Filomat 32:12 (2018), 4165–4176
https://doi.org/10.2298/FIL1812165K

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Proximal Point Algorithms Involving Cesàro Type Mean of Total Asymptotically Nonexpansive Mappings in CAT(0) Spaces

Amna Kalsoom[a], Hafiz Fukhar-ud-din[b], Sara Najib[a]

[a]*Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, 44000, Pakistan*
[b]*Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 3126, Saudi Arabia*

Abstract. In this paper, we extend the proximal point algorithm proposed by Chang et al.[8] for total asymptotically nonexpansive mapping in CAT(0) spaces. We also demonstrate the Δ-convergence and strong convergence to a common element of the set of minimizers of a convex function and the set of fixed points of the Cesàro type mean of total asymptotically nonexpansive mappings in CAT(0) spaces.

1. Introduction

Let C be a nonempty subset of a metric space X and $F(T)$, the set of fixed points of a mapping $T : C \to C$. Recall that T is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \text{for all } x,\ y \in C, \quad n \geq 1,$$

it is asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(d(T^n x, T^n y) - d(x, y)\right) \leq 0$$

and a *total asymptotically nonexpansive mapping* [1] if there exist non negative real sequences $\{k_n\}$ and $\{\varphi_n\}$ with $k_n \to 0$ and $\varphi_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + k_n \xi(d(x, y)) + \varphi_n \quad \text{for all } x,\ y \in C, \quad n \geq 1.$$

This is the most general class of mappings which includes both the classes of mappings defined above.

2010 *Mathematics Subject Classification*. 47H09, 47H10, 47J25
Keywords. CAT(0) space, convex minimization problem, resolvent identity, total asymptotically nonexpansive mapping, proximal point algorithm, strong and Δ-convergence
Received: 20 November 2017; Accepted: 24 April 2018
Communicated by Vasile Berinde
The first author would like to greatly acknowledge Start Up Research Grant program (SRGP) from Higher Education Commission (HEC) of Pakistan (No.: 21-1391 SRGP/R&D/HEC/2016) for providing funds during this project. The second author would like to acknowledge the support provided by the Deanship of Scientific Research(DSR) at King Fahd University of Petroleum & Minerals (KFUPM) for funding this work through project No. IN151014.
Email addresses: amna.kalsoom@iiu.edu.pk (Amna Kalsoom), hfdin@kfupm.edu.sa (Hafiz Fukhar-ud-din)

Examples of total asymptotically nonexpansive mappings:

(i) Let $X = \mathbb{R}$, $C = [0, \infty)$ and $T : C \to C$ be defined by $Tx = \sin x$. Then T is a total asymptotically nonexpansive[**15**].

(ii) Let $X = \mathbb{R}, C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and let $|k| < 1$. For each $x \in C$, we define $T : C \to C$ by

$$Tx = \begin{cases} kx \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive in the intermediate sense [17] and hence total asymptotically nonexpansive.

(iii) Let $X = \mathbb{R}, C = [0, 2]$ and $T : C \to C$ be defined by

$$Tx = \begin{cases} 1 & \text{if } x \in [0, 1] \\ \frac{1}{\sqrt{3}} \sqrt{4 - x^2} & \text{if } x \in [1, 2]. \end{cases}$$

Then T is a total asymptotically nonexpansive [16].

In 1975, Baillon [5] first constructed the following Cesàro mean iterative algorithm of a nonexpansive mapping T on a Hilbert space:

$$T_n x = \frac{1}{n+1} \sum_{i=0}^{n} T^i x \tag{1}$$

and proved that it weakly converges to a fixed point of T.

Shimizu and Takahashi [23] provided a strong convergence theorem modifying (1) for an asymptotically nonexpansive mapping in Hilbert spaces.

Fixed point theory for various types of mappings in CAT(0) spaces has been investigated rapidly. In 2008, Dhompongsa and Panyanak [9] studied strong and $\Delta-$convergence of the Mann and Ishikawa algorithms for nonexpansive mappings in CAT(0) spaces.

Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ a proper convex and lower semi-continuous function. One of the important optimization problems in a Hilbert space H is to find $x \in H$ such that

$$f(x) = \min_{y \in H} f(y). \tag{2}$$

The set of minimizers of f is denoted by $\arg min_{y \in H} f(y)$.

A solution to problem(2) is provided by the proximal point algorithm (shortly, the PPA) initiated by Martinet [21] in 1970. In 1976, Rockafellar [22] studied the convergence to a solution of the convex minimization problem in Hilbert spaces and also used this method.

For a proper, convex, and lower semi-continuous function f on a Hilbert space H which attains its minimum, the (PPA) is defined by:

$$\begin{aligned} x_1 &\in H, \\ x_{n+1} &= \arg min_{y \in H} \left(f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right), \text{ for all } n \geq 1, \end{aligned}$$

where $\lambda_n > 0$.

