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Some Structure Spaces of Generalized Semirings

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Abstract. In this paper, we study some structure spaces of (m,n)-semiring (S, f, g), introducing the classes of n-ary prime k-ideals, n-ary prime full k-ideals, n-ary prime ideals, maximal ideals and strongly irreducible ideals. Considering their collections, respectively, of an (m,n)-semiring (S, f, g), we construct the respective topologies on them by means of closure operator defined in terms of intersection and inclusion relation among these ideals of the (m,n)-semiring (S, f, g). The obtained topological spaces are called the structure spaces of (m,n)-semiring (S, f, g). We mainly study several principal topological axioms and properties of those structure spaces of (m,n)-semiring such as separation axioms, compactness and connectedness etc.

1. Introduction and Preliminaries

Algebraic structures play an important role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc.

The theory of semiring was first developed by H. S. Vandiver [47] and he has obtained important results of the objects. Semiring constitute a fairly natural generalization of rings, with broad applications in the mathematical foundation of computer science. Also, semiring theory has many applications to other branches. For example, automata theory, optimization theory, algebra of formal process, combinatorial optimization, Baysian networks and belief propagation.

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [4] who introduced the notion of "cubic matrix" which in turn was generalized by Kapranov, et al. in 1990 [27]. Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics and other branches of sciences.

The notion of an n-ary group was introduced in 1928 by W. Dörnte [10] (under inspiration of Emmy Noether). The idea of investigations of n-ary algebras, i.e., sets with one n-ary operation, seems to be going back to Kasner's lecture [28] at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. Such systems have many applications in different branches.

The theory of ternary algebraic system was introduced by D.H. Lehmer [32] in 1932. We note that ternary semirings are special case of (m,n)-ary rings studied by many authors such as Crombez [7], Crombez and

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Timm [8], Dudek [11], Iancu [20, 21], Pop [38, 39], Iancu, Pop [22] etc. Dudek [12] studied n-ary groups which are (n, n)-rings. Dutta and Kar [15] introduced the notion of ternary semiring which is a generalization of the notion of ternary ring introduced by Lister [34].

The concept of (m, n)-semiring as a generalization of the ordinary semiring was introduced and accordingly some properties were discussed in [2, 37–43, 48]. Some studies on topological n-ary groups and topological (m, n)-rings were studied by many author such as [6, 9, 14, 16, 35, 36, 44, 46].

In this paper, we introduce and study the structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ of all n-ary prime k-ideals of an (m, n)-semiring (S, f, g) establishing the separation axioms and compactness property of this structure space. We also study the classes of n-ary prime full k-ideals, n-ary prime ideals, maximal ideals and strongly irreducible ideals of (m, n)-semirings. By considering and investigating properties of the collections $\mathcal{T}$, $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ of all n-ary prime full k-ideals, maximal ideals, n-ary prime ideals and strongly irreducible ideals, respectively, of an (m, n)-semiring S, we construct the respective topologies on them by means of closure operator defined in terms of intersection and inclusion relation among these class of ideals of (m, n)-semiring S. The obtained topological spaces are called the structure spaces of the (m, n)-semiring S. This topological space has been studied in different algebraic structures [1, 3, 5, 17–19, 23, 24, 30]. We study a several principal topological axioms and properties of these structure spaces of (m, n)-semiring such as separation axioms, compactness etc.

First we recall some basic terms and definitions from the (m, n)-semiring theory.

The set of integers is denoted by $\mathbb{Z}$, with $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denoting the sets of positive integers and negative integers respectively and m and n used are positive integers. Let S be a nonempty set and f be a mapping $f : S^m \to S$, i.e. f is an m-ary operation. Elements of the set S are denoted by x_i, y_i where $i \in \mathbb{Z}_+$. Then (S, f) is called an m-groupoid [13].

Let f be an m-ary operation on S and $A_1, A_2, \ldots, A_m$ be nonempty subsets of S. We define $f(A_1, A_2, \ldots, A_m) = \{f(x_1, x_2, \ldots, x_m) | x_i \in A_i, i = 1, 2, \ldots, m\}$.

We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, ..., x_m$ is denoted by x_i^m where $1 \leq i \leq m$. For all $1 \leq i \leq j \leq m$, the following term: $f(x_1, ..., x_i, y_{i+1}, ..., y_j, z_{j+1}, ..., z_m)$ is represented as $f(x_1^i, y_{i+1}^j, z_{j+1}^m)$. In the case when $y_{i+1} = ... = y_j = y$ [33] is expressed as: $f(x_1^i, \overset{(j-1)}{y}, z_{j+1}^m)$.

Similarly, for subsets $A_1, A_2, \ldots, A_m$ of S we define

$$f(A_1^m) = f(A_1, A_2, \ldots, A_m) = \bigcup\{f(x_1^m) | x_i \in A_i, i = 1, \ldots, m\}.$$

An m-ary operation f is called $(i, j) -$ *associative* if

$$f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) = f(x_1^{j-1}, f(x_j^{m+j-1}), x_{m+j}^{2m-1}),$$

holds for fixed $1 \leq i < j \leq m$ and all $x_1, x_2, \ldots, x_{2m-1} \in H$. Note that (i, k)-associativity follows from (i, j)- and (j, k)-associativity. If the above condition is satisfied for all $i, j \in \{1, 2, \ldots, m\}$, then we say that f is *associative*. The m-ary operation f is called *commutative* iff for all $x_1, \ldots, x_m \in S$ and for all $\sigma \in \mathbb{S}_m, f(x_1, x_2, \ldots, x_m) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)})$. An m-ary groupoid (S, f) is called an m-ary semigroup [13] if f is an associative m-ary operation. An n-ary operation g is distributive with respect to the m-ary operation f, if for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in S, 1 \leq i \leq n$, $g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n))$.

Definition 1.1. An (m, n)-semiring is an algebraic structure (S, f, g) which satisfies the following axioms:

1. (S, f) is an m-ary semigroup,
2. (S, g) is an n-ary semigroup,
3. The n-ary operation g is distributive with respect to the m-ary operation f.

