

$$\psi_0(x,t) = f_1(x) + t f_2(x) + \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{s^2}\mathbf{g}(p,s;u,v)\right],$$

and

$$\begin{aligned} \psi_{n+1}(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n\right)\right)\right]dp\right] \\ &+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[xa(x)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxxx} - xb(x)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxtt}\right]dp\right] \\ &+\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[xc(x)\sum_{n=0}^{\infty}A_n + xd(x)\sum_{n=0}^{\infty}B_n\right]dp\right]. \end{aligned} \tag{47}$$

To illustrate this method for nonlinear singular one dimensional boussinesq equation, we take the following example.We let that $a(x) = b(x) = 1, c(x) = -4, d(x) = 2$ and $f(x,t) = -4t$.

Example 5: Consider nonlinear singular one dimensional boussinesq equation

$$\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}(x\psi_x) + \psi_{xxxx} - \psi_{xxtt} - 4\psi_t\psi_{xx} + 2\psi_x\psi_{xt} = -4t, \tag{48}$$

subject to initial condition

$$\psi(x,0) = 0, \quad \psi_t(x,0) = x^2, \tag{49}$$

The double Natural decomposition method leads to the following:

$$\psi_0(x,t) = x^2 t - \frac{2}{3}t^3,$$

and

$$\begin{aligned} \psi_{n+1}(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n\right)\right)\right]dp\right] \\ &-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[x\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxxx} - x\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxtt}\right]dp\right] \\ &-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[4x\sum_{n=0}^{\infty}A_n - 2x\sum_{n=0}^{\infty}B_n\right]dp\right], \end{aligned}$$

the first iteration is given by

$$\begin{aligned} \psi_1(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi_0}{\partial x}\right)\right]dp\right] \\ &-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[x(\psi_0)_{xxxx} - x\psi_{0xxtt}\right]dp\right] \\ &-\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[4xA_0 - 2xB_0\right]dp\right], \end{aligned}$$

where,

$$\begin{aligned}\frac{\partial}{\partial x}\left(x\frac{\partial\psi_0}{\partial x}\right) &= 4xt,\ \ x\left(\psi_0\right)_{xxxx}=0,\ \ x\psi_{0xxtt}=0,\\ A_0 &= \psi_{0t}\psi_{0xx}=2x^2t-4t^3,\ B_0=\psi_{0x}\psi_{0xt}=4x^2t,\end{aligned}$$

then,

$$\begin{aligned}\psi_1(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[4xt\right]dp\right]\\ &\quad -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[0-0\right]dp\right]\\ &\quad +\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[16xt^3\right]dp\right],\end{aligned}$$

$$\psi_1(x,t)=\frac{2}{3}t^3-\frac{4}{5}t^5,$$

In similar manner,

$$\begin{aligned}\frac{\partial}{\partial x}\left(x\frac{\partial\psi_1}{\partial x}\right) &= 0,\ \ x\left(\psi_1\right)_{xxxx}=0,\ \ x\psi_{1xxtt}=0,\\ A_1 &= \psi_{0t}\psi_{1xx}+\psi_{1t}\psi_{0xx}=4t^3-8t^5,\\ B_0 &= \psi_{0x}\psi_{1xt}+\ \psi_{1x}\psi_{0xt}=0,\end{aligned}$$

then,

$$\begin{aligned}\psi_2(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[16xt^3-32xt^5\right]dp\right]\\ &\quad -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \left[\frac{16\times 3!uv^3}{p^2s^4}-\frac{32\times 5!uv^5}{p^2s^6}\right]dp\right]\\ &= \mathbf{N}^{-1}_{p,s;u,v}\left[\frac{16\times 3!v^5}{ps^6}-\frac{32\times 5!uv^7}{ps^8}\right]\\ \psi_2(x,t) &= \frac{4}{5}t^5-\frac{16}{21}t^7.\end{aligned}$$

Similarly,

$$\begin{aligned}\psi_3(x,t) &= -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_2\right)\right]dp\right]\\ &\quad -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[x\left(\psi_2\right)_{xxxx}-x\psi_{2xxtt}\right]dp\right]\\ &\quad -\mathbf{N}^{-1}_{p,s;u,v}\left[\frac{v^2}{us^2}\int_0^p \mathbf{N}^+_{x,t}\left[4xA_2-2xB_2\right]dp\right],\end{aligned}$$