Recently, many convergence results using the (PPA) for solving optimization problems have been extended from classical linear spaces to the setting of manifolds. For numerous applications of these methods, we refer the reader to [3, 6, 10, 12, 20].

In 2013, Bačák [4] introduced the (PPA) in a CAT(0) space X as under:

$$x_1 \in X, x_{n+1} = \arg min_{y \in X} \left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right), \text{ for all } n \geq 1, \tag{3}$$

where $\lambda_n > 0$.

Ariza-Ruiz [3] established that if the set of minimizers of f is nonempty and $\Sigma_{n=1}^{\infty}\lambda_n = \infty$, then the sequence $\{x_n\}$, in (3), $\Delta-$converges to its minimizer.

In 2016, Chang-Wen-Yao [8] established strong convergence and $\Delta-$convergence theorems of the following iterative algorithm $\{x_n\}$ by using fixed points and Cesàro type mean of an asymptotically nonexpansive mapping in CAT(0) spaces:

$$\begin{aligned} &x_0 \in C,\\ &z_n = \arg min_{y\in C}[f(y) + \tfrac{1}{2\lambda_n}d^2(y, x_n)],\\ &y_n = (1-\beta_n)x_n \oplus \tfrac{\beta_n}{n+1}\bigoplus_{j=0}^{n} T^j z_n\\ &x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \quad \text{for all } n \geq 1 \end{aligned} \tag{4}$$

where $\alpha_n, \beta_n \in (0,1)$, $\lambda_n > 0$ and $d\left(\frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, x\right) = \frac{1}{n+1}\sum_{j=0}^{n} d\left(T^j z_n, x\right)$ for $x \in C$.

This paper aims to use Cesàro type mean of a total asymptotically nonexpansive mappings to study the proximal point algorithm(4) for finding a common element of the set of minimizers of a convex function and the set of fixed points of total asymptotically nonexpansive mappings through $\Delta-$convergence and stronge convergence in CAT(0) spaces.

2. Preliminaries

Let X be a metric space and $x, y \in X$. A *geodesic path* from x to y is a mapping $\theta : [0, d(x,y)] \to X$ such that $\theta(0) = x$, $\theta(d(x,y)) = y$, and $d(\theta(t), \theta(t')) = |t - t'|$ for $t, t' \in [0, d(x,y)]$. The image of θ is known as a geodesic segment in X. A metric space X is a uniquely geodesic space if any two points of X are joined by a unique geodesic segment.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space X consists of three points x_1, x_2 and x_3 in X and a geodesic segment between each pair of these points. A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}\left(\bar{x}_i, \bar{x}_j\right) = d\left(x_i, x_j\right)$ for all $i, j = 1, 2, 3$.

A geodesic space X is a CAT(0) space if for each Δ in X and $\overline{\Delta}$ in $\mathbb{R}^2$, the inequality

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

holds for all $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \overline{\Delta}$.

All Hilbert spaces are CAT(0) spaces [14] while this is not the case with Banach spaces [25].

In this paper, we write $(1-t)x \oplus ty$ for the unique point z on the geodesic segment joining x and y such that

$$d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y).$$

Let $\{x_n\}$ be a bounded sequence in a closed and convex subset C of a CAT(0) space X. For any $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ and asymptotic center $A(\{x_n\})$of $\{x_n\}$, respectively, are given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and

$$A(\{x_n\}) = \{x \in X : r(\{x_n\}) = r(x, \{x_n\})\}.$$

It is well known that, in CAT(0) spaces, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in a CAT(0) space X is said to $\Delta-$converge to a point $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case, we write $\Delta - \lim\limits_{n\to\infty} x_n = x$.

A function $f : C \to (-\infty, \infty]$ defined on a convex subset C of a CAT(0) space is convex if for any geodesic $[x, y] = \{\gamma_{x,y}(\lambda) : 0 \le \lambda \le 1\} = \{\lambda x \oplus (1-\lambda)y : 0 \le \lambda \le 1\}$ joining $x, y \in C$, the function $f \circ \gamma$ is convex, that is,

$$f\left(\gamma_{x,y}(\lambda)\right) = f(\lambda x \oplus (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y).$$

Let X be a CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in X by

$$J_\lambda(x) = \arg min_{y\in X}[f(y) + \frac{1}{2\lambda}d^2(y, x)], \quad \forall x \in X.$$

Ariza et. al [3] has shown that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\arg\min\limits_{y\in X} f(y)$ of minimizers of f.

