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The λ-Aluthge Transform of EP Matrices

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Abstract. In this work we present some relationships between an EP matrix T, its Aluthge transform $\Delta(T)$ or the λ-Aluthge transform $\Delta_\lambda(T)$ and the Moore-Penrose inverse $T^\dagger$. We prove that the λ-Aluthge transform of T is also an EP matrix, and the same thing holds for $\Delta_\lambda(T)^\dagger$ and $\Delta_\lambda(T^\dagger)$. Also, we explore the product $\Delta_\lambda(T)T$, the connections between $\Delta(T)$ and $T^\dagger$ as well as the reverse order law for generalized inverses which are associated with $\Delta_\lambda(T)$. Finally, it is verified that the ranges of T and $\Delta_\lambda(T)$ are equal in the case of EP matrices.

1. Introduction

A lot of work has been done on the Aluthge transform of operators and matrices over the recent years. The Aluthge transformation of an operator or matrix T, denoted by $\Delta(T)$, was originally studied by A. Aluthge in 1990 in the paper [1] in relation with the p-hyponormal and log-hyponormal operators. One reason for such a study is the connection between the Aluthge transform and the invariant subspace problem. Jung, Ko and Pearcy proved in [17] that T has a nontrivial invariant subspace if and only if $\Delta(T)$ does. In the same work, the nth Aluthge transform $\Delta^{(n)}(T)$ was also defined for each non-negative integer n. The λ-Aluthge transform is a generalization of the Aluthge transform (see e.g. [3, 18]).

The purpose of the Aluthge transform is converting a matrix or an operator into another matrix or operator, respectively. The original matrix (operator) and its transformation have the same spectral properties, but the second one is nearly a normal operator. Many applications of the Aluthge transform have been determined, particularly in solving the invariant subspace problem. As it was observed in [8, 10], the interest for the Aluthge transform is caused by the fact that it preserves many properties of the original operator. For example, T and $\Delta(T)$ have the same spectrum. Thus, much attention has been given to both the Aluthge transform and the λ-Aluthge transform in the recent research.

If T is a square matrix and commutes with its Moore-Penrose inverse $T^\dagger$, then T is called an EP matrix. According to another characterization, a square complex matrix T is said to be EP if A and its conjugate transpose A^* have the same range. Various characterizations of EP matrices were collected in [19]. EP

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matrices constitute a wide class of matrices, which includes the self adjoint, the normal and the invertible matrices.

The goal of this paper is to explore properties of the λ-Aluthge transform of EP matrices. Both the singular and nonsingular cases are considered. The features that are studied are in relationship with the Moore-Penrose inverse or the usual inverse of both the EP matrix and its transform. Some results are derived by making use of the Singular Value Decomposition (SVD) of EP matrices.

The paper is organized as follows. Section 2 gives the theoretical background and the basic properties of EP matrices and the λ-Aluthge transform. In section 3 at first we study the relation of an EP matrix with its λ-Aluthge transform. Afterwards, we investigate the interconnections between the inverse of an EP matrix and both its Aluthge transform as well as the inverse of its Alutghe transform. These results are then extended to singular EP matrices, replacing the inverse with the Moore-Penrose inverse. The products of these transforms are studied, and in that case it was shown that the range of a singular EP matrix and its Aluthge transform coincide. Finally, examining the reverse order law for the Moore-Penrose inverse, it is proved that it always holds for T and $\Delta_\lambda(T)$.

2. Preliminaries and Notation

The following definition of the Aluthge transform from [1] is restated for the sake of completeness.

Definition 2.1. [1] *Let T be given $n \times n$ complex matrix with the polar representation $T = U|T|$ where $|T| = (T^*T)^{\frac{1}{2}}$ and U is unitary. The Aluthge transform of T is defined by*

$$\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

The partial isometry U in the polar representation $T = U|T|$ is unique under the condition $\mathcal{N}(U) = \mathcal{N}(T)$. Despite the fact that the partial isometry U is not unique without the condition $\mathcal{N}(U) = \mathcal{N}(T)$, there exists a unitary matrix U such that $T = U|T|$. Although the unitary matrix U in the polar representation is not unique except when T is invertible, the Aluthge transform is independent of the choice of a unitary matrix U.

