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# Mathematical Analytic Techniques for Determining the Optimal Ordering Strategy for the Retailer Under the Permitted Trade-Credit Policy of Two Levels in a Supply Chain System

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**Abstract.** In this article, we explore a certain kind of two-level trade credit in order to reflect the real-life situations. With this objective in view, we consider the case when the supplier offers two-level trade credit for the retailer for settling the account. If the retailer pays off all accounts at the end of the first credit period, then he/she can utilize the sales revenue to earn interest until the inventory cycle time. On the other hand, if the retailer cannot pay off the unpaid balance at the end of the first credit period, then he/she can decide to pay off the unpaid balance either after the end of the first credit period or after the second credit period. Here, in this situation, the retailer reduces the financed loan from constant sales and revenue received gradually and he/she still can utilize the sales revenue to earn interest when he/she pays off all accounts. Maximizing the profit is used as the objective to develop the inventory model. Based upon the obtained properties of the optimal solution, two theorems are developed to determine the optimal replenishment policy. Finally, computational developments are presented in order to illustrate numerically the main theoretical results which are proven in this article by using some mathematical solution procedures.

## 1. Introduction and Motivation

The policy of trade credit is widespread and represents an important component of a company's finance. Especially small businesses, with limited financing opportunities, may be financed by their suppliers rather

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than by financial institutions (see, for details, [30]). In addition, the supplier is usually willing to offer the retailer a certain credit period without interest to promote the market competition. Furthermore, the delay in the payment by the supplier is a kind of price discount, since paying later indirectly reduces the purchase cost and it encourages the retailer to increase their quantity. Goyal [12] first developed an economic order quantity (EOQ) model when a supplier offers its retailer a permissible delay in payments. Chung and Liao [3] and Chang *et al.* [1] extended the EOQ models for deteriorating items when the supplier offers a permissible delay in payments to the purchaser if the order quantity is greater than a predetermined quantity. Chung and Liao [4] considered an inventory system for deteriorating items under the permissible delay in payment and the discounted cash-flows (DCF) approach. More recently, Musa and Sani [29] explored some inventory ordering policies of delayed deteriorating items under permissible delay in payments. Liao *et al.* [24] extended Goyal's model [12] in order to allow for deteriorating items with two warehouses under an order-size-dependent trade credit. Several interesting and relevant trade-credit results can be found in the investigations by Liao (see [18], [19] and [20]), Huang and Liao [16], Teng and Lou [39], Liao and Huang [23], Liao *et al.* (see [22] and [25]), and Yang *et al.* [42], and in the references cited in each of these earlier works (see also several other related recent works such as [2], [5] to [11], [26], [27], [33], [34] and [40]).

Huang [14], Liao [21] and Teng [36] explored some inventory models under two levels of the trade-credit policy in which the supplier offers the retailer a permissible delay period $M$, and the retailer (in turn) provides his/her customers a permissible delay period $N$ in order to stimulate the demand. Huang and Hsu [15] developed an inventory model under this two-level trade-credit policy by incorporating partial trade-credit option for the customers of the retailer. Kreng and Tan [17] and Ouyang *et al.* [32] developed an EOQ model for deteriorating items under supplier's up-stream trade credit linked to the ordering quantity. Liao *et al.* [28] explored an EOQ model for deteriorating items with capacity constraints under two levels of the trade-credit policy. Many other related developments can be found in the works by Teng and Chang [37], Ouyang and Chang [31], and Yen *et al.* [41], and indeed also in the references cited therein.

The inventory models, which were investigated in the above-mentioned works, did not consider the progressive payment scheme. Goyal *et al.* [13] explored the notion of progressive credit period as follows:

(i) If the retailer settles the outstanding amount by the first period $M_1$, then the supplier does not charge any interest.

(ii) If the retailer pays after $M_1$, but before the second period $M_2$ offered by the supplier, then the supplier charges the retailer an interest on the unpaid balance at the rate $I_{c_1}$.

(iii) If the retailer settles the account after $M_2$, then he/she will have to pay an interest at the rate $I_{c_2}$ on the unpaid balance ($I_{c_2} > I_{c_1}$).

Soni and Shah [35] explored the optimal ordering policy for the retailer under a stock-dependent demand and two progressive credits periods. Teng *et al.* [38] extended the work of Soni and Shah [35] in order to obtain the retailer's optimal ordering policy for stock-dependent demand when the supplier offers a progressive permissible delay in payments.