Lemma 2.1. *[7].Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that $\Sigma_{n=1}^{\infty} b_n < \infty$ and $\Sigma_{n=1}^{\infty} c_n < \infty$. If $a_{n+1} \le (1+b_n)a_n + c_n$ for all $n \ge 1$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.2. *[9].In a CAT(0) space X, we have the followings:*
(i) $d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z)$ for all $x, y, z \in X, t \in [0, 1]$
(ii) $d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y)$ for all $x, y, z \in X, t \in [0, 1]$.

Lemma 2.3. *[19]. Let X be a complete CAT(0) space. Then every bounded sequence in X has a $\Delta-$ convergent subsequence.*

Lemma 2.4. *[18]. If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.5. *[11]. Let X be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ a proper convex and lower semi-continuous function. Then we have the followings:*
(i) For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive
(ii) For all $x \in X$ and $\lambda > \mu > 0$, the following identity holds:

$$J_\lambda x = J_\mu(\frac{\lambda-\mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x).$$

Lemma 2.6. *[2]. (sub-differential inequality). Let X be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\lambda > 0$, the following inequality holds:*

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x, y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \le f(y).$$

Lemma 2.7. *[13].Let X be a complete CAT(0) space and C a nonempty closed and convex subset of X and $T : C \to C$ a uniformly continuous and total asymptotically nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim\limits_{n\to\infty} x_n = x$, then $Tx = x$.*

3. Convergence Theorems Involving Cesàro Type Mean of Total Asymptotically Nonexpansive Mappings

We are now in a position to prove our main results.

Theorem 3.1. *Let X be a complete $CAT(0)$ space and C its nonempty, closed and convex subset. Let $f : C \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function, $T : C \to C$ a uniformaly continuous and total asymptotically nonexpansive mapping with sequences $\{k_n\}$ and $\{\varphi_n\}$, where $k_n \to 0$ and $\varphi_n \to 0$ and a strictly increasing continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence given in (4) where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ with $0 < a \le \alpha_n, \beta_n < b < 1$ and $\{\lambda_n\}$ a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \ge 1$ and some λ. Set $L_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j$, $L_n^{'} = \frac{1}{n+1}\sum_{j=0}^{n} \varphi_j$ and $\sigma_n = \max\{k_n, \varphi_n, L_n, L_n'\}$. If $\sum_{n=0}^{\infty} \sigma_n < \infty$, $\Omega = F(T) \cap \arg min_{y\in C} f(y) \neq \phi$ and if there exist constants M and $M^* > 0$ such that $\xi(\lambda) \le M^*\lambda$ for $\lambda \ge M$, then $\{x_n\}$, $\Delta-$converges to a point in Ω.*

Proof. By the strictly increasing function ξ and the inequality $\xi(\lambda) \le M^*\lambda$ for $\lambda \ge M$, it follows that

$$\xi\left(d\left(x, y\right)\right) \le \xi\left(M\right) + M^* d\left(x, y\right). \tag{5}$$

Since $\sum_{n=0}^{\infty} \sigma_n < \infty$, therefore

$$L_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \to 0 \text{ and } L_n^{'} = \frac{1}{n+1}\sum_{j=0}^{n} \varphi_j \to 0 \quad (as\ n \to \infty). \tag{6}$$

Let $q \in \Omega$. Then $q = Tq$ and $f(q) \le f(y)$ for all $y \in C$. This implies that

$$f(q) + \frac{1}{2\lambda_n} d^2(q, q) \le f(y) + \frac{1}{2\lambda_n} d^2(y, q) \text{ for all } y \in C,$$

and hence $q = J_{\lambda_n} q$, for all $n \ge 1$, where J_{λ_n} is the Moreau-Yosida resolvent of f in X defined above. We divide the proof into five steps.
Step(I): First we prove that the $\lim_{n\to\infty} d(x_n, q)$ exists. Indeed, since $z_n = J_{\lambda_n} x_n$, by Lemma 2.5(i), J_{λ_n} is nonexpansive. Hence we have

$$d(z_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \le d(x_n, q). \tag{7}$$

Applying Lemma 2.2(i) to (4), we have that

$$\begin{aligned} d(y_n, q) &= d\left((1-\beta_n)x_n \oplus \frac{\beta_n}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right) \\ &\le (1-\beta_n) d\left(x_n, q\right) + \beta_n d\left(\frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right) \\ &\le (1-\beta_n) d\left(x_n, q\right) + \frac{\beta_n}{n+1}\sum_{j=0}^{n} d\left(T^j z_n, q\right). \end{aligned} \tag{8}$$

