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Nonlinear Sequential Fractional Differential Equations in Partially Ordered Spaces

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Abstract. In this paper, some new partially ordered Banach spaces are introduced. Based on those new partially ordered Banach spaces and applying some fixed point theorems, we present a new approach to the theory of nonlinear sequential fractional differential equations. An example illustrating our approach is also discussed.

1. Introduction

Fractional differential equations have attracted huge attention in the past few years because of their unique physical properties and their potential in the modeling of many physical phenomena and also in various field of science and engineering [11, 15–17]. During last years, the study of such kind of problems have received much attention from both theoretical and applied point of view [10, 22, 25–27].

Initial and boundary value problems of fractional order have extensively been studied by several researchers in recent years. A variety of results ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions have appeared in the literature, see [1–4, 7, 12, 14, 18, 19, 25, 28–30] and the references therein.

In this paper, we consider the initial value problem of the nonlinear sequential fractional differential equation

$$\mathcal{D}^{2\alpha}u(x) = f(x, u(x), \mathcal{D}^{\alpha}u(x)), \quad x \in (0, T], \tag{1}$$

$$\lim_{x\to 0} x^{1-\alpha}u(x) = u_0, \quad \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}u(x) = u_1, \tag{2}$$

where $0 < \alpha \leq 1$, $0 < T < \infty$. The $\mathcal{D}^{2\alpha}$ is the sequential fractional derivative presented by Miller and Ross [20]

$$\begin{cases} \mathcal{D}^{\alpha}u = D^{\alpha}u, \\ \mathcal{D}^{k\alpha}u = \mathcal{D}^{\alpha}\mathcal{D}^{(k-1)\alpha}u, \quad (k = 2, 3, \cdots), \end{cases} \tag{3}$$

where D^{α} is the classical Riemann-Liouville fractional derivative of order α.

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Problem (1)-(2) is of interest because it appears in mathematical models of physical phenomena. A classical example is Langevin equation which is widely used to describe the evolution of physical phenomena in fluctuating environments and the steady nonlinear fractional advection-dispersion equation [8, 9, 13]. Another example for an application of problem (1)-(2) is the Basset equation which describes the forces that occur when a spherical object sinks in a incompressible viscous fluid [5, 6].

Our main aim is to prove existence and uniqueness of solution for (1)-(2). This is done in Section 4. Our main tool is the fixed point theorems which are applied in the appropriate partially ordered sets as well as iterative methods, whose description can be found in [21, 24], and which we recall in Section 2. As essential tools for our reasoning we need a partially ordered set involving fractional derivatives to compare the solutions. To this end, we will introduce in Section 3 the appropriate weighted spaces of continuous functions and equip them with a partial order. The advantage and importance of this method arises from the fact that it is a constructive method that yields monotone sequences that converge to the unique solution of (1)-(2).

2. Preliminaries

Here, we recall several known definitions and properties from fractional calculus theory. For details, see [7, 17, 23].

Definition 2.1. *The Riemann-Liouville fractional integral I^α of order $\alpha > 0$ of a function $u : (0,T) \to \mathbb{R}$ is defined by*

$$I^\alpha u(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-\zeta)^{\alpha-1} u(\zeta) d\zeta,$$

provided the right-hand side is defined for almost every $x \in (0,T)$. We note that for $u \in L^1(0,T)$ we have that $I^\alpha u \in L^1(0,T)$.

Definition 2.2. *The Riemann-Liouville fractional derivative D^α of order $0 < \alpha \le 1$ of a function $u : (0,T) \to \mathbb{R}$ is defined by*

$$D^\alpha u(x) = \frac{d}{dx} I^{1-\alpha} u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\zeta)^{-\alpha} u(\zeta) d\zeta,$$

provided the right-hand side is defined for almost every $x \in (0,T)$.

Lemma 2.3. *Let $\alpha, \beta \ge 0$. If $u \in L^1(0,T)$, then $I^\alpha I^\beta u = I^{\alpha+\beta} u$ almost everywhere on $(0,T)$.*

Lemma 2.4. *Let $\alpha \ge 0$. If $u \in L^1(0,T)$, then $D^\alpha I^\alpha u = u$ almost everywhere on $(0,T)$.*

Lemma 2.5. *Assume that $u \in C(0,T] \cap L^1(0,T)$ with a fractional derivative of order $0 < \alpha \le 1$ that belongs to $C(0,T] \cap L^1(0,T)$. Then*

$$I^\alpha D^\alpha u(x) = u(x) + cx^{\alpha-1},$$

for some $c \in \mathbb{R}$.

Now we present the fixed point theorems which play main role in our discussion. For details, see [21].