In our present study, we attempt to extend the existing literature related to inventory models with delay in payments to a more general case. We shall establish an inventory system in which the supplier offers a kind of two levels of trade credit to settle the retailer's accounts. Herein, if the retailer pays off all accounts at the end of the first credit period, then he/she can utilize the sales revenue to earn interest throughout the inventory cycle. On the other hand, if the retailer cannot pay off the unpaid balance at the end of the first credit period, then he/she can decide to pay off the unpaid balance either after the end of the first credit period or after the second credit period. With this arrangement in mind, the retailer reduces the financed loan from constant sales and revenue received gradually and he/she still can utilize the sales revenue to earn interest when he/she pays off all accounts. The maximization of the average profit per unit of time is taken as the objective function to study the retailer's optimal ordering policy in this article. Based on the obtained properties of the optimal solution, two theorems are developed to determine the optimal replenishment policy. Finally, we shall present the computational developments corresponding to each of the main theoretical results which are proven in this article by using some mathematical solution procedures.

## 2. Notations, Conventions and Assumptions

The following notations, conventions and assumptions will be used in developing the mathematical model in this article.

**Assumptions Made:**

(1) The demand for the item is constant with time.

(2) Shortages are not allowed.

(3) Replenishment rate is infinite.

(4) When $M > T$, the supplier does not charge the retailer any interest and the retailer earns interest on the sales revenue up to the permissible period $M$.

(5) When $M \leqq T$, there are three cases to occur as follows. Firstly, if the retailer pays by $M$, the supplier does not charge any interest to the retailer and the retailer can continuously utilize the sales revenue to earn interest throughout the inventory cycle time $T$. Secondly, if the retailer cannot pay off the balance up to the permissible period $M$, but before the period $N$, then the supplier charges the retailer an interest at the rate of $I_1$ on the unpaid balance. At the same time, the retailer must utilize the sales revenue to pay off the remaining amount owed to the supplier. Once the retailer pays off all accounts, he/she still can utilize the sales revenue to earn interest throughout the inventory cycle time $T$. Thirdly, if the retailer pays after the period $N$, then the supplier charges the retailer an interest at the rate of $I_2$ ($I_2 > I_1$) on the unpaid balance.

(6) Time horizon is infinite.

**Notations Used:**

| | |
|---|---|
| $D$ | the demand rate per year |
| $h$ | the inventory holding cost rate excluding the interest charges rate. |
| $p$ | the selling price per unit |
| $C$ | the purchasing cost per unit $C < p$ |
| $M$ | the first offered credit period in settling the account without extra charges |
| $N$ | the second permissible credit period in settling the account with interest charge of $I_2$ on the unpaid balance and $N > M$ |
| $I_1$ | the interest charged per \$ in stocks per year by the supplier when the retailer pays during $[M, N]$ |
| $I_2$ | the interest charged per \$ in stocks per year by the supplier when the retailer pays during $[N, T]$ ($I_2 > I_1$) |
| $I_e$ | the interest earned per \$ per year ($I_1 > I_e$) |
| $S$ | the ordering cost per order |
| $T$ | the inventory cycle time (the decision variable) |
| $\pi(T)$ | the total relevant profit per year |
| $W^*$ | $\dfrac{p}{C} M + \dfrac{pI_e}{2C} N^2$ |
| $\overline{W}$ | $\dfrac{p}{C} N + \dfrac{pI_e}{2C} \left[M^2 + (N - M)^2\right]$ |

## 3. Formulation of the Mathematical Model

According to the above assumptions, conventions and notations, the elements comprising of the annual total relevant profit of the retailer are listed below:

(1) Annual ordering cost = $\dfrac{S}{T}$

(2) Annual stock-holding cost (excluding interest charged) = $\dfrac{hD}{2}T$

(3) According to the assumptions (4) and (5), there are four cases to occur in the interest payable and/or the interest earned per year: (i) $T \leqq M$; (ii) $M < T \leqq W^*$; (iii) $W^* < T \leqq \overline{W}$; and (iv) $\overline{W} < T$.