Further we calculate

$$\begin{aligned}\frac{\beta_n}{n+1}\sum_{j=0}^{n} d\left(T^j z_n, q\right) &\le \frac{\beta_n}{n+1}\sum_{j=0}^{n}\left\{d\left(z_n,q\right)+k_j\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]+\varphi_j\right\}\\ &= \frac{\beta_n}{n+1}d\left(z_n,q\right)+\frac{\beta_n}{n+1}\sum_{j=0}^{n}k_j\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]\\ &+\frac{\beta_n}{n+1}\sum_{j=0}^{n}\varphi_j\\ &\le \frac{\beta_n}{n+1}d\left(x_n,q\right)+\frac{\beta_n}{n+1}\sum_{j=0}^{n}k_j\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]\\ &+\frac{\beta_n}{n+1}\sum_{j=0}^{n}\varphi_j.\end{aligned}$$

That is,

$$\begin{aligned}\frac{\beta_n}{n+1}\sum_{j=0}^{n} d\left(T^j z_n, q\right) &\le \frac{\beta_n}{n+1}d\left(x_n,q\right)\\ &+\beta_n\ L_n\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]+\beta_n\ L_n^{'}.\end{aligned} \tag{9}$$

Using (9) into (8), we get that

$$\begin{aligned}d(y_n,q) &\le (1-\beta_n)d\left(x_n,q\right)+\frac{\beta_n}{n+1}d\left(z_n,q\right)\\ &+\beta_n\ L_n\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]+\beta_n\ L_n^{'}\\ &\le (1-\beta_n)d\left(x_n,q\right)+\beta_n d\left(x_n,q\right)\\ &+\beta_n\ L_n\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]+\beta_n\ L_n^{'}\\ &\le d\left(x_n,q\right)+\beta_n\ L_n\ \left[\xi\left(M\right)+M^* d\left(z_n,q\right)\right]+\beta_n\ L_n^{'}.\end{aligned} \tag{10}$$

With the help of Lemma 2.2(i), (5),(7) and (10), we calculate that

$$\begin{aligned}d(x_{n+1},q) &= d((1-\alpha_n)x_n\oplus\alpha_n T^n y_n,q)\\ &\le (1-\alpha_n)d(x_n,q)+\alpha_n d(T^n y_n,q)\\ &\le (1-\alpha_n)d(x_n,q)+\alpha_n\left\{d(y_n,q)+k_n\ \xi\left(d(y_n,q)\right)+\varphi_n\right\}\\ &\le (1-\alpha_n)d(x_n,q)+\alpha_n\left\{\begin{array}{c} d(y_n,q)+k_n\xi\left(M\right)\\ +k_n M^* d\left(y_n,q\right)\ +\varphi_n\end{array}\right\}\\ &\le d\left(x_n,q\right)+\alpha_n\beta_n\ L_n\xi\left(M\right)+\alpha_n\beta_n\ L_n\ M^* d\left(x_n,q\right)\\ &+\alpha_n k_n M^* d\left(x_n,q\right)+\alpha_n k_n M^*\beta_n\ L_n\ \xi\left(M\right)\\ &+\alpha_n\beta_n\ k_n L_n\ M^{*2} d\left(x_n,q\right)+\alpha_n\beta_n\ k_n L_n^{'} M^*+\alpha_n\beta_n\ L_n^{'}+\alpha_n\varphi_n\end{aligned} \tag{11}$$

continuing the process,

$$\begin{aligned} d(x_{n+1},q) &\leq d(x_n,q) + \left\{ \begin{array}{c} \alpha_n\beta_n\, L_n\, M^* + \alpha_n k_n M^* \\ +\alpha_n\beta_n\, k_n L_n\, M^{*2} \end{array} \right\} d(x_n,q) \\ &+ \alpha_n\beta_n\, L_n \xi(M) + \alpha_n k_n M^* \beta_n\, L_n\, \xi(M) + \alpha_n\beta_n\, k_n L_n^{'} M^* \\ &+ \alpha_n\beta_n\, L_n^{'} + \alpha_n\varphi_n \\ &\leq d(x_n,q) + \left\{ b^2\sigma_n\, M^* + b\sigma_n M^* + b^2 b_1\sigma_n\, M^{*2} \right\} d(x_n,q) \\ &+ b^2\xi(M)\sigma_n + b^2 M^*\xi(M) b_1\sigma_n + b^2 b_1\sigma_n M^* + b^2\sigma_n M^* + b\sigma_n \\ &\leq \left[1 + \left(b^2\, M^* + bM^* + b^2 b_1 M^{*2} \right)\sigma_n \right] d(x_n,q) \\ &+ \left(b^2\xi(M) + b^2 M^*\xi(M) b_1 + b^2 b_1 M^* + b^2 M^* + b \right)\sigma_n. \end{aligned} \tag{12}$$

