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Application of Chebyshev Polynomials to Certain Subclass of Non-Bazilević Functions

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Abstract. In this paper, we introduce a new certain subclass $\mathcal{N}(\alpha, \lambda, t)$ of Non-Bazilević analytic functions of type $(1-\alpha)$ by using the Chebyshev polynomials expansions. We investigated some basic useful characteristics for this class, also we obtain coefficient bounds and Fekete-Szegö inequalities for functions belong to this class. This class is considered a general case for some of the previously studied classes. Further we discuss its consequences.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the class of analytic functions $f(z) \in \mathcal{A}$ and normalized with the following conditions

$$f(0) = 0 \quad and \quad f'(0) = 1.$$

Let $f(z)$ and $g(z)$ are analytic functions in $\mathcal{U}$, we say that the function $f(z)$ is a subordinate to $g(z)$ in $\mathcal{U}$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \mathcal{U})$ such that $f(z) = g(w(z))$.
Furthermore, if $g(z)$ is univalent in $\mathcal{U}$, then we have the following equivalent

$$f(z) \prec g(z), (z \in \mathcal{U}) \Longleftrightarrow f(0) = g(0) \quad and \quad f(\mathcal{U}) \subset g(\mathcal{U}). \text{ (see[7])}$$

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized Taylor-Mclaurin series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

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is famous for its rich history in the geometric functions theory. Its source was in the disproof by Fekete and Szegö of the 1933 guess of Littlewood and Paley that the coefficients of odd univalent functions are limited by unity (see [9], has since received great attention, especially in many subclasses of the family of univalent functions). For that reason Fekete-Szegö functional was studied by many authors and a some assessments were found in a many subclasses of normalized univalent functions (see [3], [8], [12], [13] and [16]).

The significance of Chebyshev polynomials have progressed toward becoming progressively important in numerical analysis, in the both field theoretical and practical points of view. There are four sorts of Chebyshev polynomials.

The greater part of books and research papers dealing with orthogonal polynomials of Chebyshev, contain chiefly results of first and second kinds of Chebyshev polynomials $T_n(t)$ and $U_n(t)$ respectively and their numerous uses in different applications. Additionally, one can see those given by the papers in ([1], [2], [4], [5], [6] and [10]). The first and second kinds of the Chebyshev polynomials are well known in the case of a genuine variable t on $(-1, 1)$, which are defined as follows

$$T_n(t) = \cos n\theta.$$

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where n means degree of polynomial and $t = \cos\theta$.

Definition 1.1. *A function $f(z) \in \mathcal{A}$ given by (1), is in the aforementioned class $\mathcal{N}(\alpha, \lambda, t)$, if it satisfies the following subordination:*

$$\mathcal{N}(\alpha,\lambda,t) = \Big\{f(z) \in \mathcal{A} : (1-\lambda)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2tz+z^2}\Big\}, \tag{2}$$

where $0 \le \alpha \le 1, \lambda \ge 0, t \in (\frac{1}{2}, 1]$ *and* $z \in \mathcal{U}$.
The function $f(z)$ in this class is said to be Non-Bazilević of type $(1-\alpha)$.

We note that if $t = \cos\alpha, \alpha \in (-\pi/3, \pi/3)$, then

$$\begin{aligned} H(z,t) &:= \frac{1}{1-2\cos\alpha z + z^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\sin((n+1)\alpha)}{\sin\alpha} z^n, \quad (z \in \mathcal{U}). \end{aligned}$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \dots \quad (z \in \mathcal{U}).$$

Furthermore, from [17], we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \quad (z \in \mathcal{U}, t \in (-1,1)),$$

where

$$U_{n-1} = \frac{\sin(n\arccos t)}{\sqrt{1-t^2}}, \quad (n \in \mathbb{N} = \{1,2,3,..\})$$

denotes the second kind of the Chebyshev polynomials. It is known that

$$U_n(t) = 2tU_{n-1} - U_{n-2}(t),$$

and

$$U_1(t) = 2t,$$

$$U_2(t) = 4t^2 - 1, \tag{3}$$

$$U_3(t) = 8t^3 - 4t.$$

$$\vdots$$

The ordinary generating function for Chebyshev polynomials $T_n(t), t \in [-1, 1]$, of the first kind have the following form

$$\sum_{n=0}^{\infty} T(t)z^n = \frac{1 - tz}{1 - 2tz + z^2}, \quad (z \in \mathcal{U}).$$