Other known and important facts are the following:

$$\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|) = \mathcal{N}(|T|^{\frac{1}{2}}), \quad \mathcal{N}(U^*) = \mathcal{N}(T^*), \quad \mathcal{R}(U) = \mathcal{R}(T).$$

In addition, if $T = U|T|$ is invertible it follows that

$$\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}T|T|^{-\frac{1}{2}}.$$

So, $\Delta(T)$ is in the similarity orbit of T. For more on the Aluthge transform we refer to [17].

Definition 2.2. [18] *Let T be given $n \times n$ complex matrix with the polar representation $T = U|T| = |T^*|U$. Then the λ-Aluthge transform of T is defined by*

$$\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda},$$

for every $\lambda \in [0, 1]$.

In the case $\lambda = 1/2$, the λ-Aluthge transform $\Delta_{1/2}(T)$ becomes the Aluthge transform $\Delta(T)$ of T. Also, $\Delta_1(T) = |T|VU$ is known as Duggal's transform $\hat{T}$.

The Moore-Penrose inverse of a rectangular matrix or a singular matrix is a notion that will be used in this work. In the case when T is a complex $m \times n$ matrix of rank r, Penrose showed that there is a unique matrix satisfying the four Penrose equations, called the pseudo-inverse of T, denoted by $T^\dagger$ such that

$$TT^\dagger = (TT^\dagger)^*, \quad T^\dagger T = (T^\dagger T)^*, \quad TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \tag{1}$$

where T^* denotes the conjugate transpose of T.

It is easy to see that $TT^\dagger$ is the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{R}(T)$, denoted by P_T, and that $T^\dagger T$ is the orthogonal projection of $\mathbb{C}^m$ onto $\mathcal{R}(T^*)$ denoted by P_{T^*}. It is also well known that it holds $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$. Standard reference books on generalized inverses are [5, 9, 14].

The matrix T is an EP matrix in the case $TT^\dagger = T^\dagger T$. The set of EP matrices of rank r are usually denoted by EP_r. We take advantage of the fact that EP matrices have a simple canonical form according to the decomposition $\mathbb{C}^m = \mathcal{R}(T) \oplus \mathcal{N}(T)$. Indeed, an EP_r matrix T has the simple block matrix form

$$T = U\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}U^* = U(A \oplus 0)U^*, \tag{2}$$

where the matrix $A : \mathcal{R}(T) \to \mathcal{R}(T)$ is invertible with $\mathrm{rank}(A) = r$ and U is unitary. The generalized inverse $T^\dagger$ of the matrix T defined in (2) has the form

$$T^\dagger = U\begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = U(A^{-1} \oplus 0)U^*. \tag{3}$$

For more details about on EP matrices we refer to [9, 12, 21].

In the infinite dimensional case, an operator T is called quasinormal when $T^*TT = TT^*T$ which is equivalent to $U|T| = |T|U$. Quasinormal matrices in the finite dimensional case are normal. The next statement shows when an operator/matrix coincides with its λ-Aluthge transform.

Proposition 2.3. [8] *The matrix $T \in \mathbb{C}^{n\times n}$ is quasinormal if and only if $\Delta_\lambda(T) = T$.*

The proof of Proposition 2.3 in the case $\lambda = 1/2$ can be found in [17].

Consequently, one could say that the Aluthge transform is a measure of the normality of an operator or a square matrix, since a normal operator is equal to its Aluthge transform. Moreover, in the case of matrices, the iterated sequence of Aluthge transforms of a square matrix converges to a normal matrix (see e.g., [2]).

3. EP Matrices and the λ-Aluthge Transform

As we annotated, EP matrices form a more general set than normal matrices. On the other hand, normal matrices (and therefore all their subsets, such as symmetric, projections etc) have the property of being equal to their Aluthge transform. So, in this work we investigate the properties of a bigger family of matrices containing the normal matrices. For example, the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$$

is an EP matrix (since it is invertible), but not normal.

3.1. Properties of the λ-Aluthge Transform of an EP Matrix

Basic properties of the λ-Aluthge transform are collected in Proposition 3.1.