Applying Lemma 2.1 to (12), we obtain that $\lim_{n\to\infty} d(x_n,q)$ exists. Without loss of generality, we assume that

$$\lim_{n\to\infty} d(x_n,q) = c > 0. \tag{13}$$

In the presence of (13), we deduce that the sequences $\{z_n\}$, $\{y_n\}$, $\{T^j z_n\}$ and $\{T^n y_n\}$ are bounded. Step(II): Now we prove that

$$\lim_{n\to\infty} d(x_n,z_n) = 0.$$

By the sub-differential inequality (Lemma 2.6) we have

$$\frac{1}{2\lambda_n}\left\{ d^2(z_n,q) - d^2(x_n,q) + d^2(x_n,z_n) \right\} \leq f(q) - f(z_n).$$

Since $f(q) \leq f(z_n)$ for all $n \geq 1$, it follows from the above inequality that

$$d^2(x_n,z_n) \leq d^2(x_n,q) - d^2(z_n,q). \tag{14}$$

In order to prove that $\lim_{n\to\infty} d(x_n,z_n) = 0$, it suffices to prove that $d^2(z_n,q) \to c$.

In fact, it follows from (12) that

$$\begin{aligned} d(x_{n+1},q) &\leq (1-\alpha_n)d(x_n,q) + \alpha_n\left\{ d(y_n,q) + k_n\, \xi(d(y_n,q)) + \varphi_n \right\} \\ &\leq d(x_n,q) - \alpha_n d(x_n,q) + \alpha_n\left\{ d(y_n,q) + k_n\left[\xi(M) + M^* d(y_n,q) \right] + \varphi_n \right\}. \end{aligned}$$

Rewriting the above inequality, we have

$$\begin{aligned} d(x_n,q) &\leq \frac{1}{\alpha_n}\left[d(x_n,q) - d(x_{n+1},q) \right] + d(y_n,q)(1 + k_n\, M^*) + k_n\xi(M) + \varphi_n \\ &\leq \frac{1}{a}\left[d(x_n,q) - d(x_{n+1},q) \right] + d(y_n,q)(1 + k_n\, M^*) + k_n\xi(M) + \varphi_n. \end{aligned} \tag{15}$$

The inequality (15) together with (13) implies that

$$c = \liminf_{n\to\infty} d(x_n,q) \leq \liminf_{n\to\infty} d(y_n,q). \tag{16}$$

On the other hand, it follows from (10) and (6) that

$$\limsup_{n\to\infty} d(y_n,q) \leq \limsup_{n\to\infty} d(x_n,q) = c.$$

This together with (16) implies that

$$\lim_{n\to\infty} d(y_n, q) = c. \tag{17}$$

From (10), we also have that

$$\begin{aligned} d(y_n, q) &\leq (1-\beta_n)d(x_n, q) + \frac{\beta_n}{n+1} d(z_n, q) \\ &\quad + \beta_n L_n \left[\xi(M) + M^* d(z_n, q)\right] + \beta_n L_n' \\ &\leq (1-\beta_n)d(x_n, q) + \beta_n d(z_n, q) + \beta_n L_n \left[\xi(M) + M^* d(z_n, q)\right] + \beta_n L_n' \\ &\leq d(x_n, q) - \beta_n d(x_n, q) + \beta_n d(z_n, q) + \beta_n L_n \left[\xi(M) + M^* d(z_n, q)\right] + \beta_n L_n' \end{aligned}$$

which can be rewritten as

$$\begin{aligned} d(x_n, q) &\leq \frac{1}{\beta_n}\left[d(x_n, q) - d(y_n, q)\right] + d(z_n, q) + L_n \left[\xi(M) + M^* d(z_n, q)\right] + L_n' \\ &\leq \frac{1}{a}\left[d(x_n, q) - d(y_n, q)\right] + d(z_n, q) + L_n \left[\xi(M) + M^* d(z_n, q)\right] + L_n' \end{aligned}$$

This together with (13) and (6) provides that

$$c = \liminf_{n\to\infty} d(x_n, q) \leq \liminf_{n\to\infty} d(z_n, q). \tag{18}$$

From (7), it follows that

$$\limsup_{n\to\infty} d(z_n, q) \leq \limsup_{n\to\infty} d(x_n, q) = c. \tag{19}$$

Combining (18) and (19), we have that

$$\lim_{n\to\infty} d(z_n, q) = c. \tag{20}$$

Therefore it follows from (13)-(14) and (20) that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{21}$$

Step(III): Now we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0, \text{ and } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