The Chebyshev polynomials of the first and second kinds which they symbolized $T_n(t)$ and $U_n(t)$ respectively are connected by the following relations:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$

$$T_n(t) = U_n(t) - tU_{n-1}(t),$$

$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

Remark 1.2. *We must be remarked that the class $\mathcal{N}(\alpha, \lambda, t)$ is a generalization of many classes considered earlier. By giving specific values to the parameters α and λ in the class $\mathcal{N}(\alpha, \lambda, t)$. We acquire numerous essential subclass examined by various authors. Let us see a portion of the cases:*

(*i*) *If $\alpha = 1$ and $\lambda = 1$ in the class $\mathcal{N}(\alpha, \lambda, t)$, then we get*

$$\frac{zf'(z)}{f(z)} \prec H(z, t) := \frac{1}{1 - 2tz + z^2},$$

it reduces to the special case from the class $\mathcal{B}^{\mu}_{\Sigma}(\lambda, t)$, which introduced by Bulut, Magesh and Abirami [4].

(*ii*) *If $\alpha = 0$ in the class $\mathcal{N}(\alpha, \lambda, t)$, then we get*

$$(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z, t) := \frac{1}{1 - 2tz + z^2},$$

it reduces to the class $\mathcal{B}_{\Sigma}(\lambda, t)$, which introduced by Bulut , Magesh and Balaji [5].

(*iii*) *If $\alpha = 0$ and $\lambda = 1$ in the class $\mathcal{N}(\alpha, \lambda, t)$, then we get*

$$f'(z) \prec H(z, t) := \frac{1}{1 - 2tz + z^2},$$

it reduces to the special case from the class $\mathcal{B}^{\mu}_{\Sigma}(\lambda, t)$, which introduced by Bulut , Magesh and Abirami [4].

The aim in the present paper, we investigate the geometric properties of a new subclass $\mathcal{N}(\alpha, \lambda, t)$ by applying the Chebyshev polynomial, to provide estimates for initial coefficients of Non-Bazilević functions in $\mathcal{N}(\alpha, \lambda, t)$. In addition to that, the problem of Fekete- Szegö in this class is additionally explained.

2. Preliminaries

We need the following Lemmas to prove our main results:

Lemma 2.1. *[11]. If $w \in \mathcal{S}$, then for any complex number μ*

$$|w_2 - \mu w_1^2| \leq \max\{1; |\mu|\}.$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Lemma 2.2. *[14] Let $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$\frac{1 + A_2 z}{1 + B_2 z} \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

Lemma 2.3. *[15] Let $F(z)$ be analytic and convex in $\mathcal{U}$, $f(z) \in \mathcal{A}, g(z) \in \mathcal{A}$. If*

$$f(z) \prec F(z), \quad g(z) \prec F(z), \quad 0 \leq \lambda \leq 1,$$

then

$$\lambda f(z) + (1 - \lambda)g(z) \prec F(z).$$

3. Main Results

Theorem 3.1. *Let $f(z) \in \mathcal{A}$ belong to the $\mathcal{N}(\alpha, \lambda, t)$. Then*

$$|a_2| \leq \frac{2t}{1 + \lambda - \alpha}$$

and

$$|a_3| \leq \frac{2\alpha t^2}{(1 + \lambda - \alpha)^2} + \frac{4t^2 + 2t - 1}{1 + 2\lambda - \alpha}.$$

Proof. Let the function $f(z) \in \mathcal{N}(\alpha, \lambda, t)$. From (2), we get

$$(1 - \lambda)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda \frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} = 1 + \mathrm{U}_1(t)w(z) + \mathrm{U}_2(t)w^2(z) + \ldots \quad . \tag{4}$$