Proposition 3.1. [3, 4, 10] *Let $T \in \mathbb{C}^{n\times n}$ and $\lambda \in [0,1]$. Then:*
(a) $\Delta_\lambda(cT) = c\Delta_\lambda(T)$ for every $c \in \mathbb{C}$.
(b) When V is unitary it holds that $\Delta_\lambda(VTV^*) = V\Delta_\lambda(T)V^*$.
(c) $\Delta_\lambda(T_1 \oplus T_2) = \Delta_\lambda(T_1) \oplus \Delta_\lambda(T_2)$.

Theorem 3.2. *Let $T \in \mathbb{C}^{n\times n}$ be a singular EP matrix of the form (2). Then*

$$\Delta_\lambda(T) = U\begin{bmatrix} \Delta_\lambda(A) & 0 \\ 0 & 0 \end{bmatrix}U^* \tag{1}$$

$$\Delta_\lambda(T^\dagger) = U\begin{bmatrix} \Delta_\lambda(A^{-1}) & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{2}$$

Proof. The properties of the Aluthge transform from Proposition 3.1 imply

$$\begin{aligned}\Delta_\lambda(T) &= \Delta_\lambda\left(U(A\oplus 0)U^*\right)\\ &= U\,\Delta_\lambda(A\oplus 0)U^*\\ &= U\,(\Delta_\lambda(A)\oplus 0)\,U^*\\ &= U\begin{bmatrix}\Delta_\lambda(A) & 0\\ 0 & 0\end{bmatrix}U^*.\end{aligned}$$

The result (2) can be verified using (3). □

Theorem 3.3. *The λ-Aluthge transform $\Delta_\lambda(T)$ of an EP matrix $T \in \mathbb{C}^{n\times n}$ is also EP.*

Proof. Let T be EP matrix defined in (2). It is easy to see that $\Delta(T)$ commutes with $(\Delta(T))^\dagger$. Indeed, according to Theorem 3.2 in conjunction with (3), it follows that

$$\begin{aligned}\Delta_\lambda(T)(\Delta_\lambda(T))^\dagger &= U\begin{bmatrix}\Delta_\lambda(A) & 0\\ 0 & 0\end{bmatrix}U^*U\begin{bmatrix}(\Delta_\lambda(A))^{-1} & 0\\ 0 & 0\end{bmatrix}U^*\\ &= U\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}U^*\\ &= P_T\\ &= (\Delta_\lambda(T))^\dagger\Delta_\lambda(T).\end{aligned}$$

Therefore, $\Delta_\lambda(T)$ is also be an EP matrix. □

Remark 3.4. *We can easily see that the converse of Theorem 3.3 does not hold. For example, let*

$$T = \begin{bmatrix}1 & 2\\ -4 & -8\end{bmatrix}$$

and $\lambda = \frac{1}{2}$. Obviously,

$$\Delta(T) = \begin{bmatrix}-1.4 & -2.8\\ -2.8 & -5.6\end{bmatrix}$$

is EP while T is not.

Corollary 3.5. *When $T \in \mathbb{C}^{n\times n}$ is EP, then the following matrices are also EP:*

$$T^\dagger,\quad (\Delta_\lambda(T))^\dagger = U\begin{bmatrix}(\Delta_\lambda(A))^{-1} & 0\\ 0 & 0\end{bmatrix}U^*,\quad \Delta_\lambda(T^\dagger) = U\begin{bmatrix}\Delta_\lambda(A^{-1}) & 0\\ 0 & 0\end{bmatrix}U^*.$$

The difference between $(\Delta_\lambda(T))^\dagger$ and $\Delta_\lambda(T^\dagger)$ will be examined later in this work. Another property that should be examined is the invertibility of $\Delta(T)$ and the influence this may have to T. The statement of Lemma 3.6 can be found in [17].

Lemma 3.6. [17] *The matrix $T \in \mathbb{C}^{n\times n}$ is invertible if and only if $\Delta(T)$ is invertible, and in this case T and $\Delta(T)$ are similar.*

Since T and $\Delta(T)$ have the same spectrum, they are both singular or nonsingular. Theorem 3.7 is valid in the case when T is invertible.