It follows from Lemma 2.2(ii) and (4) that

$$\begin{aligned} d(y_n, q)^2 &= d\left((1-\beta_n)x_n \oplus \frac{\beta_n}{n+1} \bigoplus_{j=0}^{n} T^j z_n, q\right)^2 \\ &\leq (1-\beta_n)d(x_n, q)^2 + \beta_n d\left(\frac{1}{n+1} \bigoplus_{j=0}^{n} T^j z_n, q\right)^2 \\ &\quad - \beta_n(1-\beta_n)d\left(x_n, \frac{1}{n+1} \bigoplus_{j=0}^{n} T^j z_n\right)^2. \end{aligned} \tag{22}$$

We also compute

$$d\left(\frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n, q\right)^2 \leq \left\{\frac{1}{n+1}\sum_{j=0}^{n} d(z_n,q) + k_j\xi\left(d(z_n,q)\right) + \varphi_n\right\}^2$$
$$\leq \left\{\begin{array}{c} d\left(z_n,q\right) + \frac{1}{n+1}\sum_{j=0}^{n} k_j\ \left[\xi\left(M\right) + d\left(z_n,q\right)M^*\right] \\ +\frac{1}{n+1}\sum_{j=0}^{n}\varphi_j \end{array}\right\}^2 \tag{23}$$
$$\leq \left\{d\left(z_n,q\right) + L_n\ \left[\xi\left(M\right) + d\left(z_n,q\right)M^*\right] + L_n'\right\}^2$$

Denote $s_n = d\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n\right)^2$.

Substituting (23) into (22), simplifying and using $L_n \to 0$, $L_n' \to 0, d(x_n,q) \to c$, and $d(y_n,q) \to c$ (as $n \to \infty$), we have

$$a\left(1-b\right)s_n \leq \beta_n(1-\beta_n)d\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n\right)^2$$
$$\leq (1-\beta_n)d^2\left(x_n,q\right) + \beta_n\left\{d\left(z_n,q\right) + L_n\ \left[\xi\left(M\right) + d\left(z_n,q\right)M^*\right] + L_n'\right\}^2$$
$$- d^2\left(y_n,q\right)$$
$$\leq (1-\beta_n)d^2\left(x_n,q\right) + \beta_n\left\{d\left(z_n,q\right) + L_n\ \left[\xi\left(M\right) + d\left(z_n,q\right)M^*\right] + L_n'\right\}^2$$
$$- d^2\left(y_n,q\right)$$
$$\leq d^2\left(x_n,q\right) - \beta_n d^2\left(x_n,q\right) + \beta_n\left\{d\left(z_n,q\right) + L_n\ \left[\xi\left(M\right) + d\left(z_n,q\right)M^*\right] + L_n'\right\}^2$$
$$- d^2\left(y_n,q\right) \to 0 \text{ as } n \to \infty.$$

That is,

$$d\left(x_n, \frac{1}{n+1}\bigoplus_{j=0}^{n} T^j z_n\right) \to 0 \text{ as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n} d\left(x_n, T^j z_n\right)^2 = 0.$$

This implies that

$$\lim_{n\to\infty} d\left(x_n, T^j z_n\right) = 0 \text{ for each } j = 0,1,2,...,n.$$

Since T is uniformly continuous, (21) holds and

$$d(x_n,Tx_n) \leq d\left(x_n,Tz_n\right) + d\left(Tx_n,Tz_n\right),$$

therefore

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{24}$$

Step(IV): We also prove that

$$\lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0, \text{ where } \lambda_n \geq \lambda > 0.$$

It follows from (17) and Lemma 2.6(ii) that

$$\begin{aligned} d(J_\lambda x_n, x_n) &\leq d(J_\lambda x_n, z_n) + d(z_n, x_n) = d(J_\lambda x_n, J_{\lambda_n} x_n) + d(z_n, x_n) \\ &= d\left(J_\lambda x_n, J_\lambda\left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \bigoplus \frac{\lambda}{\lambda_n} x_n\right)\right) + d(z_n, x_n) \\ &\leq d\left(x_n, \left(1 - \frac{\lambda}{\lambda_n}\right) J_{\lambda_n} x_n \bigoplus \frac{\lambda}{\lambda_n} x_n\right) + d(z_n, x_n) \\ &\leq \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, J_\lambda x_n) + d(z_n, x_n) \\ &= \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, z_n) + d(z_n, x_n) \to 0. \end{aligned}$$

That is,

$$\lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0. \tag{25}$$

Step(V): Here we prove that

$$w_\Delta(x_n) = \bigcup_{\{u_n\}\subset\{x_n\}} \{A(\{u_n\})\} \subset \Omega.$$

Let $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. It follows from Lemma 2.3 that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$. In view of (24) and (25), we get that

$$\lim_{n\to\infty} d(v_n, Tv_n) = 0 = \lim_{n\to\infty} d(J_\lambda v_n, v_n).$$

Therefore by Lemma 2.7, $v \in \Omega$. Also, by (13), the limit $\lim_{n\to\infty} d(x_n, v)$ exists. Hence by Lemma 2.4, $u = v$. This shows that $w_\Delta(x_n) \subset \Omega$.