Replacing the value of $f(z)$ and $f'(z)$ with their equivalent series expressions in (4), we have

$$(1 - \lambda)\Big(\frac{z + \sum_{n=2}^{\infty} a_n z^n}{z}\Big)\Big(\frac{z}{z + \sum_{n=2}^{\infty} a_n z^n}\Big)^{\alpha} + \lambda(1 + \sum_{n=2}^{\infty} n a_n z^{n-1})\Big(\frac{z}{z + \sum_{n=2}^{\infty} a_n z^n}\Big)^{\alpha}$$

$$= 1 + \mathrm{U}_1(t)w(z) + \mathrm{U}_2(t)w^2(z) + \ldots \quad . \tag{5}$$

By using the binomial expansion on the left-hand side of (5) subject to the condition

$$\Big|\sum_{n=2}^{\infty} a_n z^n\Big| < \alpha.$$

Upon simplification, we obtain

$$(1 - \lambda)(1 + \sum_{n=2}^{\infty} a_n z^{n-1})\Big(\frac{1}{1 + \sum_{n=2}^{\infty} \alpha a_n z^{n-1}}\Big) + \lambda(1 + \sum_{n=2}^{\infty} n a_n z^{n-1})\Big(\frac{1}{1 + \sum_{n=2}^{\infty} \alpha a_n z^{n-1}}\Big)$$

$$= 1 + U_1(t)w(z) + U_2(t)w^2(z) + \dots \quad , \tag{6}$$

$$(1-\lambda)(1+\sum_{n=2}^{\infty} a_n z^{n-1}) + \lambda(1+\sum_{n=2}^{\infty} n a_n z^{n-1}) = \Big[1 + U_1(t)w(z) + U_2(t)w^2(z) + \dots.\Big](1+\sum_{n=2}^{\infty} \alpha a_n z^{n-1}). \tag{7}$$

Using the series expansion of $1+\sum_{n=2}^{\infty} \alpha a_n z^{n-1}$, also for some analytic function w such that $w(0) = 0$ and

$$|w(z)| = |c_1 z + c_2 z^2 + c_3 z^3 + \dots| < 1, \quad (z \in \mathcal{U}) \tag{8}$$

where

$$|c_j| \le 1, \quad j \in \mathbb{N} = 1, 2, 3, \dots \quad . \tag{9}$$

From the equalities (8) and (9), we obtain that

$$1 + a_2 z^1 + a_3 z^2 + a_4 z^3 + \dots + \lambda a_2 z^1 + 2\lambda a_3 z^2 + \dots = \Big[1 + U_1(t)c_1 z + (U_1(t)c_2 + U_2(t)c_1^2)z^2 + \dots.\Big]$$

$$\times(1 + \alpha a_2 z^1 + (\alpha a_3 + \frac{\alpha(\alpha-1)}{2!}a_2^2)z^2 + \dots),$$

$$1 + a_2 z^1 + a_3 z^2 + a_4 z^3 + \dots + \lambda a_2 z^1 + 2\lambda a_3 z^2 + \dots = 1 + U_1(t)c_1 z^1 + (U_1(t)c_2 + U_2(t)c_1^2)z^2 + \alpha a_2 z^1$$

$$+\alpha a_2 U_1(t)c_1 z^2 + \alpha a_2(U_1(t)c_2 + U_2(t)c_1^2)z^3 + (\alpha a_3 + \frac{\alpha(\alpha-1)}{2!}a_2^2)z^2 + \dots \tag{10}$$

It follows from (10), we get

$$a_2 + \lambda a_2 = \alpha a_2 + U_1(t)c_1,$$

$$a_2 = \frac{U_1(t)c_1}{1+\lambda-\alpha}. \tag{11}$$

From (3) and (11), we have

$$|a_2| \le \frac{2t}{1+\lambda-\alpha}. \tag{12}$$

Now, in order to find the bound on $|a_3|$, from (10), we have

$$a_3(1+2\lambda-\alpha) = \alpha U_1(t)a_2 c_1 + \frac{\alpha(\alpha-1)}{2!}a_2^2 + U_1(t)c_2 + U_2(t)c_1^2. \tag{13}$$