Theorem 3.7. *Let the EP matrix $T \in \mathbb{C}^{n\times n}$ be invertible. It holds that*

$$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1} \Longleftrightarrow \lambda = \frac{1}{2} \text{ and } T = \Delta_\lambda(T).$$

Proof. Let $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$. It is known that if $T = U|T|$ is the polar decomposition of T and $T = W\Sigma V^*$ is the SVD decomposition of T, then

$$U = WV^*, \quad |T| = V\Sigma V^*.$$

So,

$$\begin{aligned}\Delta_\lambda(T) &= |T|^\lambda U|T|^{1-\lambda}\\ &= V\Sigma^\lambda V^* \, WV^* \, V\Sigma^{1-\lambda}V^*\\ &= V\Sigma^\lambda V^* W\Sigma^{1-\lambda}V^*.\end{aligned} \tag{3}$$

Therefore, it follows that

$$(\Delta_\lambda(T))^{-1} = V\Sigma^{-(1-\lambda)}W^*V\Sigma^{-\lambda}V^*. \tag{4}$$

On the other hand we have that $T^{-1} = V\Sigma^{-1}W^*$. Further, the polar decomposition of T^{-1} is defined by

$$T^{-1} = VW^*|T^{-1}| = U^*|T^{-1}|, \qquad |T^{-1}| = W\Sigma^{-1}W^*.$$

Thereafter,

$$\begin{aligned}\Delta_\lambda(T^{-1}) &= |T^{-1}|^\lambda U^*|T^{-1}|^{1-\lambda}\\ &= W\Sigma^{-\lambda}W^* \, VW^* \, W\Sigma^{-(1-\lambda)}W^*\\ &= W\Sigma^{-\lambda}W^*V\Sigma^{-(1-\lambda)}W^*.\end{aligned} \tag{5}$$

Consequently, according to (4) and (5), $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ implies $\lambda = \frac{1}{2}$ and $W = V$. Then, (3) implies

$$\begin{aligned}\Delta_{\frac{1}{2}}(T) = \Delta(T) &= V\Sigma^{\frac{1}{2}}V^*V\Sigma^{\frac{1}{2}}V^*\\ &= V\Sigma V^* = T.\end{aligned}$$

The converse statement can be verified using

$$\begin{aligned}T = U|T| &= WV^*V\Sigma V^* = W\Sigma V^*,\\ \Delta(T) &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}V^*.\end{aligned}$$

Then $T = \Delta(T)$ implies $V = W$, and further $\Delta(T^{-1}) = (\Delta(T))^{-1}$. □

Example 3.8. *Consider the invertible (therefore EP) matrix*

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Then,

$$\begin{aligned}\Delta(A) &= \begin{bmatrix} 2.7 & 1.8487 \\ -0.0487 & 3.3 \end{bmatrix}, \quad (\Delta(A))^{-1} = \begin{bmatrix} 0.3667 & -0.2054 \\ 0.0054 & 0.3 \end{bmatrix},\\ \Delta(A^{-1}) &= \begin{bmatrix} 0.3 & -0.2054 \\ 0.0054 & 0.3667 \end{bmatrix}.\end{aligned}$$

The statement of Corollary 3.9 can be derived by combining Theorem 3.7 with Proposition 3.5.

Corollary 3.9. *Let $T \in \mathbb{C}^{n\times n}$ be a singular EP matrix. Then it holds*

$$\Delta_\lambda(T^\dagger) = (\Delta_\lambda(T))^\dagger \iff T = \Delta_\lambda(T).$$

In Theorem 3.10 we will examine the same question for the case of a random singular matrix T.

Theorem 3.10. *Let T a singular square $n \times n$ matrix. Then, it holds that*

$$\Delta_\lambda(T) = T \Longrightarrow (\Delta_\lambda(T))^\dagger = \Delta_\lambda(T^\dagger) = T^\dagger.$$

Proof. It is known that if $T = U|T|$ is the polar decomposition of T and $T = W\Sigma V^*$ is the SVD of T, then

$$U = WV^*, \quad |T| = V\Sigma V^*.$$

In addition, acording to Proposition 2.3, the condition $\Delta_\lambda(T) = T$ implies that the corresponding operator is quasinormal. But, in the finite dimensional case, quasinormality coincides with normality. Therefore, $TT^* = T^*T$ holds in the finite dimensional case. Since T^*T and TT^* are having the same eigenvectors, it follows that the SVD of T satisfies $W = V$. So, it follows that

$$\begin{aligned} \Delta_\lambda(T) &= |T|^\lambda U |T|^{1-\lambda} \\ &= V\Sigma^\lambda V^* \, VV^* \, V\Sigma^{1-\lambda} V^* \\ &= V\Sigma V^*. \end{aligned}$$

Hence, $(\Delta_\lambda(T))^\dagger = V\Sigma^\dagger V^*$.