Let (S, f, g) be an (m, n)-semiring. Then m-ary semigroup (S, f) has an *identity element* 0 if $x = f(\underbrace{0, ..., 0}_{i-1}, x, \underbrace{0, ..., 0}_{m-i})$ for all $x \in S$ and $1 \leq i \leq m$. We call 0 as zero (or pseudo zero) element of (m, n)-semiring (S, f, g). Similarly, n-ary semigroup (S, g) has an identity element) 1 if $y = g(\underbrace{1, ..., 1}_{j-1}, y, \underbrace{1, ..., 1}_{n-j})$ for all $y \in S$ and $1 \leq j \leq n$. We call

1 as an identity element of (m,n)-semiring (S,f,g). We therefore call 0 the f-identity, and 1 the g-identity. If (S,f,g) be an (m,n)-semiring with an f-identity element 0 and g-identity element 1, then 0 is said to be *multiplicatively absorbing* if it is absorbing in (S,g), i.e., if $g(0,x_1^{n-1}) = g(x_1^{n-1},0) = 0$, for all $x_1,x_2,...,x_{n-1} \in S$. An element x of an m-ary semigroup (S,f) is called an idempotent if $f(\underbrace{x,....,x}_{m}) = x$. The (m,n)-semiring (S,f,g) is called: (a) *additively idempotent* if (S,f) is an idempotent m-ary semigroup, i.e., if $f(\underbrace{x,....,x}_{m}) = x$ for all $x \in S$; (b) is called *multiplicatively idempotent* if (S,g) is an idempotent n-ary semigroup, i.e., if $g(\underbrace{y,y,...,y}_{n}) = y$ for all $y \in S, y \neq 0$.

Let (S,f,g) be an (m,n)-semiring. An m-ary sub-semigroup R of S is called an *sub-(m,n)-semiring* of S if $g(a_1^n) \in R$ for all $a_1,...,a_n \in R$. A subset I of S is called an i-ideal ($i \in 1,2,...,n$) of (m,n)-semiring (S,f,g) if (I,f) is an sub-(m,n)-semiring of (S,f,g) and $g(\overset{(i-1)}{S},I,\overset{(n-i)}{S}) \subseteq I$. If I is an i-ideal for all $i \in 1,2,...,n$, then I is said to be an ideal of (m,n)-semiring S.

Different interesting examples of (m,n)-semirings can be found in [2, 37–43, 48].

Definition 1.2. Let $\{A_i : i \in I\}$ be a family of ideals of an (m,n)-semiring (S,f,g), where I is finite or infinite. Then $f(A_1,...,A_i), i \in I$ and $i = k(m-1)+1, k = 1,2,...$, is the set $\{a \in S : a \in f(A_1,...,A_{i_0}), i_0 \in I_0, i_0 = k(m-1)+1, k = 1,2,...$ for some finite subset I_0 of $I\}$.

In this case it is noticed that $f(A_1,...,A_i), i \in I$ and $i = k(m-1)+1, k = 1,2,...$ is an ideal of S. If I is empty, then we take $f(A_1,...,A_i) = \{0\}$.

For an (m,n)-semiring S, let $E^+(S) = \{x \in S : x = f(x,...,x)\}$. A k-ideal I of S is said to be full if $E^+(S) \subseteq I$. An ideal I of an (m,n)-semiring S is called *proper* if $I \subset S$ holds, where $\subset$ denotes proper inclusion, and a proper ideal I is called *maximal* if there is no ideal A of S satisfying $I \subset A \subset S$. A proper ideal P of an (m,n)-semiring S is called an *n-ary prime ideal* of S if for ideals $A_1,...,A_n$ of S, $g(A_1,...,A_n) \subseteq P$ implies that $A_1 \subseteq P$ or $A_2 \subseteq P$ or ... or $A_n \subseteq P$. Equivalently, a proper ideal P of an (m,n)-semiring S is called an *n-ary prime ideal* of S if for all $a_1^n \in S$, $g(a_1^n) \in P$ implies that $a_1 \in P$ or $a_2 \in P$ or ... or $a_n \in P$. An ideal I of an (m,n)-semiring (S,f,g) is called a *k-ideal* if $f(x,y_1,....,y_{m-1}) \in I; x \in S, y_i \in I, i \in \{1,2,...,m-1\}$ imply that $x \in I$.

Throughout this paper, S will always denote an (m,n)-semiring with f-identity 0 (or multiplicatively absorbing 0) and unless otherwise stated an (m,n)-semiring means an (m,n)-semiring with f-identity 0.

2. Structure Space of n-ary Prime k-Ideals

In this section, we introduce the notion of the structure space of (m,n)-semirings formed by the class of n-ary prime k-ideals. We first consider the collection $\mathcal{A}$ of all n-ary prime k-ideals of an (m,n)-semiring (S,f,g) and then we give a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ by means of the closure operator defined in terms of the intersection and inclusion relation among these ideals of the (m,n)-semiring (S,f,g). We call the topological space $(\mathcal{A};\tau_{\mathcal{A}})$- the structure space of the (m,n)-semiring (S,f,g). We establish and study the separation axioms and compactness property of this structure space extending the results obtained in [25].

Let we denote by $\mathcal{A}$ the collection of all n-ary prime k-ideals of S. For any subset (that is, sub-collection) $A \in \mathcal{A}$ we define $\overline{A} = \{I \in \mathcal{A} : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}$. It is clear that $\overline{\emptyset} = \emptyset$.

Theorem 2.1. *Let A,B be any two subsets of $\mathcal{A}$. Then*

1. $A \subseteq \overline{A}$;
2. $\overline{\overline{A}} = \overline{A}$;
3. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$;
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1) Since $\bigcap\limits_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ for all α, then we have $A \subseteq \overline{A}$.

(2) We have $\overline{\overline{A}} \subseteq \overline{A}$ by (1). Let we prove the $\overline{A} \subseteq \overline{\overline{A}}$. Let $I_\beta \in \overline{\overline{A}}$. Then we have $\bigcap\limits_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. Since $I_\alpha \in \overline{A}$, then we have $\bigcap\limits_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$ for all $\alpha \in \Lambda$. Thus $\bigcap\limits_{I_\gamma \in A} I_\gamma \subseteq \bigcap\limits_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. Thus $I_\beta \in \overline{A}$ and so $\overline{A} \subseteq \overline{\overline{A}}$. Therefore $\overline{\overline{A}} = \overline{A}$.

(3) Let us suppose that $A \subseteq B$ and let $I_\alpha \in \overline{A}$. We have $\bigcap\limits_{I_\beta \in A} I_\beta \subseteq I_\alpha$. Since $A \subseteq B$, it follows that $\bigcap\limits_{I_\beta \in B} I_\beta \subseteq \bigcap\limits_{I_\beta \in A} I_\beta \subseteq I_\alpha$, that is, $I_\alpha \in \overline{B}$. Therefore $\overline{A} \subseteq \overline{B}$.