By using (11) in (13), we get

$$a_3 = \Big\{U_1(t)c_2 + \Big[\frac{\alpha U_1^2(t)}{1+\lambda-\alpha} + \frac{\alpha(\alpha-1)U_1^2(t)}{2(1+\lambda-\alpha)^2} + U_2(t)\Big]c_1^2\Big\}\frac{1}{1+2\lambda-\alpha}, \tag{14}$$

$$= \Big\{U_1(t)c_2 + \Big[\frac{[2\alpha(1+\lambda-\alpha) + \alpha(\alpha-1)]U_1^2(t)}{2(1+\lambda-\alpha)^2} + U_2(t)\Big]c_1^2\Big\}\frac{1}{1+2\lambda-\alpha}.$$

Furthermore, by applying (3) in (14), we obtain

$$|a_3| \le \frac{2t}{1+2\lambda-\alpha} + \frac{4\alpha t^2 - 4\alpha^2 t^2 + 8\alpha\lambda t^2}{2(1+\lambda-\alpha)^2(1+2\lambda-\alpha)} + \frac{4t^2-1}{1+2\lambda-\alpha}, \tag{15}$$

$$= \frac{2\alpha t^2}{(1+\lambda-\alpha)^2} + \frac{4t^2+2t-1}{1+2\lambda-\alpha}.$$

The proof is complete. □

Theorem 3.2. *Let $f(z)$ given by (1) belongs to the class $\mathcal{N}(\alpha,\lambda,t)$ and for any $\mu\in\mathbb{C}$. Then*

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{2t}{1+2\lambda-\alpha}, & \mu\in[\mu_1,\mu_2],\\ \left|\frac{4t^2-1}{(1+2\lambda-\alpha)}+\frac{2\alpha t^2}{(1+\lambda-\alpha)^2}-\mu\frac{4t^2}{(1+\lambda-\alpha)^2}\right| & \mu\notin[\mu_1,\mu_2],\end{cases}\tag{16}$$

where

$$\mu_1=\frac{(1+\lambda-\alpha)^2(4t^2-2t-1)+(2\alpha t^2-2\alpha^2t^2+4\alpha\lambda t^2)}{4t^2(1+2\lambda-\alpha)},$$

$$\mu_2=\frac{(1+\lambda-\alpha)^2(4t^2+2t-1)+(2\alpha t^2-2\alpha^2t^2+4\alpha\lambda t^2)}{4t^2(1+2\lambda-\alpha)}.$$

All of the inequalities are sharp.

Proof:

From (11) and (14), we have

$$|a_3-\mu a_2^2|=\left|\frac{1}{1+2\lambda-\alpha}\left\{U_1(t)c_2+\left[\frac{\alpha U_1^2(t)}{1+\lambda-\alpha}+\frac{\alpha(\alpha-1)U_1^2(t)}{2(1+\lambda-\alpha)^2}+U_2(t)\right]c_1^2\right\}-\mu\frac{U_1^2(t)c_1^2}{(1+\lambda-\alpha)^2}\right|.$$

$$|a_3-\mu a_2^2|=\frac{U_1(t)}{1+2\lambda-\alpha}\left|c_2+\left[\frac{U_2(t)}{U_1(t)}+\frac{\alpha U_1(t)}{1+\lambda-\alpha}+\frac{\alpha(\alpha-1)U_1(t)}{2(1+\lambda-\alpha)^2}-\mu\frac{U_1(t)(1+2\lambda-\alpha)}{(1+\lambda-\alpha)^2}\right]c_1^2\right|.$$

Then, in view of Lemma 2.1 for all $\mu\in\mathbb{C}$, we conclude that

$$|a_3-\mu a_2^2|\le\frac{U_1(t)}{1+2\lambda-\alpha}\max\left\{1,\frac{1}{U_1(t)}\left|U_2(t)+\frac{\alpha U_1^2(t)}{1+\lambda-\alpha}+\frac{\alpha(\alpha-1)U_1^2(t)}{2(1+\lambda-\alpha)^2}-\mu\frac{U_1^2(t)(1+2\lambda-\alpha)}{(1+\lambda-\alpha)^2}\right|\right\}.\tag{17}$$