On the other hand, let $T^\dagger = V\Sigma^\dagger V^*$ and the polar decomposition of $T^\dagger$ be $T^\dagger = \hat{U}|T^\dagger|$. Then $\hat{U} = VV^*$ and $|T^\dagger| = V\Sigma^\dagger V^*$. Also, the following is satisfied:

$$\begin{aligned} \Delta_\lambda(T^\dagger) &= V\left(\Sigma^\lambda\right)^\dagger V^* V V^* V \left(\Sigma^{1-\lambda}\right)^\dagger V^* \\ &= V\Sigma^\dagger V^*. \end{aligned}$$

Consequently, the assumption $\Delta_\lambda(T) = T$ implies

$$\begin{aligned} \Delta_\lambda(T^\dagger) &= (\Delta_\lambda(T))^\dagger \\ &= V\Sigma^\dagger V^* \\ &= T^\dagger, \end{aligned}$$

which completes the proof. □

For a fixed matrix T, denote by $\mathbb{R}_T$ the set of EP_r matrices B having the property $\mathcal{R}(B) = \mathcal{R}(T)$. The set $\mathbb{R}_T$ is described in [6] as in Proposition 3.11.

Proposition 3.11. [6] *Let T be an EP_r matrix, with the decomposition* (2). *The set $\mathbb{R}_T$ consists of all the matrices*

$$B = U \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where E is any nonsingular $r \times r$ matrix.

Proposition 3.12. *When A and B are EP_r matrices such that $\mathcal{R}(A) = \mathcal{R}(B)$ then $\Delta_\lambda(A)\Delta_\lambda(B)$ and $\Delta_\lambda(AB)$ are also EP_r matrices.*

Proof. Using Proposition 3.11 we have that

$$A = U \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad B = U \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Therefore,

$$AB = U\begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$$

is EP_r, and so

$$\Delta_\lambda(AB) = U\begin{bmatrix} \Delta(A_1B_1) & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

In addition,

$$\Delta_\lambda(A)\Delta_\lambda(B) = U\begin{bmatrix} \Delta_\lambda(A_1)\Delta_\lambda(B_1) & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

which is also an EP_r matrix. □

Corollary 3.13. *Let T be a singular EP matrix having the form* (2). *If $T = W\Sigma V^*$ is the SVD of T, the following identities hold:*

$$\begin{aligned} \Delta(T)VW^*\Delta(T^{-1}) &= VW^* \\ \Delta(T^{-1})WV^*\Delta(T) &= WV^*. \end{aligned}$$

Proof. Let $T = U|T|$ be the polar decomposition of T and $T = W\Sigma V^*$ be the SVD of T, with $U = WV^*$ and $|T| = V\Sigma V^*$. Then, $T^{-1} = V\Sigma^{-1}W^*$ is the SVD of T^{-1}, its polar decomposition is defined by $T^{-1} = VW^*|T^{-1}|$ and $|T^{-1}| = W\Sigma^{-1}W^*$.

In addition, the Aluthge transform satisfies

$$\begin{aligned} \Delta(T) &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}V^* \\ \Delta(T^{-1}) &= W\Sigma^{-\frac{1}{2}}W^*V\Sigma^{-\frac{1}{2}}W^*. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta(T)\, VW^*\, \Delta(T^{-1}) &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}V^*\, VW^*\, W\Sigma^{-\frac{1}{2}}W^*V\Sigma^{-\frac{1}{2}}W^* \\ &= VW^*. \end{aligned}$$

Following the same procedure we can prove

$$\Delta(T^{-1})WV^*\Delta(T) = WV^*,$$

which completes the proof. □

The identity $\Delta(T)\Delta(T)^\dagger = P_T$ is verified in Theorem 3.3. But, what happens in the case of $\Delta(T)\Delta(T^\dagger)$? Proposition 3.14 gives the answer to that question. This proposition is a continuation of Corollary 3.13.