**Case 1.** $T \leqq M$

In this case, since the credit period $M$ is not less than $T$, the retailer has no interest paid, so that the interest payable in each order cycle is zero. In addition, the retailer uses the sales revenue to earn interest during the interval $[0, M]$. For this situation, there are two elements to discuss in the interest earned as follows: Firstly, the retailer's interest earned is $\dfrac{pI_eDT^2}{2}$ in the interval $[0, T]$; Secondly, the retailer earns interest given by

$$\left(pI_eDT + \frac{pI_e^2DT^2}{2}\right)(M - T)$$

during the period from $T$ to $M$. Consequently, the total interest earned per year is given by

$$\frac{pI_eDT}{2} + \left[\left(pI_eD + \frac{pI_e^2DT}{2}\right)(M - T)\right] = pI_eD\left[\frac{T}{2} + \left(1 + \frac{I_eT}{2}\right)(M - T)\right].$$

**Case 2.** $M < T \leqq W^*$

In this case, since $T \leqq W^*$, that is, the retailer has enough money in his/her accounts to pay off the total purchase cost at time $M$ and he/she decides to pay off the balance, so no interest payable is paid for the items as well. In addition, there are three elements to discuss in the interest earned as follows: Firstly, during the period from 0 to $M$, the retailer earns interest on the average sales revenue; Secondly, after time $M$, the retailer still earns interest on the average sales revenue by selling the items until the end of the replenishment cycle time; Thirdly, the retailer utilizes the sales revenue to earn interest during the period from $M$ to $T$. Hence, in this case, the total interest earned is given by

$$\frac{pI_eDM^2}{2T} + \frac{pI_eD(T - M)^2}{2T} + \frac{I_e}{T}\left(pDM + \frac{pI_eD}{2}M^2 - CDT\right)(T - M).$$

**Case 3.** $W^* < T \leqq \overline{W}$

In this case, since $W^* < T$, that is, the retailer's accumulated sales revenue is less than the total purchase cost at time $M$. On the other hand, we have $T \leqq \overline{W}$, that is, the retailer decides to gradually reduce the amount of the loan due to constant sales and the revenue received and he/she pays off the total purchase cost before or on $N$. Thereafter, the supplier starts to charge the retailer the unpaid balance given by

$$L_1 = CDT - pDM - \frac{pI_eDM^2}{2}$$

with the interest rate $I_1$ at time $M$, so the interest payable per year in this case is given by

$$\frac{I_1}{2pDT}\left[CDT - pDM\left(1 + \frac{I_eM}{2}\right)\right]^2.$$

Additionally, there are two elements to discuss in the interest earned as follows: Firstly, the retailer's earned interest on the average sales revenue is given by $\frac{pI_eDM^2}{2}$ during the interval $[0, M]$; Secondly, the retailer continuously sells products and uses the revenue to earn interest during the period from

$$M + \left( \frac{CDT - pDM - \frac{pI_eDM^2}{2}}{pD} \right)$$

to $T$. Furthermore, the total interest earned is given by

$$\frac{pI_eDM^2}{2T} + \frac{pI_eD}{2T} \left( T - M - \frac{CDT - pDM - \frac{pI_eDM^2}{2}}{pD} \right)^2 .$$

**Case 4.** $\overline{W} < T$

In this case, we have $\overline{W} < T$, that is, the retailer has not enough money in his/her account to pay off the total purchase cost at time $M$ and he/she decides to pay off the balance after time $N$. Furthermore, there are two elements to discuss in the interest payable as follows: Firstly, the supplier starts to charge the retailer the unpaid balance $L_1$ with interest at the rate $I_1$ during the interval $[M, N]$; Secondly, the retailer pays the amount given by

$$\left[ pD(N - M) + \frac{pI_eD}{2}(N - M)^2 \right]$$

at $N$, so the supplier starts to charge the retailer the unpaid balance given by

$$L_2 = CDT - \left[ pDM + \frac{pI_eD}{2} M^2 \right] - \left( pD(N - M) + \frac{pI_eD}{2} \left[ M^2 + (N - M)^2 \right] \right)^2$$

with interest at the rate $I_2$ at time $N$. Furthermore, the interest charged per year in this case is given by

$$\frac{I_1(N - M)D}{T} \left[ CT - pM \left( 1 + \frac{I_eM}{2} \right) \right] + \frac{I_2D}{2pT} \left( CT - pN - \frac{pI_e}{2} \left[ M^2 + (N - M)^2 \right] \right)^2 .$$

Similarly, the retailer earns interest on the average sales revenue during the interval $[0, M]$, so that the total interest earned is given by $\frac{pI_eD}{2T} M^2$.