Definition 2.6. *Let $(X, \preceq)$ is a partially ordered set and $A : X \to X$. We say that A is non-decreasing if $x \preceq y$ implies $A(x) \preceq A(y)$.*

Definition 2.7. *Let $(X, \preceq)$ is a partially ordered set and $x, y \in X$. We say that x and y are comparable if either $x \preceq y$ or $y \preceq x$ (or both, in which case $x = y$).*

Theorem 2.8. (**Partially Fixed Point Theorem**). *Let $(X, \preceq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Furthermore, let $A : X \to X$ be a continuous and non-decreasing mapping such that*

$$\exists\, 0 \le k < 1 : d(A(x), A(y)) \le kd(x, y), \quad \forall\; y \preceq x, \tag{4}$$

$$\exists x_0 \in X : x_0 \preceq A(x_0). \tag{5}$$

Then A has a fixed point.

Theorem 2.9. *Assume the hypotheses of Theorem 2.8, except the continuity of A. Moreover, we assume that if a non-decreasing sequence $x_n \to x$ in X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to x. Then A has a fixed point.*

Theorem 2.10. *Let all the conditions of Theorem 2.8 (resp. Theorem 2.9) be fulfilled and let the following condition holds:*

For every $x, y \in X$, there exists $z \in X$ which is comparable to x and y.

Then A has a unique fixed point $\bar{x}$. Moreover, for every $x \in X$, $\lim_{n\to\infty} A^n(x) = \bar{x}$.

3. Partially Ordered Spaces

Hereafter we suppose α and $\mathcal{D}^\alpha$ are as in (3). Now we introduce the following weighted continuous spaces and equip them with a partially order.

Definition 3.1. *We introduce the weighted spaces of continuous functions*

$$C_{1-\alpha}[0, T] = \{u \in C(0, T] : x^{1-\alpha}u \in C[0, T]\},$$

with the norm $\|u\|_{C_{1-\alpha}[0,T]} = \max_{0\le x\le 1} |x^{1-\alpha}u(x)|$.

Definition 3.2. *We define the following spaces of functions*

$$C^\alpha_{1-\alpha}[0, T] = \{u \in C_{1-\alpha}[0, T] : \mathcal{D}^\alpha u \in C_{1-\alpha}[0, T]\},$$

with the norm

$$\|u\|_{C^\alpha_{1-\alpha}[0,T]} = \|u\|_{C_{1-\alpha}[0,T]} + \|\mathcal{D}^\alpha u\|_{C_{1-\alpha}[0,T]}.$$

We show below in Theorem 3.6 that $C^\alpha_{1-\alpha}[0, T]$ is complete.

Lemma 3.3. *Let $\alpha > 0$ and $0 < \gamma < 1$. If $u \in C_\gamma[0, T]$, then $I^\alpha u \in C_{\gamma-\alpha}[0, T]$. Moreover, the following inequality holds:*

$$|I^\alpha u(x)| \le \|u\|_{C_\gamma[0,T]} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} x^{\alpha-\gamma},$$

for every $x \in (0, T]$.

Proof. See [15]. □

Consequently, from Lemma 3.3, we have the following property.

Lemma 3.4. *For $\alpha > 0$ and $0 < \gamma < 1$, I^α is linear and continuous from $C_\gamma[0, T]$ to $C_\gamma[0, T]$. Precisely, the following inequality holds:*

$$\|I^\alpha u\|_{C_\gamma[0,T]} \le T^\alpha \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \|u\|_{C_\gamma[0,T]}.$$

Lemma 3.5. *Assume that $u \in C^{\alpha}_{1-\alpha}[0,T]$ and $\lim_{x\to 0} x^{1-\alpha}u(x) = u_0$. Then*

$$I^{\alpha}D^{\alpha}u(x) = u(x) - u_0 x^{\alpha-1},$$

for every $x \in (0,T]$.

Proof. Since $C_{1-\alpha}[0,T] \subseteq C(0,T] \cap L^1(0,T)$, from Lemma 2.5, we have $I^{\alpha}D^{\alpha}u(x) = u(x) + cx^{\alpha-1}$ for some $c \in \mathbb{R}$. Therefore, $c = x^{1-\alpha}I^{\alpha}D^{\alpha}u(x) - x^{1-\alpha}u(x)$. Taking the limit as $x \to 0$, we have $c = -u_0$. It is important to note that $\lim_{x\to 0} x^{1-\alpha}I^{\alpha}D^{\alpha}u(x) = 0$, because of Lemma 3.3. □