Proposition 3.14. *Let T be a singular square EP matrix, having the form* (2). *If $A = W\Sigma V^*$ is the SVD of A, then it holds:*

$$\Delta(T)\Delta(T^\dagger) = U\begin{bmatrix} \left(\Delta(A^{-1})WV^*\right)^{-1} WV^*\Delta(A^{-1}) & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Proof. Simple verification shows

$$\Delta(T)\Delta(T^\dagger) = U\begin{bmatrix} \Delta(A)\Delta(A^{-1}) & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Since $\Delta(A)VW^*\Delta(A^{-1}) = VW^*$, it follows that

$$\Delta(A) = VW^*\left(\Delta(A^{-1})\right)^{-1} WV^*.$$

By multiplying the last equality from the right side with $\Delta(A^{-1})$ one can obtain

$$\begin{aligned}\Delta(A)\Delta(A^{-1}) &= VW^*\left(\Delta(A^{-1})\right)^{-1}WV^*\Delta(A^{-1})\\ &= \left(\Delta(A^{-1})WV^*\right)^{-1}WV^*\Delta(A^{-1}),\end{aligned}$$

which competes the proof. □

Finally, we present a result on the ranges of T and $\Delta_\lambda(T)$. It was proved in [16] that the range of $\Delta(T)$ is not dense nor closed in the infinite dimensional case. We investigate the relation between the range of the original matrix T and its λ-Aluthge transform in the case of singular EP matrices.

Theorem 3.15. *If $T \in \mathbb{C}^{n\times n}$ is a singular EP matrix then it holds that $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$.*

Proof. The assumption that T is EP implies $\mathcal{N}(T) = \mathcal{N}(T^*)$. By the definition of the Aluthge transform, we further obtain $\mathcal{N}(T) \subseteq \mathcal{N}(\Delta_\lambda(T))$. So, we only have to prove the inclusion $\mathcal{N}(\Delta(T)) \subseteq \mathcal{N}(T)$ under the assumption $\mathcal{N}(T) = \mathcal{N}(T^*)$. To that end, let us choose an $x \in \mathcal{N}(\Delta(T))$. Then, $Tx = |T|^\lambda U|T|^{1-\lambda}x = 0$. Therefore, since

$$\mathcal{N}(|T|^\lambda) = \mathcal{N}(T) = \mathcal{N}(T^*) = \mathcal{N}(|T^*|^\lambda)$$

it follows that $Tx = |T^*|^\lambda U|T|^{1-\lambda}x = 0$, and further $\mathcal{N}(\Delta_\lambda(T)) \subseteq \mathcal{N}(T)$.

By taking orthogonal complements in the relation $\mathcal{N}(T) = \mathcal{N}(\Delta_\lambda(T))$, taking into account that both T and $\Delta_\lambda(T)$ are EP, we have that $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$, which was our original intention. □

3.2. The Reverse Order Law in Relation to Aluthge Transform

In this subsection we will present results concerning the reverse order law and its interconnection with the Aluthge transform. The assumption that both T and $\Delta(T)$ are EP matrices makes the problem much simpler than in the general case. A natural question to ask about the Aluthge transform and the Moore-Penrose inverse is whether the reverse order law for an EP matrix T and its Aluthge transform holds or not. In general, the reverse order law for generalized inverses holds under certain conditions (see e.g., [7, 15]). In the case of EP matrices, the following statement is valid.

Proposition 3.16. [11] *Let $A, B\mathbb{C}^{n\times n}$ be two EP matrices. If $\mathcal{R}(A) = \mathcal{R}(B)$ then $(AB)^\dagger = B^\dagger A^\dagger$.*

Combining Proposition 3.11 and Proposition 3.16, we get the following statement.