(4) Let we prove that $\overline{A \cup B \cup C_1 \cup .... \cup C_{n-2}} = \overline{A} \cup \overline{B} \cup \overline{C_1} \cup .... \cup \overline{C_{n-2}}$ for any subset $C_i \in \mathcal{A}, i = 1, 2, ..., n-2$. It is clear that $\overline{A} \cup \overline{B} \cup \overline{C_1} \cup .... \cup \overline{C_{n-2}} \subseteq \overline{A \cup B \cup C_1 \cup .... \cup C_{n-2}}$.
Let $I_\alpha \in \overline{A \cup B \cup C_1 \cup .... \cup C_{n-2}}$. Then $\bigcap_{I_\beta \in A \cup B \cup C_1 \cup .... \cup C_{n-2}} I_\beta \subseteq I_\alpha$. Clearly,

$$\bigcap_{I_\beta \in A \cup B \cup C_1 \cup .... \cup C_{n-2}} I_\beta = \left(\bigcap_{I_\beta \in A} I_\beta\right) \bigcap \left(\bigcap_{I_\beta \in B} I_\beta\right) \bigcap \left(\bigcap_{I_\beta \in C_1} I_\beta\right) \bigcap .... \bigcap \left(\bigcap_{I_\beta \in C_{n-2}} I_\beta\right).$$

Since $\left(\bigcap\limits_{I_\beta \in A} I_\beta\right), \left(\bigcap\limits_{I_\beta \in B} I_\beta\right), \left(\bigcap\limits_{I_\beta \in C_1} I_\beta\right), ..., \left(\bigcap\limits_{I_\beta \in C_{n-2}} I_\beta\right)$ are ideals of S, then we have

$$\begin{aligned} & g\left(\left(\bigcap_{I_\beta \in A} I_\beta\right), \left(\bigcap_{I_\beta \in B} I_\beta\right), \left(\bigcap_{I_\beta \in C_1} I_\beta\right), ..., \left(\bigcap_{I_\beta \in C_{n-2}} I_\beta\right)\right) \\ \subseteq & \left(\bigcap_{I_\beta \in A} I_\beta\right) \bigcap \left(\bigcap_{I_\beta \in B} I_\beta\right) \bigcap \left(\bigcap_{I_\beta \in C_1} I_\beta\right) \bigcap .... \bigcap \left(\bigcap_{I_\beta \in C_{n-2}} I_\beta\right) \\ = & \bigcap_{I_\beta \in A \cup B \cup C_1 \cup .... \cup C_{n-2}} I_\beta \subseteq I_\alpha. \end{aligned}$$

Since I_α is an n-ary prime ideal of S, it follows that $\bigcap\limits_{I_\beta \in A} I_\beta \subseteq I_\alpha$ or $\bigcap\limits_{I_\beta \in B} I_\beta \subseteq I_\alpha$ or $\bigcap\limits_{I_\beta \in C_1} I_\beta \subseteq I_\alpha$ or....or $\bigcap\limits_{I_\beta \in C_{n-2}} I_\beta \subseteq I_\alpha$, that is, $I_\alpha \in \overline{A}$ or $I_\alpha \in \overline{B}$ or $I_\alpha \in \overline{C_1}$ or .... or $I_\alpha \in \overline{C_{n-2}}$, i.e. $I_\alpha \in \overline{A} \cup \overline{B} \cup \overline{C_1} \cup .... \cup \overline{C_{n-2}}$. Thus $\overline{A \cup B \cup C_1 \cup ... \cup C_{n-2}} \subseteq \overline{A} \cup \overline{B} \cup \overline{C_1} \cup .... \cup \overline{C_{n-2}}$, therefore $\overline{A \cup B \cup C_1 \cup ... \cup C_{n-2}} = \overline{A} \cup \overline{B} \cup \overline{C_1} \cup .... \cup \overline{C_{n-2}}$. Thus since $\overline{\emptyset} = \emptyset$, taking $C_i = \emptyset, i = 1, 2, ..., n-2$, we get $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

Definition 2.2. The closure operator $A \to \overline{A}$ gives a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$. This topology $\tau_{\mathcal{A}}$ is called the *hull-kernel topology* and the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called the *structure space* of the (m, n)-semiring (S, f, g).

Let I be a k-ideal of the (m, n)-semiring (S, f, g). We denote $\Delta(I) = \{I' \in \mathcal{A} : I \subseteq I'\}$ and $C\Delta(I) = \mathcal{A} \backslash \Delta(I) = \{I' \in \mathcal{A} : I \nsubseteq I'\}$.

The following results hold for the (m, n)-semiring (S, f, g) and their proofs are the same as in [25], so we omit them.

Proposition 2.3. *Let (S, f, g) be an (m, n)-semiring. Any closed set in $\mathcal{A}$ is of the form $\Delta(I)$, where I is a k-ideal of S.*

Corollary 2.4. *Let (S, f, g) be an (m, n)-semiring. Any open set in $\mathcal{A}$ is of the form $C\Delta(I)$, where I is a k-ideal of S.*

If $a \in S$, then we denote $\Delta(I) = \{I \in \mathcal{A} : a \in I\}$ and $C\Delta(a) = \mathcal{A} \backslash \Delta(a) = \{I \in \mathcal{A} : a \notin I\}$.

Proposition 2.5. *Let (S, f, g) be an (m, n)-semiring. $\{C\Delta(a) : a \in S\}$ is an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.*

Theorem 2.6. *The structure $(\mathcal{A}, \tau_{\mathcal{A}})$ on the (m,n)-semiring (S, f, g) is a T_0-space.*

Theorem 2.7. *The structure $(\mathcal{A}, \tau_{\mathcal{A}})$ on the (m,n)-semiring (S, f, g) is a T_1-space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.*

Corollary 2.8. *Let $\mathcal{M}$ be the set of all maximal k-ideals of an (m,n)-semirings (S, f, g) with g-identity. Then $(\mathcal{M}, \tau_{\mathcal{M}})$ is a T_1-space, where $\tau_{\mathcal{M}}$ is the induced topology on $\mathcal{M}$ from $(\mathcal{A}, \tau_{\mathcal{A}})$.*

Theorem 2.9. *Let (S, f, g) be an (m,n)-semiring. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorf space if and only if for any two distinct pair of elements I, J of $\mathcal{A}$, there exist $a, b \in S$ such that $a \notin I, b \notin J$ and there does not exist any element K of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$.*

Corollary 2.10. *Let (S, f, g) be an (m,n)-semiring. If $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorf space, then no n-ary prime k-ideal contains any other n-ary prime k-ideal. If $(\mathcal{A}, \tau_{\mathcal{A}})$ contains more than one element, then there exist $a, b \in S$ such that $\mathcal{A} = C\Delta a \cup C\Delta(b) \cup \Delta(I)$, where I is the k-ideal generated by a, b.*

Theorem 2.11. *Let (S, f, g) be an (m,n)-semiring. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space if and only if for any $I \in \mathcal{A}$ and $a \notin I$, $a \in S$, there exist a k-ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.*

Theorem 2.12. *Let (S, f, g) be an (m,n)-semiring. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space if and only if for any collection $\{a_\alpha\}_{\alpha\in\Lambda}$ of S there exists a finite subcollection $\{a_i : i = 1, 2, ..., n\}$ in S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.*

Corollary 2.13. *Let (S, f, g) be an (m,n)-semiring. If S is finitely generated, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space.*

An (m,n)-semiring (S, f, g) is called a *k-Noetherian (m,n)-semiring* if it satisfies the ascending chain condition on k-ideals of S i.e. if $I_1 \subseteq I_2 \subseteq ... \subseteq I_t \subseteq ....$ is an ascending chain of k-ideals of S, then there exists a positive integer s such that $I_t = I_s$, for all $t \geq s$.