Finally, we show that the sequence $\{x_n\}$, $\Delta-$converges to a point in Ω. To this end, it suffices to show that $w_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_\Delta(x_n) \subset \Omega$ and $\{d(x_n, u)\}$ converges by (13), we have $x = u$. Hence $w_\Delta(x_n) = \{x\}$. This completes the proof. □

Recall that a mapping $T : C \to C$ is semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$), there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ strongly converges to a point $x \in C$.

Our strong convergence theorems are as under.

Theorem 3.2. *Under the assumptions of Theorem 3.1 if, in addition, T or J_λ is semi-compact, then the sequence $\{x_n\}$ defined by (4) strongly converges to a point in Ω.*

Proof. It follows from (24) and (25) that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, J_\lambda(x_n)). \tag{26}$$

By the assumption, we assume T is semi-compact. Then by (26), there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x \in C$. Hence, from (26), we have

$$d(x, Tx) = \lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0 \text{ and } d(x, J_\lambda x) = \lim_{i\to\infty} d(x_{n_i}, J_\lambda x_{n_i}) = 0.$$

Therefore $x \in \Omega$. As by (13) the limit $\lim_{n\to\infty} d(x_n, x)$ exists and also $\{x_{n_i}\}$ strongly converges to $x \in \Omega$, therefore $\{x_n\}$ strongly converges to $x \in \Omega$. □

Theorem 3.3. *Under the assumptions of Theorem 3.1 if, in addition, there exists a nondecreasing function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$, $g(r) > 0$ *for all* $r > 0$ *such that*

$$g(d(x,\Omega)) \leq d(x, J_\lambda x) + d(x, Tx) \text{ for all } x \in C, \tag{27}$$

then the sequence $\{x_n\}$ *defined by (4) strongly converges to a point* $x \in \Omega$.

Proof. With the help of (24)-(25) and (27), we have $\lim_{n\to\infty} g(d(x_n,\Omega)) = 0$. Since g is nondecreasing with $g(0) = 0$ and $g(r) > 0$ for $r > 0$, we have that

$$\lim_{n\to\infty} d(x_n,\Omega) = 0.$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, it follows from (12) that for any $q \in \Omega$, we have that

$$d(x_{n+1}, q) \leq (1 + \mu_n)\, d(x_n, q) + \nu_n \tag{28}$$

where $\mu_n = \left(b^2 M^* + bM^* + b^2 b_1 M^{*2}\right)\sigma_n$
and $\nu_n = \left(b^2 \xi(M) + b^2 M^* \xi(M)\, b_1 + b^2 b_1 M^* + b^2 M^* + b\right)\sigma_n$.

By taking $\inf_{p\in\Omega}$ on both sides of (28), we obtain that

$$d(x_{n+1},\Omega) \leq (1 + \mu_n)\, d(x_n,\Omega) + \nu_n \tag{29}$$

Assume that $\sum_{n=1}^{\infty} \mu_n = \mu$ and hence $\prod_{n=1}^{\infty}(1 + \mu_n) = \mu$. For $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $d(x_{n_0},\Omega) < \frac{\varepsilon}{4\mu+4}$ and $\sum_{n=n_0}^{\infty} t_n < \frac{\varepsilon}{4s}$.

Let $m > n \geq n_0$ and $p \in F$. With the help of (29), we obtain

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m,\Omega) + d(x_n,\Omega) \\
&\leq \prod_{i=n_0}^{m-1}(1+\mu_i)\, d(x_{n_0}, F) + \prod_{i=n_0}^{m-1}(1+\mu_i)\sum_{n=n_0}^{m-1}\nu_i \\
&\quad + \prod_{i=n_0}^{n-1}(1+\mu_i)\, d(x_{n_0},\Omega) + \prod_{i=n_0}^{n-1}(1+\mu_i)\sum_{n=n_0}^{n-1}\nu_i \\
&\leq \prod_{i=n_0}^{\infty}(1+\mu_i)\, d(x_{n_0},\Omega) + \prod_{i=n_0}^{\infty}(1+\mu_i)\sum_{n=n_0}^{\infty}\nu_i \\
&\quad + \prod_{i=n_0}^{\infty}(1+\mu_i)\, d(x_{n_0},\Omega) + \prod_{i=n_0}^{\infty}(1+\mu_i)\sum_{n=n_0}^{\infty}\nu_i \\
&< 2\left[(1+\mu)\frac{\varepsilon}{4s+4} + \mu\frac{\varepsilon}{4\mu}\right] = \varepsilon.
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = q$. Then $d(q,\Omega) = d(\lim_{n\to\infty} x_n,\Omega) = \lim_{n\to\infty} d(x_n,\Omega) = 0$. As Ω is closed, so we obtain $q \in \Omega$. Hence $\{x_n\}$ strongly converges to a point of Ω. □

References

[1] Ya. I. Alber, C. E. Chidume, H. Zegeye, Approximating fixed points of total asymptotically nonexpansive mappings, Fixed Point Theory and Applications 2006 (2006) 1–20 article ID 10673.