Finally, by using equalities (3), we get

$$|a_3-\mu a_2^2|\le\frac{2t}{1+2\lambda-\alpha}\max\left\{1,\left|\frac{4t^2-1}{2t}+\frac{2\alpha t}{(1+\lambda-\alpha)}+\frac{\alpha(\alpha-1)t}{(1+\lambda-\alpha)^2}-\mu\frac{2(1+2\lambda-\alpha)t}{(1+\lambda-\alpha)^2}\right|\right\}.\tag{18}$$

$$=\frac{2t}{1+2\lambda-\alpha}\max\left\{1,\left|\frac{4t^2-1}{2t}+\frac{2\alpha(1+\lambda-\alpha)t+\alpha(\alpha-1)t}{(1+\lambda-\alpha)^2}-\mu\frac{2(1+2\lambda-\alpha)t^2}{(1+\lambda-\alpha)^2}\right|\right\}.$$

$$=\frac{2t}{1+2\lambda-\alpha}\max\left\{1,\left|\frac{4t^2-1}{2t}+\frac{\alpha(1+2\lambda-\alpha)t}{(1+\lambda-\alpha)^2}-\mu\frac{2(1+2\lambda-\alpha)t}{(1+\lambda-\alpha)^2}\right|\right\}.$$

Because $t>0$, we have

$$\left|\frac{4t^2-1}{2t}+\frac{\alpha(1+2\lambda-\alpha)t}{(1+\lambda-\alpha)^2}-\mu\frac{2(1+2\lambda-\alpha)t}{(1+\lambda-\alpha)^2}\right|\le 1$$

$$\Longleftrightarrow\left\{\frac{(1+\lambda-\alpha)^2(4t^2-2t-1)+(2\alpha t^2-2\alpha^2t^2+4\alpha\lambda t^2)}{4t^2(1+2\lambda-\alpha)}\le\mu\le\frac{(1+\lambda-\alpha)^2(4t^2+2t-1)+(2\alpha t^2-2\alpha^2t^2+4\alpha\lambda t^2)}{4t^2(1+2\lambda-\alpha)}\right\}$$

$\Longleftrightarrow \quad \{\mu_1 \le \mu \le \mu_2\}$. Moreover, in this case

$$|a_3 - \mu a_2^2| = \left|\frac{4t^2-1}{(1+2\lambda-\alpha)} + \frac{2\alpha t^2}{(1+\lambda-\alpha)^2} - \mu\frac{4t^2}{(1+\lambda-\alpha)^2}\right|$$

The proof is complete □

If we taking $\lambda = 0$ in Theorem 3.2, we get the next Corollary.

Corollary 3.3. *Let $f(z)$ which defined by (1) belongs to the class $\mathcal{N}(\alpha, t)$. Then*

$$|a_2| \le \frac{2t}{1-\alpha}$$

and

$$|a_3| \le \frac{2\alpha t^2}{(1-\alpha)^2} + \frac{4t^2+2t-1}{1-\alpha}.$$

and for any $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{2t}{1-\alpha}, & \mu \in [\mu_1, \mu_2], \\ \left|\frac{4t^2-1}{(1-\alpha)} + \frac{2\alpha t^2}{(1-\alpha)^2} - \mu\frac{4t^2}{(1-\alpha)^2}\right| & \mu \notin [\mu_1, \mu_2], \end{cases} \tag{19}$$

where

$$\mu_1 = \frac{(1-\alpha)^2(4t^2-2t-1) + (2\alpha t^2 - 2\alpha^2 t^2)}{4t^2(1-\alpha)},$$

$$\mu_2 = \frac{(1-\alpha)^2(4t^2+2t-1) + (2\alpha t^2 - 2\alpha^2 t^2)}{4t^2(1-\alpha)}.$$

All of the inequalities are sharp.

If $\alpha = 0$ in Theorem 3.2, we get the next Corollary.