Theorem 2.14. *Let (S, f, g) be an (m,n)-semiring. If S is k-Notherian, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact.*

Corollary 2.15. *Let (S, f, g) be an (m,n)-semiring. If S is k-Notherian and $(\mathcal{A}, \tau_{\mathcal{A}})$ is second countable, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is comapct.*

3. Structure Space of *n*-ary Prime Full *k*-Ideals

In this section, we introduce and study the notion of the structure space of (m,n)-semirings (S, f, g) formed by the class of n-ary prime full k-ideals. Let C be the collection of all n-ary prime full k-ideals of the (m,n)-semirings (S, f, g). Then we see that C is a subset of $\mathcal{A}$ and hence (C, τ_C) is a topological space, where τ_C is a subspace topology.

Now we have the following results :

Theorem 3.1. *Let (S, f, g) be an (m,n)-semiring. Then (C, τ_C) is a compact space if $E^+(S) \neq \{0\}$.*

Proof. Let $\{\Delta(I_i) : i \in \Lambda\}$ be any collection of closed sets in C with finite intersection property. Let I be the proper n-ary prime k-ideal which is also full k-ideal generated by $E^+(S)$. Since any n-ary prime full k-ideal J of S contains $E^+(S)$, then J contains I. Hence $I \in \bigcap_{i\in\Lambda} \Delta(I_i) \neq \emptyset$. Consequently, (C, τ_C) is compact. □

Similarly, we can prove that (C, τ_C) is a connected space if $E^+(S) \neq \{0\}$.

Remark 3.2. Let $\{I_\alpha\}$ be a collection of n-ary prime k-ideals of an (m,n)-semirings (S, f, g). Then $\bigcap I_\alpha$ is a k-ideal of S but it may not be a n-ary prime k-ideal of S, in general.

But in particular, we have :

Proposition 3.3. *Let (S, f, g) be an (m, n)-semiring and $\{I_\alpha\}$ be a collection of n-ary prime k-ideals of S such that $\{I_\alpha\}$ forms a chain. Then $\bigcap I_\alpha$ is a n-ary prime k-ideal of S.*

We give the following definition in the structure space $(\mathcal{A}, \tau_{\mathcal{A}})$.

Definition 3.4. Let (S, f, g) be an (m, n)-semiring. The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ of S is called *irreducible* if for any decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup .... \cup \mathcal{A}_n$, where $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ are closed subsets of $\mathcal{A}$, we have either $\mathcal{A} = \mathcal{A}_1$ or $\mathcal{A} = \mathcal{A}_2$ or ... or $\mathcal{A} = \mathcal{A}_n$.

Now we have the following result :

Theorem 3.5. *Let (S, f, g) be an (m, n)-semiring having a collection of n-ary prime k-ideals $\{I_\alpha\}$ and A be a closed subset of $\mathcal{A}$. Then A is irreducible if and only if $\bigcap\limits_{I_\alpha \in A} I_\alpha$ is a n-ary prime k-ideal of S.*

Proof. Let A be irreducible. Let $P_1, P_2, ..., P_n$ be n k-ideals of S such that $g(P_1, ..., P_n) \subseteq \bigcap\limits_{I_\alpha \in A} I_\alpha$. Then $g(P_1, ..., P_n) \subseteq I_\alpha$ for all α. Since I_α is n-ary prime, we have $P_1 \subseteq I_\alpha$ or ... or $P_n \subseteq I_\alpha$ which implies for $I_\alpha \in A$, $I_\alpha \in \{\overline{P_1}\}$ or .... or $I_\alpha \in \{\overline{P_n}\}$. Hence $A = (A \cap \overline{P_1}) \cup ... \cup (A \cap \overline{P_n})$. Since A is irreducible and $(A \cap \overline{P_1}), ..., (A \cap \overline{P_n})$ are closed, it follows that $A = A \cap \overline{P_1}$ or ... or $A = A \cap \overline{P_n}$ and hence $A \subseteq \overline{P_1}$ or ... or $A \subseteq \overline{P_n}$. This implies that $P_1 \subseteq \bigcap\limits_{I_\alpha \in A} I_\alpha$ or .... or $P_n \subseteq \bigcap\limits_{I_\alpha \in A} I_\alpha$. Consequently, $\bigcap\limits_{I_\alpha \in A} I_\alpha$ is a n-ary prime k-ideal of S.

Conversely, let us suppose that $\bigcap\limits_{I_\alpha \in A} I_\alpha$ be a n-ary prime k-ideal of S. Let $A = A_1 \cup ... \cup A_n$, where $A_1, ..., A_n$ are closed subsets of A. Then $\bigcap\limits_{I_\alpha \in A} I_\alpha \subseteq \bigcap\limits_{I_\alpha \in A_1} I_\alpha, ..., \bigcap\limits_{I_\alpha \in A} I_\alpha \subseteq \bigcap\limits_{I_\alpha \in A_n} I_\alpha$. We have

$$\bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_1 \cup ... \cup A_n} I_\alpha = \left(\bigcap_{I_\alpha \in A_1} I_\alpha\right) \cap .... \cap \left(\bigcap_{I_\alpha \in A_n} I_\alpha\right).$$

Also we have

$$g\left(\left(\bigcap_{I_\alpha \in A_1} I_\alpha\right), ..., \left(\bigcap_{I_\alpha \in A_n} I_\alpha\right)\right) \subseteq \left(\bigcap_{I_\alpha \in A_1} I_\alpha\right) \cap ... \cap \left(\bigcap_{I_\alpha \in A_n} I_\alpha\right).$$

Since $\bigcap\limits_{I_\alpha \in A} I_\alpha$ is n-ary prime and $\left(\bigcap\limits_{I_\alpha \in A_1} I_\alpha\right) \cap ... \cap \left(\bigcap\limits_{I_\alpha \in A_n} I_\alpha\right) = \bigcap\limits_{I_\alpha \in A} I_\alpha$, it follows that

$$\bigcap_{I_\alpha \in A_1} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha \text{ or ... or } \bigcap_{I_\alpha \in A_n} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha.$$

So we find that

$$\bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_1} I_\alpha \text{ or .... or } \bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_n} I_\alpha.$$

Let $I_\beta \in A$. Then we have

$$\bigcap_{I_\alpha \in A_1} I_\alpha \subseteq I_\beta \text{ or .... or } \bigcap_{I_\alpha \in A_n} I_\alpha \subseteq I_\beta.$$

Since $A_1, ..., A_n \subseteq A$, so $I_\alpha \subseteq I_\beta$ for all $I_\alpha \in A_1$ or ... or $I_\alpha \subseteq I_\beta$ for all $I_\alpha \in A_n$. Thus $I_\beta \in \overline{A_1} = A_1$ or ... or $I_\beta \in \overline{A_n} = A_n$, since $A_1, ..., A_n$ are closed. So we find that $A = A_1$ or ... or $A = A_n$. $\square$

4. Structure Space of Maximal Ideals

In this section, the structure space of all maximal ideals of an (m, n)-semirings (S, f, g) with g-identity e is considered and studied.