Theorem 3.6. *The space $C^{\alpha}_{1-\alpha}[0,T]$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $C^{\alpha}_{1-\alpha}[0,T]$, then $\{u_n\}$ and $\{\mathcal{D}^{\alpha}u_n\}$ are Cauchy sequences in $C_{1-\alpha}[0,T]$. It follows that

$$u_n \xrightarrow{C_{1-\alpha}} u \quad , \quad \mathcal{D}^{\alpha}u_n \xrightarrow{C_{1-\alpha}} u^{(\alpha)}.$$

As $\{u_n\} \subseteq C^{\alpha}_{1-\alpha}[0,T]$, using Lemma 3.5 we get

$$I^{\alpha}\mathcal{D}^{\alpha}u_n(x) = u_n(x) + \Big(\lim_{x\to 0} x^{1-\alpha}u_n(x)\Big)x^{\alpha-1},$$

for every $n \in \mathbb{N}$. Now, from the continuity of Riemann-Liouville fractional integral operator from $C_{\gamma}[0,T]$ to $C_{\gamma}[0,T]$, we deduce

$$\begin{aligned} I^{\alpha}u^{(\alpha)} &= u(x) + \Big(\lim_{n\to\infty}\lim_{x\to 0} x^{1-\alpha}u_n(x)\Big)x^{\alpha-1} \\ &= u(x) + \Big(\lim_{x\to 0}\lim_{n\to\infty} x^{1-\alpha}u_n(x)\Big)x^{\alpha-1}, \end{aligned}$$

the last equality following from the uniformly convergent of $\{x^{1-\alpha}u_n\}$. Therefore, we have

$$I^{\alpha}u^{(\alpha)} = u(x) + \Big(\lim_{x\to 0} x^{1-\alpha}u(x)\Big)x^{\alpha-1}.$$

Finally, using Lemma 2.4, we immediately get that $u^{(\alpha)} = \mathcal{D}^{\alpha}u$, and hence the result. □

Definition 3.7. *We define the following order relation for $C^{\alpha}_{1-\alpha}[0,T]$,*

$$u \preceq v \iff x^{1-\alpha}u(x) \le x^{1-\alpha}v(x), \quad x^{1-\alpha}\mathcal{D}^{\alpha}u(x) \le x^{1-\alpha}\mathcal{D}^{\alpha}v(x), \quad x \in [0,T].$$

Lemma 3.8. $(C^{\alpha}_{1-\alpha}[0,T], \preceq)$ *is a partially ordered set and every pair of elements has a lower bound and an upper bound.*

Proof. It is easy to see that $C^{\alpha}_{1-\alpha}[0,T]$ is a partially ordered set. Now we prove that every pair of elements in $C^{\alpha}_{1-\alpha}[0,T]$ has a lower bound and an upper bound. Let $u, v \in C^{\alpha}_{1-\alpha}[0,T]$ and define

$$\underline{w}(x) = I^{\alpha}\min\{\mathcal{D}^{\alpha}u(\cdot), \mathcal{D}^{\alpha}v(\cdot)\}(x) + \min\{u_0, v_0\}x^{\alpha-1},$$

and

$$\overline{w}(x) = I^{\alpha}\max\{\mathcal{D}^{\alpha}u(\cdot), \mathcal{D}^{\alpha}v(\cdot)\}(x) + \max\{u_0, v_0\}x^{\alpha-1},$$

where $u_0 = \lim_{x\to 0} x^{1-\alpha}u(x)$ and $v_0 = \lim_{x\to 0} x^{1-\alpha}v(x)$. Then from Lemma 3.3 and Lemma 2.4, we have $\underline{w} \in C^{\alpha}_{1-\alpha}[0,T]$ and $\overline{w} \in C^{\alpha}_{1-\alpha}[0,T]$. On the other hand, it is easy to see that $\lim_{x\to 0} x^{1-\alpha}\underline{w}(x) = \min\{u_0, v_0\}$ and $\lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}\underline{w}(x) = \min\{u_1, v_1\}$ where

$$u_1 = \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}u(x), \qquad v_1 = \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}v(x).$$