Corollary 3.4. *Let $f(z)$ which defined by (1) belongs to the class $\mathcal{N}(\lambda, t)$. Then*

$$|a_2| \le \frac{2t}{1+\lambda}$$

and

$$|a_3| \le \frac{4t^2+2t-1}{1+2\lambda}.$$

and for any $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{2t}{1+2\lambda}, & \mu \in [\mu_1, \mu_2], \\ \left|\frac{4t^2-1}{(1+2\lambda)} - \mu\frac{4t^2}{(1+\lambda)^2}\right| & \mu \notin [\mu_1, \mu_2], \end{cases} \tag{20}$$

where

$$\mu_1 = \frac{(1+\lambda)^2(4t^2-2t-1)}{4t^2(1+2\lambda)},$$

$$\mu_2 = \frac{(1+\lambda)^2(4t^2+2t-1)}{4t^2(1+2\lambda)}.$$

All of the inequalities are sharp.

Taking $\lambda = 1$ in Theorem 3.2, we get the next Corollary.

Corollary 3.5. *Let $f(z)$ which defined by (1) belongs to the class $\mathcal{N}(\alpha, t)$. Then*

$$|a_2| \le \frac{2t}{2-\alpha}$$

and

$$|a_3| \le \frac{2\alpha t^2}{(2-\alpha)^2} + \frac{4t^2+2t-1}{3-\alpha}.$$

and for any $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{2t}{3-\alpha}, & \mu \in [\mu_1, \mu_2], \\ \left|\frac{4t^2-1}{(3-\alpha)} + \frac{2\alpha t^2}{(2-\alpha)^2} - \mu \frac{4t^2}{(2-\alpha)^2}\right| & \mu \notin [\mu_1, \mu_2], \end{cases} \tag{21}$$

where

$$\mu_1 = \frac{(2-\alpha)^2(4t^2-2t-1) + (6\alpha t^2 - 2\alpha^2 t^2)}{4t^2(3-\alpha)},$$

$$\mu_2 = \frac{(2-\alpha)^2(4t^2+2t-1) + (6\alpha t^2 - 2\alpha^2 t^2)}{4t^2(3-\alpha)}.$$

All of the inequalities are sharp.

Taking $\alpha = 1$ in Theorem 3.2, we get the next Corollary.

Corollary 3.6. *Let $f(z)$ which defined by (1) belongs to the class $\mathcal{N}(\lambda, t)$. Then*

$$|a_2| \le \frac{2t}{\lambda}$$

and

$$|a_3| \le \frac{2t^2}{\lambda^2} + \frac{4t^2+2t-1}{2\lambda}.$$

and for any $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{t}{\lambda}, & \mu \in [\mu_1, \mu_2], \\ \left|\frac{4t^2-1}{2\lambda} + \frac{2t^2}{\lambda^2} - \mu \frac{4t^2}{\lambda^2}\right| & \mu \notin [\mu_1, \mu_2], \end{cases} \tag{22}$$

where

$$\mu_1 = \frac{\lambda^2(4t^2 - 2t - 1) + 4\lambda t^2}{8\lambda t^2},$$

$$\mu_2 = \frac{\lambda^2(4t^2 + 2t - 1) + 4\lambda t^2}{8\lambda t^2}.$$

All of the inequalities are sharp.

Taking $\alpha = 1$ and $\lambda = 1$ in Theorem 3.2, we get the next Corollary.

Corollary 3.7. *Let $f(z)$ which defined by (1) belongs to the class $\mathcal{N}(t)$. Then*

$$|a_2| \leq 2t$$

and

$$|a_3| \leq 4t^2 + t - \frac{1}{2}.$$

and for any $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} t, & \mu \in [\mu_1, \mu_2], \\ \left|\frac{4t^2-1}{2} + 2t^2 - 4\mu t^2\right| & \mu \notin [\mu_1, \mu_2], \end{cases} \tag{23}$$

where

$$\mu_1 = \frac{8t^2 - 2t - 1}{8t^2},$$

$$\mu_2 = \frac{8t^2 + 2t - 1}{8t^2}.$$

Theorem 3.8. *A function $f(z)$ be defined by (1) belong to the class $\mathcal{N}(\alpha, \lambda, t)$ if there exists a function $q \in \mathcal{N}(\alpha, \lambda, t)$, $q(z) \prec H(z, t)$, and $\alpha = 1$ such that*

$$f(z) = \exp \int_0^z \frac{q(u) + \lambda - 1}{\lambda u} du. \tag{24}$$

Proof: If $f \in \mathcal{N}(\alpha, \lambda, t)$, then there exists a function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in \mathcal{U}$ such that

$$(1 - \lambda)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda \frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} = H(w(z), t) := q(z).$$