Example 4.1. Let S denote the $(2, 3)$-semiring of positive even integers with the usual addition and ternary multiplication. If $M = \{x \in S : x > 2\}$, then M is maximal ideal in S.

Let $\mathcal{M}$ be the set of all maximal ideals in an (m, n)-semirings (S, f, g). We shall define two topologies on $\mathcal{M}$. For every $x \in S$, we denote by Δ_x the set of all maximal ideals containing x and by Ω_x the set $\mathcal{M} - \Delta_x$, i.e. the set of all maximal ideals not containing x. Let I be an ideal of S, we denote by Δ_I the set of all maximal ideals containing I.

We choose the family $\{\Delta_x : x \in S\}$ as a subbase for open sets of $\mathcal{M}$. We shall refer to the resulting topology on $\mathcal{M}$ as Δ-topology (in symbol, $\Delta_{\mathcal{M}}$). Similarly, we shall take the family $\{\Omega_x : x \in S\}$ as a subbase for open sets of $\mathcal{M}$ (in symbol, $\Omega_{\mathcal{M}}$).

Theorem 4.2. *The topological space $(\mathcal{M}, \Delta_{\mathcal{M}})$ is a T_2-space.*

Proof. Let $M_1, ..., M_s$, where $s = k(m-1)+1, k = 1, 2, ...,$ be distinct elements of $\mathcal{M}$. Then we have $f(M_1, ..., M_s) = S$. Therefore there are $a_1, ..., a_s$ such that $e = f(a_1, ..., a_s)$ and $a_1 \in M_1, ..., a_s \in M_s$, so we have $\Delta_{a_1} \ni M_1, ..., \Delta_{a_s} \ni M_s$ and $\Delta_{a_1} \cap ... \cap \Delta_{a_s} = \emptyset$. □

Theorem 4.3. *The topological space $(\mathcal{M}, \Omega_{\mathcal{M}})$ is a T_1-space.*

Proof. Let M_1 be an element of $\mathcal{M}$, and $M_1 \neq M_2 \in \mathcal{M}$, then there is an element a such that $a \in M_2$ and $a \notin M_1$. Therefore $M_2 \notin \Omega_a$ and $M_2 \notin \bigcap\limits_{x \notin M_1} \Omega_x$. This implies $M_1 = \bigcap\limits_{x \notin M_1} \Omega_x$. □

Theorem 4.4. *The closed sets for $(\mathcal{M}, \Omega_{\mathcal{M}})$ are expressed by sets Δ_I, where I is an ideal of S.*

Proof. Let I be an ideal of S and $\{a_\lambda\}$ a generator of I, then we have

$$\Delta_I = \bigcap_{\lambda} \Delta_{a_\lambda}.$$

Therefore, the closed sets for the topological space $(\mathcal{M}, \Omega_{\mathcal{M}})$ have the form $\Delta_{I_1} \cup \Delta_{I_2} \cup ... \cup \Delta_{I_s}$, where I_i are ideals of S and $s = k(m-1)+1, k = 1, 2, ...$

Let $I = \bigcap\limits_{i=1}^{s} I_i$, if $M \in \Delta_{I_i}$ for some i, then $M \supset I_i$ and $M \supset I$. This implies $\Delta_I \ni M$ and we have $\bigcup\limits_{i=1}^{s} \Delta_{I_i} \subset \Delta_I$. Let us suppose that there is a maximal ideal M such that $M \in \Delta I - \bigcup\limits_{i=1}^{s} \Delta_{I_i}$, then $M \in \Delta_I$ and $M \notin \bigcup\limits_{i=1}^{s} \Delta_{I_i}$. Hence $M \supset I$ and M does not contain every $I_i (i = 1, 2, ..., s)$. Therefore, since M is a maximal ideal, there are elements $a_i \in I_i$ and $b_i \in M$ such that

$$e = f(g(a_1, a_2, ..., a_s), b_1, ..., b_{s-1})$$

where $f(b_1, ..., b_s) \in M$ and $g(a_1, a_2, ..., a_s) \in I$. This implies $f(I, M, ..., M) = S$. Hence, by $I \subset M$, we have $M = S$, which is a contradiction. Thus we have:

$$\bigcup_{i=1}^{s} \Delta_{I_i} = \Delta_I$$

and this completes the proof. □

By Theorem 4.4, we prove the following result.

Theorem 4.5. *The space $(\mathcal{M}, \Omega_{\mathcal{M}})$ is a compact T_1-space.*

Proof. Let $\{\Delta_{I_\lambda}\}$ be a family of closed sets in $(\mathcal{M}, \Omega_{\mathcal{M}})$ with the finite intersection property, where I_λ are ideals in S. Therefore, any finite family of I_λ does not contain the (m, n)-semiring S. Hence the ideal I generated by $\{I_\lambda\}$ does not contain the g-identity e of S. This shows that I is contained in a maximal ideal M. Hence $M \in \bigcap\limits_{\lambda} \Delta_{I_\lambda}$. Therefore, since $\bigcap\limits_{\lambda} \Delta_{I_\lambda}$ is non-empty, $(\mathcal{M}, \Omega_{\mathcal{M}})$ is a compact space. □

5. Structure Space of n-ary Prime Ideals

In this section, the structure space $\mathcal{B}$ of all n-ary prime ideals of an (m,n)-semiring (S,f,g) with g-identity e is considered and the relation of $\mathcal{B}$ and the structure space $\mathcal{M}$ of all maximal ideals of S is investigated. Throughout the section, S will denote a commutative (m,n)-semiring with g-identity element e.

An ideal P of S is n-ary prime if and only if for $a_1,...,a_n \in S$, $g(a_1,...,a_n) \in P$ implies $a_1 \in P$ or ... or $a_n \in P$. Since S has a g-identity e, any maximal ideal is n-ary prime, therefore $\mathcal{B} \supseteq \mathcal{M}$. We notice here that a maximal ideal in a commutative (m,n)-semiring without a g-identity may not be n-ary prime. In Example 4.1, the ideal M is maximal but it is not 3-ary prime since $2 \notin M$ and $2 \cdot 2 \cdot 2 \in M$.