As mentioned earlier, the total annual profit for the retailer can be expressed as follows:

$$\pi(T) = \begin{cases} \pi_1(T) & \text{if} \quad 0 < T \leqq M & \text{(1a)} \\ \pi_2(T) & \text{if} \quad M < T \leqq W^* & \text{(1b)} \\ \pi_3(T) & \text{if} \quad W^* < T \leqq \overline{W} & \text{(1c)} \\ \pi_4(T) & \text{if} \quad \overline{W} < T, & \text{(1d)} \end{cases}$$

where

$$\pi_1(T) = (p - C)D - \frac{S}{T} - \frac{hDT}{2} + pI_eD \left[ \frac{T}{2} + \left( 1 + \frac{I_eT}{2} \right)(M - T) \right], \tag{2}$$

$$\begin{aligned}\pi_2(T) &= (p-C)D - \frac{S}{T} - \frac{hDT}{2} + \frac{pI_eDM^2}{2T} + \frac{pI_eD}{2T}(T-M)^2 \\ &\quad + \frac{I_e}{T}\left(pDM + \frac{pI_eDM^2}{2} - CDT\right)(T-M) \\ &= -\frac{S}{T} - \frac{hDT}{2} + \frac{1}{T}\left[\left(pDM + \frac{pI_eDM^2}{2} - CDT\right)[1+I_e(T-M)]\right] \\ &\quad + \frac{pD}{T}(T-M) + \frac{pI_eD}{2T}(T-M)^2, \end{aligned} \tag{3}$$

$$\begin{aligned}\pi_3(T) &= (p-C)D - \frac{S}{T} - \frac{hDT}{2} + \frac{pI_eDM^2}{2T} - \frac{I_1}{2pDT}\left[CDT - pDM\left(1+\frac{I_eM}{2}\right)\right]^2 \\ &\quad + \frac{pI_eD}{2T}\left(T - M - \frac{CDT - pDM - \frac{pI_eDM^2}{2}}{pD}\right)^2 \end{aligned} \tag{4}$$

and

$$\begin{aligned}\pi_4(T) &= (p-C)D - \frac{S}{T} - \frac{hDT}{2} + \frac{pI_eDM^2}{2T} - \frac{I_1(N-M)}{T}\left[CDT - pDM\left(1+\frac{I_eM}{2}\right)\right] \\ &\quad - \frac{I_2}{2pDT}\left(CDT - pDN - \frac{pI_eD}{2}\left[M^2 + (N-M)^2\right]\right)^2. \end{aligned} \tag{5}$$

## 4. Determination of the Optimal Inventory Cycle Time $T$

First of all, it follows from Eqs. (2) to (5) that

$$\pi_1(M) = \pi_2(M), \quad \pi_2(W^*) = \pi_3(W^*) \qquad \text{and} \qquad \pi_3(\overline{W}) > \pi_4(\overline{W}).$$

Hence, clearly, $\pi(T)$ is continuous on $(0,\infty)$ except at $T = \overline{W}$.

Next, by using Eqs. (2) to (5), we also find that

$$\pi_1'(T) = \frac{1}{2T^2}\left[2S - D(h + pI_e - pI_e^2M)T^2 - 2pI_e^2DT^3\right], \tag{6}$$

$$\pi_2'(T) = \frac{1}{2T^2}\left[2S + pI_e^2DM^3 - D(h - pI_e + 2CI_e)T^2\right], \tag{7}$$

$$\begin{aligned}\pi_3'(T) &= \frac{1}{2T^2}\Bigg[2S + \frac{pI_e^2DM^4}{4}(I_1 - I_e) + pDM^2(I_1 - I_e) + pDI_1I_eM^3 \\ &\quad + D\left(\frac{C^2(I_e - I_1)}{p} - (h - pI_e + 2CI_e)\right)T^2\Bigg] \end{aligned} \tag{8}$$

and

$$\pi_4^{'}(T) = \frac{1}{2T^2}\Bigg\{2S - hDT^2 - pI_eDM^2 - 2I_1(N-M)pDM\left(1 + \frac{I_eM}{2}\right) - \frac{I_2}{pD}\left[C^2D^2T^2 - \left(pDN + \frac{pI_eD}{2}\left[M^2 + (N-M)^2\right]\right)^2\right]\Bigg\}, \tag{9}$$