Since $\alpha = 1$, we have

$$(1 - \lambda) + \lambda \frac{zf'(z)}{f(z)} = H(w(z), t) := q(z). \tag{25}$$

$$\lambda \frac{zf'(z)}{f(z)} = q(z) + \lambda - 1. \tag{26}$$

Now, $q(z) \prec H(z,t)$ and the equality (26) can be easily obtained

$$\left\{\log f(z)\right\}' = \frac{q(z)+\lambda-1}{\lambda z}.$$

Then, we get the integration in the equation (24). Thus, it is complete the proof of Theorem.

Corollary 3.9. *Let $\lambda \in \mathbb{C}, 0 \leq \alpha \neq 1$ and $Re\{\lambda\} \geq 0$. Then*

$$\mathcal{N}(\alpha,\lambda,t) \subset \mathcal{N}(\alpha,0,t).$$

Proof: Let $f(z) \in \mathcal{N}(\alpha,\lambda,t)$. Then

$$(1-\lambda)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2tz+z^2}.$$

If we impose the value of $\lambda = 0$, we have

$$\Big(\frac{f(z)}{z}\Big)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2tz+z^2}.$$

Thus mean that $f(z) \in \mathcal{N}(\alpha,0,t)$. Therefore, we get $\mathcal{N}(\alpha,\lambda,t) \subset \mathcal{N}(\alpha,0,t)$.

Theorem 3.10. *Let $0 \leq \lambda_1 \leq \lambda_2, \alpha \neq 1$, and $\frac{1}{2} < t_1 \leq t_2 \leq 1$ .Then*

$$\mathcal{N}(\alpha,\lambda_2,t_2) \subset \mathcal{N}(\alpha,\lambda_1,t_1).$$

Proof: Suppose that $f(z) \in \mathcal{N}(\alpha,\lambda_2,t_2)$, we get $f(z) \in \mathcal{A}$ and

$$(1-\lambda_2)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda_2\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2t_2z+z^2}. \tag{27}$$

Since $\frac{1}{2} < t_1 \leq t_2 \leq 1$. Therefore, it follows form Lemma 2.2 we get

$$(1-\lambda_2)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda_2\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2t_1z+z^2}, (z \in \mathcal{U}). \tag{28}$$

That is $f(z) \in \mathcal{N}(\alpha,\lambda_2,t_1)$. Furthermore, Theorem 3.10 is proved when we impose $\lambda_1 = \lambda_2 \geq 0$.

When $\lambda_2 > \lambda_1 \geq 0$, then we can see form Corollary 3.9, that $f(z) \in \mathcal{N}(\alpha,0,t_1)$,

then

$$\Big(\frac{f(z)}{z}\Big)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2t_1z+z^2}. \tag{29}$$

But

$$(1-\lambda_1)\Big(\frac{f(z)}{z}\Big)^{1-\alpha} + \lambda_1\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha} = (1+\frac{\lambda_1}{\lambda_2})\Big(\frac{f(z)}{z}\Big)^{1-\alpha}$$

$$-\frac{\lambda_1}{\lambda_2}\Big[(1+\lambda_2)\Big(\frac{f(z)}{z}\Big)^{1-\alpha}-\lambda_2\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha}\Big].$$

Clearly that $h(z)=\frac{1}{1-2t_1z+z^2}$, is analytic and convex in $\mathcal{U}$, therefore, from

Lemma 2.3 and differential subordination (28) and (29), we get

$$(1-\lambda_1)\Big(\frac{f(z)}{z}\Big)^{1-\alpha}+\lambda_1\frac{zf'(z)}{f(z)}\Big(\frac{f(z)}{z}\Big)^{1-\alpha}\prec\frac{1}{1-2t_1z+z^2}.$$

We conclude that $f(z)\in\mathcal{N}(\alpha,\lambda_1,t_1)$. Thus we get

$$\mathcal{N}(\alpha,\lambda_2,t_2)\subset\mathcal{N}(\alpha,\lambda_1,t_1).$$

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