[2] L. Ambrosio, N. Gigli, G. Savare, Gradient flows in metric spaces and in the space of probablity measures, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel (2008).

[3] D. Ariza-Ruiz, L. Leustean, G. Lez, Firmly nonexpansive mappings in classes of geodesic spaces, Transactions of the American Mathematical Society 366 (2014) 4299–4322.

[4] M. Bačák, The proximal point algorithm in metric spaces, Israel Journal of Mathematica 194 (2013) 689–701.
[5] J. B. Baillon, Un théoréme de type ergodique pour les contractions non linéairs dans un espaces de Hilbert, (French), C. R. Acad. Sci. Paris Sér 280 (1975) 1511–1514.
[6] O. A. Boikanyo, G. Morosanu, A proximal point algorithm converging strongly for general errors, Optimization Letters 4 (2010) 635–641.
[7] S. S. Chang, Y. J. Cho, H. Zhou, Demiclosed principal and weak convergence problems for asymptotically nonexpansive mappings. Journal of the Korean Mathematical Society 38 (2001) 1245–1260.
[8] S.S. Chang, C.F. Wen and J. C. Yao, Proximal point algorithms involving Ces aro type mean of asymptotically nonexpansive mappings in CAT(0) spaces, Journal of Nonlinear Sciences and Applications 9 (2016) 4317–4328.
[9] S. DhompongsB. Panyanak, On Δ-convergence theorems in CAT(0) spaces, Computers and Mathematics with Applications 56 (2008) 2572–2579.
[10] B. Halpern, Fixed points of nonexpanding maps, Bulletin of the American Mathematical Society 73 (1967) 957–961.
[11] J. Jost, Convex functionals and generalized harmonic maps into spaces of nonpositive curvature, Commentarii Mathematici Helvetici 70 (1995) 659–673.
[12] S. Kamimura, W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, Journal of Approximation Theory 106 (2000) 226–240.
[13] E. Karapinar, H. Salahifard, S. M. Vaezpour, Demiclosedness principle for total asymptotically nonexpansive mappings in CAT(0) spaces, Journal of Applied Mathematics 2014 (2014) Article ID 738150.
[14] M. A. Khamsi and A. R. Khan, Inequalities in metric spaces with applications, Nonlinear Analysis 74 (2011) 4036–4045.
[15] M. A. A.Khan, H. Fukhar-ud-din, A. Kalsoom, Existence and higher arity iteration for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, Fixed Point Theory Applications (2016), 2016:3.
[16] G. E. Kim, Strong convergence to a fixed point of a total asymptotically nonexpansive mapping, Fixed Point Theory Applications (2013) 2013:302.
[17] G. E. Kim, T. H. Kim, Mann and Ishikawa Iterations with Errors for Non-Lipschitzian Mappings in Banach Spaces, Computers and Mathematics with Applications 42 (2001) 1565–1570.
[18] W. A. Kirk, Geodesic geometry and fixed point theory, Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta vol 64 (Univ. Sevilla Secr. Publ., Seville, 2003), 195–225.
[19] W. A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Analysis 68 (2008) 3689–3696.
[20] G. Marino and H. K. Xu, Convergence of generalized proximal point algorithm, Communications on Pure and Applied Analysis 3 (2004) 791–808.
[21] B. Martinet, Réularisation d'inéquations variationnelles par approximations successives, (French), Rev. Franaise Informat. Recherche Opérationnelle 4 (1970) 154–158.
[22] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM Journal on Control and Optimization 14 (1976) 877–898.
[23] T. Shimizu, W. Takahashi, Strong convergence theorem for asymptotically nonexpansive mappings, Nonlinear Analysis 26 (1996) 265–272.
[24] J. Tang, S. S. Chang and J. Dong, Strong convergence theorems of Cesàro-type means for nonexpansive mapping in CAT(0) space, Fixed Point Theory and Applications 2015 (2015) 13 pages.
[25] I. Uddin, J. J. Nieto and J. Ali, One-step iteration scheme for multivalued nonexpansive mappings in CAT(0) spaces, Mediterranean Journal of Mathematics 13 (2016) 1211–1225.