therefore,

$$\begin{aligned}\frac{\partial}{\partial x}\left(x\frac{\partial\psi_2}{\partial x}\right) &= 0,\ \ x\left(\psi_2\right)_{xxxx}=0,\ \ x\psi_{2xxtt}=0,\\ A_2 &= \psi_{0t}\psi_{2xx}+\psi_{1t}\psi_{1xx}+\psi_{2t}\psi_{0xx}=8t^5-\frac{32}{3}t^7,\\ B_2 &= \psi_{0x}\psi_{2xt}+\ \psi_{1x}\psi_{1xt}+\ \psi_{2x}\psi_{0xt}=0,\end{aligned}$$

then, we have

$$\psi_3(x,t) = \frac{16}{21}t^7 - \frac{16}{27}t^9.$$

The series solutions are therefore is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= \psi_0+\psi_1+\psi_2+....\\ &= x^2t-\frac{2}{3}t^3+\frac{2}{3}t^3-\frac{4}{5}t^5+\frac{4}{5}t^5-\frac{16}{21}t^7+\frac{16}{21}t^7-\frac{16}{27}t^9+....\end{aligned}$$

and hence the exact solution become

$$\psi(x,t) = x^2t.$$

Conclusion 3.1. *In this article, we have successfully employed the Double Natural Decomposition Method (DNDM) to obtain analytic solutions of singular linear and nonlinear Boussinesq equation. Moreover, two examples were given to validate our method.*

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References

[1] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1998), pp. 501-544.
[2] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Methods, Kluver Academic Publishers, Boston, 1994.
[3] A.M. Wazwaz, A reliable modification of Adomian's decomposition method, Appl. Math. Comput. 102 (1999), 77-86.
[4] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. 111(1) (2000), 33-51.
[5] Z.H.Khan and W.A.Khan.N-transform properties and applications. NUST Jour of Engg Sciences,1 (1) (2008), 127-133.
[6] Hassan Eltayeb, Adem Kilicman and Said Mesloub, Application of the double Laplace Adomian decomposition method for solving linear singular one dimensional thermo-elasticity coupled system, J. Nonlinear Sci. Appl, 10 (2017), 278-289.
[7] Hassan Eltayeb Gadain and Imed Bachar, On a nonlinear singular one-dimensional parabolic equation and double Laplace decomposition method, Adv. in Mech. Eng , 9(1)(2017) 1-7.
[8] Hassan Eltayeb Gadain, Application of double Laplace decomposition method for solving singular one dimensional system of hyperbolic equations, J. Nonlinear Sci. Appl., 10 (2017), 111-121.
[9] A.-M. Wazwaz, New travelling wave solutions to the Boussinesq and the KleinGordon equations, Commun Nonlinear Sci Numer Simulat, 13(5)(2008), 889-901.
[10] A. M.Wazwaz, Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method, Chaos, Solitons Fract, 12(8)(2001),1549-1556.
[11] A. Biswas, D. Milovic, and A. Ranasinghe, Solitary waves of Boussinesq equation in a power law media, Commun Nonlinear Sci Numer Simulat, 14(11)(2009) 3738-3742.
[12] Mahmoud S. Rawashdeh and Shehu Maitama, Solving coupled system of nonlinear pde's using the Natural decomposition method, Inter. J. Pure Appl. Math, 92(5)(2014), 757-776.
[13] S. K. Q. Al-Omari, On the application of Natural transforms, Inter. J. Pure Appl. Math, 85 (2013), 729-744.
[14] Adem Kiliciman and Maryam Omran, On double Natural transform and its applications, J. Nonlinear Sci. Appl., 10 (2017), 1744-1754.
[15] Shehu Maitama, Mahmoud S. Rawashdeh and Surajo Sulaiman, An analytical method for solving linear and nonlinear Schrodinger equations, P. J. of Math, 6(1)(2017), 59-67.
[16] J. M. Tchuenche, N. S. Mbare, An application of the double Sumudu transform, Appl. Math. Sci. (Ruse), 1 (2007), 31-39.
[17] Adem Kiliciman and Hassan Eltayeb, A note on defining singular integral as distribution and partial differential equations with convolution term, Mathematical and Computer Modelling 49 (2009) 327-336.
[18] Sadek Gala, Zhengguang Guo and Maria Alessandra Ragusa, A remark on the regularity criterion of Boussinesq equations with zero heat conductivity, Applied Mathematics Letters 27 (2014) 70-73.
[19] Sadek Gala, Qiao Liu and Maria Alessandra Ragusa, Logarithmically improved regularity criterion for the nematic liquid crystal flows in $B^{-1}\infty,\infty$ space, Computers and Mathematics with Applications 65 (2013) 1738-1745.
[20] Jeffrey C. Lagarias, Euler's constant Euler's work and modern developments, Bulletin Amer. Math. Soc. 50 (2013), No. 4, 527-628.