To introduce a topology $\tau_{\mathcal{B}}$ on $\mathcal{B}$, we shall take $\tau_x = \{P : x \notin P, P \in \mathcal{B}\}$ for every $x \in S$ as an open base of $\mathcal{B}$.

Now we have the following result:

Theorem 5.1. *Let $\mathcal{U}$ be a subset of $\mathcal{B}$. Then*

$$\overline{\mathcal{U}} = \{P' \in \mathcal{B} : \bigcap_{P\in\mathcal{U}} P \subset P'\},$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ w.r.t. the topology τ.

Proof. Let $P' \in \{P' \in \mathcal{B} : \bigcap_{P\in\mathcal{U}} P \subset P'\}$ and let τ_x be a neighbourhood of P'. Then $x \notin P'$, and hence $x \notin \bigcap_{P\in\mathcal{U}} P$. Therefore, there is a n-ary prime ideal $P \in \mathcal{U}$ such that $x \notin P$ and $P \in \tau_x$. This shows that $P \in \overline{\mathcal{U}}$. Thus we have proved that $\{P' \in \mathcal{B} : \bigcap_{P\in\mathcal{U}} P \subset P'\} \subseteq \overline{\mathcal{U}}$.

If a n-ary prime ideal P' is not in $\{P' \in \mathcal{B} : \bigcap_{P\in\mathcal{U}} P \subset P'\}$, then $\bigcap_{P\in\mathcal{U}} P - P' \neq \emptyset$. Hence, for $x \in \bigcap_{P\in\mathcal{U}} P - P'$, we have $x \in P$ for $P \in \mathcal{U}$ and $x \notin P'$. This shows $P \notin \tau_x$ for $P \in \mathcal{U}$ and $P' \in \tau_x$. Therefore $\tau_x \cap \mathcal{U} = \emptyset$ and hence $P' \notin \overline{\mathcal{U}}$. This completes the proof. □

A similar argument for $\mathcal{M}$ relative to Ω-topology implies the following result :

Theorem 5.2. *Let $\mathcal{U}$ be a subset of $\mathcal{M}$. Then*

$$\overline{\mathcal{U}} = \{M' \in \mathcal{M} : \bigcap_{M\in\mathcal{U}} M \subset M'\},$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ w.r.t. the topology $\Omega_{\mathcal{M}}$.

In a similar way to the proof of Theorem 2.1, we can prove the following result.

Theorem 5.3. *The closure operator $\mathcal{U} \to \overline{\mathcal{U}}$ of $\mathcal{B}$ satisfies the following relations:*

1. $\mathcal{U} \subseteq \overline{\mathcal{U}}$.
2. $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.
3. $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$.

Proof. We shall prove only the last relation (3). Let $\mathcal{V}_1, ..., \mathcal{V}_{n-2} \in \mathcal{B}$. By Theorem 5.1, $\mathcal{U}, \mathcal{V}_1, ..., \mathcal{V}_{n-2} \subseteq \mathcal{B}$ implies $\overline{\mathcal{U}} \subseteq \overline{\mathcal{B}}, \overline{\mathcal{V}_1} \subseteq \overline{\mathcal{B}}, ..., \overline{\mathcal{V}_{n-2}} \subseteq \overline{\mathcal{B}}$ and hence $\overline{\mathcal{U}} \cup \overline{\mathcal{V}_1} \cup ... \cup \overline{\mathcal{V}_{n-2}} \cup \overline{\mathcal{B}} \subseteq \overline{\mathcal{U} \cup \mathcal{V}_1 \cup ... \cup \mathcal{V}_{n-2} \cup \mathcal{B}}$. Let $P \notin \overline{\mathcal{U}} \cup \overline{\mathcal{V}_1} \cup ... \cup \overline{\mathcal{V}_{n-2}} \cup \overline{\mathcal{B}}$. Then $P \notin \overline{\mathcal{U}}$, $P \notin \overline{\mathcal{V}_1}$, ..., $P \notin \overline{\mathcal{V}_{n-2}}$ and $P \notin \overline{\mathcal{B}}$. Hence $P \not\supset \bigcap_{P'\in\mathcal{U}} P' = P_{\mathcal{U}}$, $P \not\supset \bigcap_{P'\in\mathcal{V}_1} P' = P_{\mathcal{V}_1}$, ..., $P \not\supset \bigcap_{P'\in\mathcal{V}_{n-2}} P' = P_{\mathcal{V}_{n-2}}$ and $P \not\supset \bigcap_{P'\in\mathcal{B}} P' = P_{\mathcal{B}}$. The sets $P_{\mathcal{U}}$, $P_{\mathcal{V}_1}$, ...,$P_{\mathcal{V}_{n-2}}$ and $P_{\mathcal{B}}$ are ideals. If $g(P_{\mathcal{U}}, P_{\mathcal{V}_1}, ..., P_{\mathcal{V}_{n-2}}, P_{\mathcal{B}}) \subset P$, for any elements $a, b_1, ..., b_{n-2}, c$ such that $a \in P_{\mathcal{U}} - P, b_i \in P_{\mathcal{V}_i} - P (i = 1, 2, ..., n-2), c \in P_{\mathcal{B}} - P$, we have $g(a, b_1^{n-2}, c) \in P$ and since P is a n-ary prime ideal, $a \in P$ or $b_1 \in P$ or... or $b_{n-2} \in P$ or $c \in P$, which is a contradiction. Therefore, $P \not\supset g(P_{\mathcal{U}}, P_{\mathcal{V}_1}, ..., P_{\mathcal{V}_{n-2}}, P_{\mathcal{B}}) \supseteq P_{\mathcal{U}} \cap P_{\mathcal{V}_1} \cap ... \cap P_{\mathcal{V}_{n-2}} \cap P_{\mathcal{B}} = P_{\mathcal{U} \cup \mathcal{V}_1 \cup ... \cup \mathcal{V}_{n-2} \cup \mathcal{B}}$. Hence $P \notin \overline{\mathcal{U} \cup \mathcal{V}_1 \cup ... \cup \mathcal{V}_{n-2} \cup \mathcal{B}}$. So we find that $P \in \overline{\mathcal{U} \cup \mathcal{V}_1 \cup ... \cup \mathcal{V}_{n-2} \cup \mathcal{B}}$ implies that $P \in \overline{\mathcal{U}} \cup \overline{\mathcal{V}_1} \cup ... \cup \overline{\mathcal{V}_{n-2}} \cup \overline{\mathcal{B}}$ i.e. $\overline{\mathcal{U} \cup \mathcal{V}_1 \cup ... \cup \mathcal{V}_{n-2} \cup \mathcal{B}} \subseteq \overline{\mathcal{U}} \cup \overline{\mathcal{V}_1} \cup .... \cup \overline{\mathcal{V}_{n-2}} \cup \overline{\mathcal{B}}$. Since $\overline{\emptyset} = \emptyset$, taking $\mathcal{V}_i = \emptyset (i = 1, ..., n-2)$, we have $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$. □