so that we have

$$\pi_1^{''}(T) = -\frac{2S}{T^3} - pI_e^2D < 0, \tag{10}$$

$$\pi_2^{''}(T) = -\frac{2S}{T^3} - \frac{pI_e^2DM^3}{T^3} < 0, \tag{11}$$

$$\pi_3^{''}(T) = \frac{2S}{T^3} - \frac{pDI_e^2M^4(I_1 - I_e)}{4T^3} - \frac{pDM^2(I_1 - I_e)}{T^3} - \frac{pDI_1I_eM^3}{T^3} < 0 \tag{12}$$

and

$$\pi_4^{''}(T) = -\frac{2S}{T^3} + \frac{pI_eDM^2}{T^3} + \frac{2I_1(N-M)}{T^3}\,pDM\left(1 + \frac{I_eM}{2}\right) - \frac{I_2pD\left(N + \frac{I_e}{2}\left[M^2 + (N-M)^2\right]\right)^2}{T^3}. \tag{13}$$

If we now define $G$ by

$$G = -2S + pI_eDM^2 + 2I_1(N-M)pDM\left(1 + \frac{I_eM}{2}\right) - I_2pD\left(N + \frac{I_e}{2}\left[M^2 + (N-M)^2\right]\right)^2, \tag{14}$$

then Eqs. (6) to (14) imply the following results.

**Lemma 1.** *Each of the following convexity and monotonicity properties holds true.*

(1) *The functions $\pi_1(T)$, $\pi_2(T)$ and $\pi_3(T)$ are concave on $T > 0$.*

(2) *The function $\pi_4(T)$ is concave on $T > 0$ if $G \leqq 0$. Furthermore, the function $\pi_4(T)$ is decreasing on $T > 0$ if $G > 0$.*

*Proof.* We prove the assertions of Lemma 1 as follows.

(1) Eqs. (10), (11) and (12) imply that the functions $\pi_1(T)$, $\pi_2(T)$ and $\pi_3(T)$ are concave on $T > 0$.

(2) If $G \leqq 0$, Eq. (13) implies that the function $\pi_4(T)$ is concave on $T > 0$.

(3) If $G > 0$, Eq. (9) reveals that $\pi_4^{'}(T) < 0$ when $T > 0$, and hence that the function $\pi_4(T)$ is decreasing on $T > 0$.

Incorporating the above arguments, we have completed the proof of Lemma 1. □

In addition to the results asserted by Lemma 1, we have the following results:

**Lemma 2.** *Each of the following assertions holds true.*

(1) *If* $h - pI_e + 2CI_e > 0$, *then* $T_2^*$ *exists.*

(2) *If* $h - pI_e + 2CI_e \leqq 0$, *then*

(a) $\pi_2^{'}(T) > 0$ *and* $\pi_2(T)$ *is increasing on* $T > 0$.

(b) $T_2^*$ *does not exist.*

*Proof.* The proof of Lemma 2 follows readily from Eqs. (7) and (11). □

Next, by applying Eqs. (6) to (9), we get

$$\pi_1^{'}(M) = \frac{\delta_1}{2M^2} < \pi_2^{'}(M) = \frac{\delta_2}{2M^2}, \tag{15}$$

$$\pi_2^{'}(W^*) = \pi_3^{'}(W^*) = \frac{\delta_3}{2W^{*2}}, \tag{16}$$

$$\pi_3^{'}(\overline{W}) = \frac{\delta_4}{2\overline{W}^2} \tag{17}$$

and

$$\pi_4^{'}(\overline{W}) = \frac{\delta_5}{2\overline{W}^2}, \tag{18}$$

where

$$\delta_1 = 2S - D(h + pI_e + pI_e^2 M)M^2, \tag{19}$$

$$\delta_2 = 2S + pI_e^2 DM^3 - D(h - pI_e + 2CI_e)M^2, \tag{20}$$

$$\delta_3 = 2S + pI_e^2 DM^3 - D(h - pI_e + 2CI_e)W^{*2}, \tag{21}$$

$$\begin{aligned}\delta_4 = 2S &+ \frac{pI_e^2 DM^4}{4}(I_1 - I_e) + pDM^2(I_1 - I_e) + pDI_1 I_e M^3 \\ &+ D\left(\frac{C^2(I_e - I_1)}{p} - (h + 2CI_e - pI_e)\right)\overline{W}^2\end{aligned} \tag{22}$$

and

$$\delta_5 = 2S - pI_e DM^2 - 2I_1(N - M)pDM\left(1 + \frac{I_e M}{2}\right) - Dh\overline{W}^2. \tag{23}$$

Furthermore, it is easily observed that

$$\delta_1 < \delta_2, \quad \delta_2 > \delta_3 > \delta_4 \qquad \text{and} \qquad \delta_2 > \delta_5.$$

In addition, we let $T_i^*$ denote the optimal solution of $\pi_i(T)$ on $T > 0$ if $T_i^*$ exists for $i = 1, 2, 3, 4$.