Silmilarly, we have $\lim_{x\to 0} x^{1-\alpha}\overline{w}(x) = \max\{u_0, v_0\}$ and $\lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}\overline{w}(x) = \max\{u_1, v_1\}$. Owing to the monotonicity of Riemann-Liouville fractional integral I^{α} and using Lemma 3.5, we have

$$\begin{aligned}\underline{w}(x) &\leq I^{\alpha}\mathcal{D}^{\alpha}u(x) + \min\{u_0, v_0\}x^{\alpha-1}\\ &= u(x) - u_0x^{\alpha-1} + \min\{u_0, v_0\}x^{\alpha-1}\\ &\leq u(x),\end{aligned}$$

on $(0, T]$. Similarly, we get $\underline{w}(x) \leq v(x)$ on $(0, T]$. The same argument implies $\overline{w}(x) \geq v(x)$ and $\overline{w}(x) \geq v(x)$ on $(0, T]$. Finally, from Lemma 2.4 and utilizing the fact that $\mathcal{D}^{\alpha}x^{\alpha-1} = 0$, we have

$$\mathcal{D}^{\alpha}\underline{w}(x) = \min\{\mathcal{D}^{\alpha}u(\cdot), \mathcal{D}^{\alpha}v(\cdot)\}(x),$$

and

$$\mathcal{D}^{\alpha}\overline{w}(x) = \max\{\mathcal{D}^{\alpha}u(\cdot), \mathcal{D}^{\alpha}v(\cdot)\}(x),$$

on $(0, T]$. Therefore, we get

$$\mathcal{D}^{\alpha}\underline{w}(x) \leq \mathcal{D}^{\alpha}u(x), \qquad \mathcal{D}^{\alpha}\underline{w}(x) \leq \mathcal{D}^{\alpha}v(x),$$

and

$$\mathcal{D}^{\alpha}\overline{w}(x) \geq \mathcal{D}^{\alpha}u(x), \qquad \mathcal{D}^{\alpha}\overline{w}(x) \geq \mathcal{D}^{\alpha}v(x),$$

on $(0, T]$. Therefore $\underline{w}$ and $\overline{w}$ are a lower bound and an upper bound of $\{u, v\}$, respectively. □

4. Weighted Cauchy Type Problem

In this section, we intend to give an existence and uniqueness result for the initial value problem (1)-(2).

Definition 4.1. *A function $\underline{u} \in C^{\alpha}_{1-\alpha}[0, T]$ is called a lower solution of the initial value problem* (1)-(2), *if $\mathcal{D}^{2\alpha}\underline{u}(x) \leq f(., \underline{u}(.), \mathcal{D}^{\alpha}\underline{u}(.))(x)$ for every $x \in (0, T]$ and*

$$\lim_{x\to 0} x^{1-\alpha}\underline{u}(x) \leq u_0, \quad \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x) \leq u_1.$$

To prove the main results, we need the following assumptions:

(H1) $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that for every $u \in C^{\alpha}_{1-\alpha}[0, T]$, $f(., u(.), \mathcal{D}^{\alpha}u(.))(x) \in C_{\gamma}[0, T]$ for some $0 \leq \gamma < 1$.

(H2) f is non-decreasing in all its arguments except for the first argument and

$$f(x, u, v) - f(x, \tilde{u}, \tilde{v}) \leq L_1(u - \tilde{u}) + L_2(v - \tilde{v}),$$

for some $L_1, L_2 > 0$ whenever $x \in (0, T]$ and $u \geq \tilde{u}, v \geq \tilde{v}$.

Theorem 4.2. *Assume that* (H1)-(H2) *hold. Then there exists $0 < \delta \leq T$ such that the existence of a lower solution for* (1)-(2) *in $C^{\alpha}_{1-\alpha}[0, \delta]$ provides the existence of a unique solution $u \in C^{\alpha}_{1-\alpha}[0, \delta]$ for* (1)-(2).

Proof. We choose $\delta > 0$ such that the inequality

$$L = \max\{L_1, L_2\}\left(\delta^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(3\alpha)} + \delta^{\alpha}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\right) < 1,$$

holds. Now we define $A : C^{\alpha}_{1-\alpha}[0, \delta] \to C^{\alpha}_{1-\alpha}[0, \delta]$ by

$$Au(x) = u_0x^{\alpha-1} + u_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}x^{2\alpha-1} + I^{2\alpha}f(\cdot, u(\cdot), \mathcal{D}^{\alpha}u(\cdot))(x).$$

It is obvious that u is a solution of problem (1)-(2) if and only if u is a fixed point of the operator A.