Theorem 5.4. *The topological space $(\mathcal{B}, \tau_{\mathcal{B}})$ is a T_0-space.*

Proof. It is sufficient to prove that $\overline{P_1} = \overline{P_2}$ implies $P_1 = P_2$. By $P_2 \in \overline{P_1}$, we find that $P_2 \supset P_1$. Similarly, $P_1 \supset P_2$ and we have $P_1 = P_2$. □

Theorem 5.5. *The topological space $(\mathcal{B}, \tau_{\mathcal{B}})$ is a compact T_1-space.*

Proof. Let $\mathcal{U}_\lambda$ be a family of closed sets such that $\bigcap_\lambda \mathcal{U}_\lambda = \emptyset$. Then we have $f(P_{\mathcal{U}_\lambda}) = S$, where $P_{\mathcal{U}_\lambda} = \bigcap_{P \in \mathcal{U}_\lambda} P$. Indeed: Let us suppose that $f(P_{\mathcal{U}_\lambda}) \neq S$. Then there is a maximal ideal M containing $f(P_{\mathcal{U}_\lambda})$. Therefore $f(P_{\mathcal{U}_\lambda}) \subset M$ for every λ. Hence $M \in \mathcal{U}_\lambda$ for every λ, and we have $\bigcap_\lambda \mathcal{U}_\lambda \ni M$, which is a contradiction. By $f(P_{\mathcal{U}_\lambda}) = S$, we find that $e = f(a_1^m)$, $a_i \in P_{\mathcal{U}_{\lambda_i}}, i = 1, 2, ..., m$. Hence $f\left(P_{\mathcal{U}_{\lambda_1}}, ..., P_{\mathcal{U}_{\lambda_m}}\right) = S$. If $\bigcap_{i=1}^m \mathcal{U}_{\lambda_i} \neq \emptyset$, then for a n-ary prime ideal P of $\bigcap_{i=1}^m \mathcal{U}_{\lambda_i}$, we have $P \supset P_{\mathcal{U}_{\lambda_i}} (i = 1, 2, ..., m)$ and hence $P \supset f\left(P_{\mathcal{U}_{\lambda_1}}, ..., P_{\mathcal{U}_{\lambda_m}}\right)$. Therefore we have $\bigcap_{i=1}^m \mathcal{U}_{\lambda_i} = \emptyset$. □

By the $\mathcal{B}$-*radical* $r(\mathcal{B})$ of the (m, n)-semiring S, we mean the intersection of all prime ideals of S, that is, $\bigcap_{P \in \mathcal{B}} P$. By the $\mathcal{M}$-*radical* $r(\mathcal{M})$ of S, we mean the intersection of all maximal ideals of S, that is, $\bigcap_{M \in \mathcal{M}} M$. From $\mathcal{M} \subseteq \mathcal{B}$, we have $r(\mathcal{B}) \subseteq r(\mathcal{M})$. In the following result we give a condition to be $r(\mathcal{B}) = r(\mathcal{M})$.

Theorem 5.6. *The subset $\mathcal{M}$ of $\mathcal{B}$ is dense in $\mathcal{B}$, if and only if, $r(\mathcal{B}) = r(\mathcal{M})$.*

Proof. Let $\overline{\mathcal{M}} = \mathcal{B}$ for the topology τ. Then we have

$$\{P : \bigcap_{M \in \mathcal{M}} M \subset P\} = \mathcal{B}.$$

Hence

$$r(\mathcal{M}) = \bigcap_{M \in \mathcal{M}} M \subseteq \bigcap_{P \in \mathcal{B}} P = r(\mathcal{B}).$$

Since $r(\mathcal{B}) \subseteq r(\mathcal{M})$, therefore we have $r(\mathcal{B}) = r(\mathcal{M})$.

Conversely, if $P \in \mathcal{B} - \overline{\mathcal{M}}$, then $P \in \mathcal{B}$ and $P \notin \overline{\mathcal{M}}$. Therefore, there is a neighbourhood τ_x of P such that $\tau_x \cap \mathcal{M} = \emptyset$. Hence $r(\mathcal{B}) = \bigcap_{P \in \mathcal{B}} P$ is a proper subset of $\bigcap_{M \in \mathcal{M}} M$. Therefore $r(\mathcal{B}) \neq r(\mathcal{M})$, which completes the proof. □

Definition 5.7. The (m, n)-semiring (S, f, g) is said to be $\mathcal{M}$ – *semisimple*, if $r(\mathcal{M})$ is the zero ideal (0).

From Theorem 5.6, we have the following result.

Theorem 5.8. *If the (m, n)-semiring (S, f, g) is $\mathcal{M}$-semisimple, then $\mathcal{M}$ is dense in $\mathcal{B}$.*

6. Structure Space of Strongly Irreducible Ideals

In this section, the structure space $\mathcal{S}$ of all strongly irreducible ideals of a commutative (m, n)-semiring (S, f, g) with g-identity e is investigated.

An ideal I of an (m, n)-semiring (S, f, g) is called *irreducible* if and only if $A \cap B = I$ for two ideals A, B implies $A = I$ or $B = I$. An ideal I of an (m, n)-semiring S is called *strongly irreducible*, if and only if $A \cap B \subset I$ for any two ideals A, B implies $A \subset I$ or $B \subset I$. From $g(\overset{(n-2)}{S}, A, B) \subset A \cap B$ for any two ideals A, B, it follows that any n-ary prime ideals are strongly irreducible and any strongly irreducible ideals are irreducible.

Let $\mathcal{S}$ be the set of all strongly irreducible ideals of the (m, n)-semiring (S, f, g). From the above, it is clear that $\mathcal{M} \subset \mathcal{B} \subset \mathcal{S}$. To give a topology $\sigma_{\mathcal{S}}$ on $\mathcal{S}$, we shall take $\sigma_x = \{S' \in \mathcal{S} : x \notin S'\}$ for every $x \in S$ as an open base of $\mathcal{S}$.

Theorem 6.1. *Let $\mathcal{U}$ be a subset of $\mathcal{S}$. Then we have*

$$\overline{\mathcal{U}} = \{S' \in \mathcal{S} : \bigcap_{I \in \mathcal{U}} I \subset S'\}$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ w.r.t. the topology $\sigma_{\mathcal{S}}$.

Proof. Let $S' \in \{S' \in \mathcal{S} : \bigcap_{I \in \mathcal{U}} I \subset S'\}$. Let σ_x be an open base of S', then, by the definition of the topology $\sigma_{\mathcal{S}}$, $x \notin S'$. Hence, we have $x \notin \bigcap_{I \in \mathcal{U}} I$. It follows from this that there is a strongly irreducible ideal I of $\mathcal{U}$ such that $x \notin I$. Hence $I \in \sigma_x$. Therefore $S' \in \overline{\mathcal{U}}$ and $\{S' \in \mathcal{S} : \bigcap_{I \in \mathcal{U}} I \subset S'\} \subset \overline{\mathcal{U}}$.