By means of the above arguments, we are led to the following results.

**Lemma 3.** *If* $\delta_5 \geqq 0$, *then each of the following assertions holds true*:

(1) $G < 0$;

(2) *$T_4^*$ exists;*

(3) *$T_4^* \geqq \overline{W}$;*

(4) *$\pi_4(T)$ is concave on $T > 0$.*

*Proof.* Our demonstration of Lemma 3 runs as follows.
First of all, if $\delta_5 \geqq 0$, then

$$-pI_eDM^2 - 2I_1(N-M)pDM\left(1+\frac{I_eM}{2}\right) - Dh\overline{W}^2 \geqq -2S \tag{24}$$

and

$$\begin{aligned} G &= -2S + pI_eDM^2 + 2I_1(N-M)pDM\left(1+\frac{I_eM}{2}\right) \\ &\quad - I_2pD\left(N^2+\frac{I_e}{2}\left[M^2+(N-M)^2\right]\right)^2 \\ &\leqq D\left(h+\frac{C^2}{p}I_2\right)\overline{W} < 0. \end{aligned} \tag{25}$$

Eqs. (24) and (25) demonstrate that Lemma 3 holds true. □

Obviously, if $T_i^*$ exists for $i = 1, 2, 3, 4$, we find from Eqs. (10) to (13) that

$$\pi_i'(T)\begin{cases} > 0 & \text{if} \quad 0 < T \leqq T_i^* & \text{(26a)} \\ = 0 & \text{if} \quad T = T_i^* & \text{(26b)} \\ < 0 & \text{if} \quad T > T_i^*, & \text{(26c)} \end{cases}$$

which means that the function $\pi_i(T)$ is increasing on $(0, T_i^*]$ and decreasing on $[T_i^*, \infty)$ (for $i = 1, 2, 3, 4$). The objective is to find the optimal inventory cycle time $T^*$ in order to maximize the average profit per unit time $\pi(T)$ of the retailer. In fact, Eqs. 1(a) to 1(d) imply that

$$\pi(T^*) = \max\left\{\pi_1(T_1^*), \pi_2(T_2^*), \pi_3(T_3^*), \pi_4(T_4^*)\right\}, \tag{27}$$

where

$$\pi_1(T_1^*) = \max\{\pi_1(T) : 0 < T \leqq M\}, \tag{28}$$

$$\pi_2(T_2^*) = \max\{\pi_2(T) : M \leqq T \leqq W^*\}, \tag{29}$$

$$\pi_3(T_3^*) = \max\left\{\pi_3(T) : W^* \leqq T \leqq \overline{W}\right\} \tag{30}$$

and

$$\pi_4(T_4^*) = \max\left\{\pi_4(T) : \overline{W} \leqq T\right\}. \tag{31}$$

From the mathematical analytic arguments which we have detailed above, we can easily deduce the following results.

**Theorem 1.** *Suppose that $\delta_4 \leqq \delta_5$. Then the optimal inventory cycle time $T^*$ that maximizes the annual total profit $\pi(T)$ is given as follows*:

(1) *If* $\delta_5 \geqq 0,\ \delta_4 \geqq 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $\overline{W}$ *or* $T_4^*$ *associated with the most profit.*

(2) *If* $\delta_5 \geqq 0,\ \delta_4 \geqq 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$, $\overline{W}$ *or* $T_4^*$ *associated with the most profit.*

(3) *If* $\delta_5 \geqq 0,\ \delta_4 < 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_3^*$ *or* $T_4^*$ *associated with the most profit.*

(4) *If* $\delta_5 \geqq 0,\ \delta_4 < 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$, $T_3^*$ *or* $T_4^*$ *associated with the most profit.*

(5) *If* $\delta_5 \geqq 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_2^*$ *or* $T_4^*$ *associated with the most profit.*

(6) *If* $\delta_5 \geqq 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$, $T_2^*$ *or* $T_4^*$ *associated with the most profit.*

(7) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_3^*$.

(8) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$ *or* $T_3^*$ *associated with the most profit.*

(9) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_2^*$.

(10) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$ *or* $T_2^*$ *associated with the most profit.*

(11) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 < 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$.