Proposition 3.17. *When T is an EP matrix then it holds*

$$(T\,\Delta_\lambda(T))^\dagger = (\Delta_\lambda(T))^\dagger\, T^\dagger$$

and

$$(\Delta_\lambda(T)T)^\dagger = T^\dagger(\Delta_\lambda(T))^\dagger.$$

Proof. In view of Proposition 3.16, the reverse order law is valid if the condition $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$ is satisfied. But, from Proposition 3.11 it is possible to conclude that this condition is satisfied. □

4. Concluding Remarks

In this work we presented some interconnections between the λ-Aluthge transform $\Delta_\lambda(T)$ of a matrix, its inverse and the Moore-Penrose inverse. We focused in the case of EP matrices but the general square matrix was also explored. Some results on the products of EP matrices, the reverse order law and their associations with the λ-Aluthge transform are also presented.

Further work on this subject may include the extension of the above results in infinite dimensional Hilbert spaces. Moreover, the relations between the numerical ranges $W(T), W(\Delta_\lambda(T))$ in the case of EP matrices can be explored, especially in relation to the general inclusion $W(\Delta(T)) \subseteq W(T)$. Also, various properties between them are studied in the literature (see e.g. [20]). In addition, a very similar transformation to the Alutghe transform, called the Duggal transform of an operator/matrix T and denoted by $\widehat{T}$ (see [13]), could be examined in relation to the class of EP matrices. Also, the corresponding properties of the generalized inverse $\widehat{T}^{\dagger}$ can be analysed.

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.

[2] J. Antezana, E.R. Pujlas, D. Stojanoff, *The iterated Aluthge transforms of a matrix converge*, Advances in Mathematics, **226**(2) (2011), 1591-1620

[3] J. Antezana, P. Massey, D. Stojanoff, *λ-Aluthge transforms and Schatten ideals*, Linear Algebra Appl. **405** (2005), 177–199.

[4] J. Antezana, E. Pujals, D. Stojanoff, *Convergence of the iterated Aluthge transform sequence for diagonalizable matrices II: λ-Aluthge transforms*, Integr. Equ. Oper. Theory **62** (2008), 465–488.

[5] A. Ben-Israel and T. N. E. Grenville. *Generalized Inverses: Theory and Applications*, Springer-Verlag, Berlin, 2002.

[6] T. S. Baskett, I. J. Katz, *Theorems on products of* EP_r *matrices*, Linear Algebra Appl. **2** (1969), 87–103.

[7] R. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math. **24**(4) (1973), 489–495.

[8] F. Botelho, L. Molnár, G. Nagy, *Linear bijections on von Neumann factors commuting with λ-Aluthge transform*, Bull. London Math. Soc. **48** (2016), 74–84.

[9] S. L. Campbell, C. D. Meyer, *Generalized Inverses of Linear Transformations*, Dover Publications Inc., New York, 1991.

[10] F. Chabbabi, *Product commuting maps with the λ-Aluthge transform*, J. Math. Anal. Appl. **449** (2017), 589–600.

[11] D. Djordjevic, *Products of EP operators in Hilbert spaces*, Proc. Amer. Math. Soc. **129**(6) (2000), 1727–1731.

[12] D. Drivaliaris S. Karanasios, D. Pappas, *Factorizations of EP operators*, Linear Algebra Appl. **429** (2008), 1555-1567.

[13] C. Foias, Il Bong Jung, E. Ko, C. Pearcy, *Complete contractivity of maps associated with the Aluthge and Duggal transforms*, Pacific J. Math. **209**(2) (2003), 249–259.

[14] C. W. Groetsch (1977). Generalized inverses of linear operators, Marcel Dekker Inc. New York.

[15] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. **34** (1982), 43–52.

[16] G. Ji, Y. Pang, Z. Li, *On the Range of the Aluthge Transform*, Integr. Equ. Oper. Theory **57** (2007), 209–215.

[17] I. B. Jung, E. Ko, C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory **37** (2000), 437-448.

[18] K. Okubo, *On weakly unitarily invariant norm and the Aluthge transformation*, Linear Algebra Appl. **371** (2003), 369–375.

[19] Y. Tian, H. Wang, *Characterizations of EP matrices and weighted-EP matrices* Linear Algebra Appl. **434** (2011), 1295–1318.

[20] T. Yamazaki, *On numerical range of the Aluthge transformation*, Linear Algebra Appl. **341** (2002), 111–117.

[21] N. Zhang, Y. Wei, *Solving EP singular linear systems*, Int. J. Comput. Math. **81** (2004), 1395–1405.