To prove that $\{S' \in \mathcal{S} : \bigcap_{I \in \mathcal{U}} I \subset S'\} \supset \overline{\mathcal{U}}$, take a strongly irreducible ideal S' such that $S' \notin \{S' \in \mathcal{S} : \bigcap_{I \in \mathcal{U}} I \subset S'\}$. Then $\bigcap_{I \in \mathcal{U}} I - S' \neq \emptyset$. For an element $x \in \bigcap_{I \in \mathcal{U}} I - S'$, we have $x \in I$ $(I \in \mathcal{U})$ and $x \in S'$. Hence $S' \in \sigma_x$ and $I \notin \sigma_x$ for all I of $\mathcal{U}$. Therefore $\mathcal{U} \cap \sigma_x = \emptyset$ and then we have $S' \notin \overline{\mathcal{U}}$. Hence $\{S' \in \mathcal{S} : \bigcap_{I \in \mathcal{U}} I \subset S'\} \supset \overline{\mathcal{U}}$. This completes the proof. □

In a similar way to the proof of Theorem 5.3, we can easily prove the following result.

The closure operator $\mathcal{U} \to \overline{\mathcal{U}}$ of $\mathcal{S}$ satisfies the following relations :

1. $\mathcal{U} \subseteq \overline{\mathcal{U}}$.
2. $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.
3. $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$.

Theorem 6.2. *Let (S, f, g) be an (m, n)-semiring. The topological space $(\mathcal{S}, \sigma_{\mathcal{S}})$ is compact T_0-space.*

Proof. To prove that $\mathcal{S}$ is a T_0-space, it is sufficient to verify that $\overline{S_1} = \overline{S_2}$ implies $S_1 = S_2$. For this, we shall use the condition (1). Then $S_2 \in \overline{S_1}$ and by the definition of closure operation, we have $S_1 \subset S_2$. Similarly we have $S_1 \supset S_2$ and $S_1 = S_2$. Therefore we have proved that $\mathcal{S}$ is a T_0-space.

We shall prove now that $\mathcal{S}$ is a compact space. Let $\mathcal{U}_t$ be a family of closed sets with empty intersection. Let $S_{\mathcal{U}_t} = \bigcap_{S' \in \mathcal{U}_t} S'$, such that $f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right) \neq S$ where $i = k(m-1)+1, k = 1, 2, ...$ Then there is a maximal ideal M containing the ideal $f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right)$. Therefore we have $S_{\mathcal{U}_t} \subset M$ for every t. By the definition of $S_{\mathcal{U}_t}, M \in \mathcal{U}_t$ for every t. Hence $M \in \bigcap_t \mathcal{U}_t$, which contradicts our hypothesis of $\mathcal{U}_t$. Therefore $f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right) = S$. So we find that $e = f(a_1, ..., a_i)$, $a_i \in S_{\mathcal{U}_{t_i}}$, $i = k(m-1)+1, k = 1, 2, ...$ Hence we have $S = f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right)$. If $\bigcap_{i=1}^{j} \mathcal{U}_{t_i} \neq \emptyset$, $j = k(m-1)+1, k = 1, 2, ...$, for every strongly irreducible ideal S' of $\bigcap_{i=1}^{j} \mathcal{U}_{t_i}$, $S' \supset S_{\mathcal{U}_{t_i}}, i = 1, 2, ..., j, j = k(m-1)+1, k = 1, 2, ...$ and $S' \supset f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right)$, $i = k(m-1)+1, k = 1, 2, ...$ If $\bigcap_{i=1}^{j} \mathcal{U}_{t_i} = S$, $j = k(m-1)+1, k = 1, 2, ...$, we can prove easily that $\mathcal{S}$ is a compact space. If $\bigcap_{i=1}^{j} \mathcal{U}_{t_i}$ contains a proper strongly irreducible ideal S', we have $S' \supset f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right)$, $i = k(m-1)+1, k = 1, 2, ...$ which is a contradiction to $S = f\left(S_{\mathcal{U}_{t_1}}, ..., S_{\mathcal{U}_{t_i}}\right)$. Therefore $\bigcap_{i=1}^{j} \mathcal{U}_{\mathcal{U}_{t_i}} = \emptyset$, $j = k(m-1)+1, k = 1, 2, ...$ Hence $\mathcal{S}$ is a compact space. This completes the proof. □

By the $\mathcal{S}$ – *radical* $r(\mathcal{S})$ of an (m, n)-semiring (S, f, g), we mean the intersection of all strongly irreducible ideals of it, i.e., $\bigcap_{S' \in \mathcal{S}} S'$. From $\mathcal{M} \subset \mathcal{B} \subset \mathcal{S}$, we have $r(\mathcal{M}) \supset r(\mathcal{B}) \supset r(\mathcal{S})$.

Theorem 6.3. *The subset $\mathcal{B}$ of $\mathcal{S}$ is dense in $\mathcal{S}$, if and only if $r(\mathcal{B}) = r(\mathcal{S})$.*

Proof. Let $\overline{\mathcal{B}} = \mathcal{S}$ for the topology $\sigma_{\mathcal{S}}$, then we have

$$\{S' : \bigcap_{P \in \mathcal{B}} P \subset S'\} = \mathcal{S}.$$

Hence, we have

$$r(\mathcal{B}) = \bigcap_{P \in \mathcal{B}} P \subset \bigcap_{S' \in \mathcal{S}} S' = r(\mathcal{S}).$$

On the other hand, $r(\mathcal{B}) \supset r(\mathcal{S})$. This shows $r(\mathcal{S}) = r(\mathcal{B})$.

Conversely, suppose that $\mathcal{S} - \overline{\mathcal{B}} \neq \emptyset$, then there is a strongly irreducible ideal S' such that $S' \notin \overline{\mathcal{B}}$ and $S' \in \mathcal{S}$. Therefore there is a neighbourhood σ_x of S' which does not meet $\mathcal{B}$. Hence $r(\mathcal{S}) = \bigcap_{S' \in \mathcal{S}} S'$ is a proper subset of $\bigcap_{P \in \mathcal{B}} P$, and we have $r(\mathcal{S}) \neq r(\mathcal{B})$. □

Corollary 6.4. *The subset $\mathcal{M}$ of $\mathcal{S}$ is dense in $\mathcal{S}$, if and only if $r(\mathcal{M}) = r(\mathcal{S})$.*

Corollary 6.5. *Let (S, f, g) be an (m, n)-semiring. If S is $\mathcal{M}$-semisimple, then $\mathcal{M}$ and $\mathcal{B}$ are dense in $\mathcal{S}$.*

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