*Proof.* The results asserted by Theorem 1 follow readily from Eqs. (19) to (23), (26a), (26b) and (26c). □

**Theorem 2.** *Suppose that $\delta_4 > \delta_5$. Then the optimal inventory cycle time $T^*$ that maximizes the annual total profit $\pi(T)$ is given as follows*:

(1) *If* $\delta_5 \geqq 0,\ \delta_4 \geqq 0,\ \delta_3 \geqq 0, \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $\overline{W}$ *or* $T_4^*$ *associated with the most profit.*

(2) *If* $\delta_5 \geqq 0,\ \delta_4 \geqq 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$, $\overline{W}$ *or* $T_4^*$ *associated with the most profit.*

(3) *If* $\delta_5 < 0,\ \delta_4 \geqq 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $\overline{W}$.

(4) *If* $\delta_5 < 0,\ \delta_4 \geqq 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$ *or* $\overline{W}$ *associated with the most profit.*

(5) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_3^*$.

(6) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 \geqq 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$ *or* $T_3^*$ *associated with the most profit.*

(7) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 \geqq 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_2^*$.

(8) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 \geqq 0$ *and* $\delta_1 < 0$, *then* $T^*$ *is* $T_1^*$ *or* $T_2^*$ *associated with the most profit.*

(9) *If* $\delta_5 < 0,\ \delta_4 < 0,\ \delta_3 < 0,\ \delta_2 < 0$ *and* $\delta_1 \geqq 0$, *then* $T^*$ *is* $T_1^*$.

*Proof.* The results asserted by Theorem 2 follow immediately from Eqs. (19) to (23), (26a), (26b) and (26c). □

## 5. Computational Developments

In this section, we present some potentially useful numerical results which are based upon the theoretical results derived in Section 4 (see Table 1 *and* Table 2 below).
Table 1 below would result upon setting $I_1 = 0.111$, $I_2 = 0.12$, $h = 2$ and $M = 0.1$ in Theorem 1.

Table 1. The optimal replenishment policy used Theorem 1.

| | $D$ | $S$ | $C$ | $p$ | $I_e$ | $N$ | $W^*$ | $\overline{W}$ | $\delta_5$ | $\delta_4$ | $\delta_3$ | $\delta_2$ | $\delta_1$ | $T^*$ | $\pi(T^*)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | 30 | 1.100000 | 36 | 36.000135 | 0.11 | 0.1000100 | 0.1005502 | 0.1005602 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | $T_4^*$=0.1106600 | 92.0508 |
| (2) | 30 | 0.900000 | 36 | 36.0001 | 0.11 | 0.1000100 | 0.1005502 | 0.1005602 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | <0 | $\overline{W}$=0.1005602 | 93.9434 |
| (3) | 30 | 1.9000000 | 36 | 36.0001 | 0.11 | 0.1500000 | 0.1005502 | 0.15069000 | $\geq 0$ | <0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $T_3^*$=0.14555000 | 85.8298 |
| (4) | 30 | 0.8975000 | 36 | 36.001 | 0.11 | 0.1000100 | 0.1005502 | 0.1005602 | $\geq 0$ | <0 | $\geq 0$ | $\geq 0$ | <0 | $T_3^*$=0.10055930 | 93.9683 |
| (5) | 30 | 0.3693278 | 4 | 4.0001 | 0.11 | 0.1000001 | 0.100552517 | 0.10056260 | $\geq 0$ | <0 | <0 | $\geq 0$ | $\geq 0$ | $T_2^*$=0.10055250 | 93.9699 |
| (6) | 30 | 0.864320 | 34 | 34.0001 | 0.11 | 0.1000001 | 0.10055029 | 0.10055039 | $\geq 0$ | <0 | <0 | $\geq 0$ | <0 | $T_1^*$=0.10054975 | 93.9701 |
| (7) | 30 | 1.000000 | 36 | 36.0001 | 0.11 | 0.1500000 | 0.10055030 | 0.150690000 | <0 | <0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $T_3^*$=0.10655000 | 92.9750 |
| (8) | 30 | 0.897400 | 36 | 36.0001 | 0.11 | 0.1000100 | 0.100550279 | 0.100560279 | <0 | <0 | $\geq 0$ | $\geq 0$ | <0 | $T_3^*$=0.10055430 | 93.9693 |
| (9) | 30 | 0.571500 | 36 | 36.0001 | 0.05 | 0.1000100 | 0.100250278 | 0.100260280 | <0 | <0 | <0 | $\geq 0$ | $\geq 0$ | $T_2^*$=0.10025 | 93.9880 |
| (10) | 30 | 0.890000 | 36 | 36.0001 | 0.11 | 0.1000100 | 0.100550000 | 0.100560000 | <0 | <0 | <0 | $\geq 0$ | <0 | $T_1^*$=0.09900000 | 94.04316 |
| (11) | 100 | 0.001000 | 36 | 36.0001 | 0.11 | 0.1000100 | 0.100550000 | 0.100560000 | <0 | <0 | <0 | <0 | <0 | $T_1^*$=0.00200000 | 38.51837 |

Table 2 below would result upon setting $I_1 = 0.111$, $I_2 = 0.12$, $h = 2$ and $M = 0.1$ in Theorem 2.

Table 2. The optimal replenishment policy used Theorem 2.

| | $D$ | $S$ | $C$ | $p$ | $h$ | $I_1$ | $I_2$ | $I_e$ | $M$ | $N$ | $W^*$ | $\overline{W}$ | $\delta_5$ | $\delta_4$ | $\delta_3$ | $\delta_2$ | $\delta_1$ | $T^*$ | $\pi(T^*)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1) | 50 | 10 | 25 | 35 | 4 | 0.04 | 0.12 | 0.03 | 0.12 | 0.15 | 0.1683 | 0.2103 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | $T_4^*$=0.2765 | 436.1152 |
| (2) | 40 | 0.0207 | 11.499 | 11.5 | 0.01 | 0.0991 | 0.0992 | 0.099 | 0.03 | 0.030001 | 0.030047163 | 0.030048163 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | <0 | $\overline{W}$=0.030048163 | 0.027103 |
| (3) | 100 | 10 | 25 | 35 | 4 | 0.04 | 0.12 | 0.03 | 0.12 | 0.15 | 0.1683 | 0.2103 | <0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\overline{W}$=0.2103 | 914.5920 |
| (4) | 60000 | 0.01 | 0.5 | 2 | 0.12 | 0.13 | 0.14 | 0.11 | 0.001 | 0.00105 | 0.00400022 | 0.00420022 | <0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | <0 | $\overline{W}$=0.00420022 | 89999.65 |
| (5) | 150 | 10 | 25 | 35 | 4 | 0.04 | 0.12 | 0.03 | 0.12 | 0.15 | 0.1683 | 0.2103 | <0 | <0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $T_3^*$=0.1733 | 1398 |
| (6) | 30 | 0.00075 | 2 | 10 | 0.5 | 0.09 | 0.10 | 0.08 | 0.01 | 0.01003 | 0.05002 | 0.05017 | <0 | <0 | $\geq 0$ | >0 | <0 | $T_3^*$=0.05003 | 240.0181 |
| (7) | 200 | 10 | 25 | 35 | 4 | 0.04 | 0.12 | 0.03 | 0.12 | 0.15 | 0.1683 | 0.2103 | <0 | <0 | <0 | $\geq 0$ | $\geq 0$ | $T_2^*$=0.15 | 1884.6 |
| (8) | 300 | 10 | 25 | 35 | 4 | 0.04 | 0.12 | 0.03 | 0.12 | 0.15 | 0.1683 | 0.2103 | <0 | <0 | <0 | $\geq 0$ | <0 | $T_1^*$=0.00150 | 2863.7 |
| (9) | 1000 | 10 | 25 | 35 | 4 | 0.04 | 0.12 | 0.03 | 0.12 | 0.15 | 0.1683 | 0.2103 | <0 | <0 | <0 | <0 | <0 | $T_1^*$=0.063 | 9808.3 |

## 6. Concluding Remarks and Observations

In this paper, we have considered a potentially useful kind of two-level trade credit to reflect the real-life situations in which the supplier offers two levels of trade credit and the retailer can decide to pay off the unpaid balance either after the period $M$, but before the period $N$, or after the period $N$ when the retailer cannot pay off the unpaid balance at the end of the period $M$. Our main results (Theorem 1 and Theorem 2 in Section 4) characterize the optimal solution under different sets of conditions. Finally, we have presented the computational developments corresponding to each of the main theoretical results which are proven in this article by using some mathematical solution procedures.

The mathematical model, which we have considered in this article, can indeed be further extended in several ways. For example, we can extend the constant demand rate to be a function of the time or the price. Additionally, we can extend the model to include trade credits linked to the order quantity.

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