First, we show that for any $u \in C^{\alpha}_{1-\alpha}[0,\delta]$, we have $Au \in C^{\alpha}_{1-\alpha}[0,\delta]$. Since

$$Au(x) = u_0 x^{\alpha-1} + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + I^{2\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x),$$

and

$$\mathcal{D}^{\alpha} Au(x) = u_1 x^{\alpha-1} + I^{\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x),$$

it is sufficient to verify that $I^{\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x) \in C_{1-\alpha}[0,\delta]$. By Lemma 3.3, $I^{\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x) \in C_{\gamma-\alpha}[0,\delta] \subset C_{1-\alpha}[0,\delta]$ which implies $Au \in C^{\alpha}_{1-\alpha}[0,\delta]$. So the operator A is well defined. For simplicity, we define the function $N_f : [0,T] \to \mathbb{R}$ as follows

$$N_f u(x) := f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x).$$

Now let $u, \tilde{u} \in C^{\alpha}_{1-\alpha}[0,\delta]$ with $u \le \tilde{u}$. From the non-decreasing assumption of f in all its arguments except for the first and using the monotonicity of Riemann-Liouville fractional integral operator, we obtain

$$\begin{aligned} Au(x) &= u_0 x^{\alpha-1} + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + I^{2\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x) \\ &\le u_0 x^{\alpha-1} + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + I^{2\alpha} f(\cdot, \tilde{u}(\cdot), \mathcal{D}^{\alpha} \tilde{u}(\cdot))(x) \\ &= A\tilde{u}(x), \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}^{\alpha} Au(x) &= u_1 x^{\alpha-1} + I^{\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x) \\ &\le u_1 x^{\alpha-1} + I^{\alpha} f(\cdot, u(\cdot), \mathcal{D}^{\alpha} u(\cdot))(x) \\ &= \mathcal{D}^{\alpha} A\tilde{u}(x), \end{aligned}$$

on $(0,\delta]$. On the other hand, for any $x \in (0,\delta]$ and for any $u \in C^{\alpha}_{1-\alpha}[0,\delta]$, we can prove that

$$\left| x^{1-\alpha} I^{2\alpha} N_f u(x) \right| \le \|N_f u\|_{C_{\gamma}[0,T]} \frac{\Gamma(1-\gamma)}{\Gamma(1+2\alpha-\gamma)} x^{1+\alpha-\gamma} \underset{x\to 0}{\longrightarrow} 0,$$

and

$$\left| x^{1-\alpha} I^{\alpha} N_f u(x) \right| \le \|N_f u\|_{C_{\gamma}[0,T]} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} x^{1-\gamma} \underset{x\to 0}{\longrightarrow} 0.$$

Therefore,

$$\lim_{x\to 0} x^{1-\alpha} Au(x) = u_0 = \lim_{x\to 0} x^{1-\alpha} A\tilde{u}(x),$$

and

$$\lim_{x\to 0} x^{1-\alpha} \mathcal{D}^{\alpha} Au(x) = u_1 = \lim_{x\to 0} x^{1-\alpha} \mathcal{D}^{\alpha} A\tilde{u}(x).$$

This proves that A is a non-decreasing operator. Also, for $\tilde{u} \le u$, we have

$$\begin{aligned} \left\| Au(x) - A\tilde{u}(x) \right\|_{C^{\alpha}_{1-\alpha}[0,\delta]} &= \left\| I^{2\alpha}\Big(N_f u(\cdot)(x) - N_f \tilde{u}(\cdot)(x)\Big)(x) \right\|_{C^{\alpha}_{1-\alpha}[0,\delta]} \\ &\le L_1 \left\| I^{2\alpha}\Big(u(\cdot) - \tilde{u}(\cdot)\Big)(x) \right\|_{C^{\alpha}_{1-\alpha}[0,\delta]} + L_2 \left\| I^{2\alpha} \mathcal{D}^{\alpha}\Big(u(\cdot) - \tilde{u}(\cdot)\Big)(x) \right\|_{C^{\alpha}_{1-\alpha}[0,\delta]} \\ &= L_1 \left\| I^{2\alpha}\Big(u(\cdot) - \tilde{u}(\cdot)\Big)(x) \right\|_{C_{1-\alpha}[0,\delta]} + L_1 \left\| I^{\alpha}\Big(u(\cdot) - \tilde{u}(\cdot)\Big)(x) \right\|_{C_{1-\alpha}[0,\delta]} \\ &\quad + L_2 \left\| I^{2\alpha} \mathcal{D}^{\alpha}\Big(u(\cdot) - \tilde{u}(\cdot)\Big)(x) \right\|_{C_{1-\alpha}[0,\delta]} + L_2 \left\| I^{\alpha} \mathcal{D}^{\alpha}\Big(u(\cdot) - \tilde{u}(\cdot)\Big)(x) \right\|_{C_{1-\alpha}[0,\delta]} \end{aligned}$$

$$
\begin{aligned}
&\leq L_1\delta^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}\left\|u-\tilde{u}\right\|_{C_{1-\alpha}[0,\delta]}+L_1\delta^{\alpha}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left\|u-\tilde{u}\right\|_{C_{1-\alpha}[0,\delta]}\\
&\quad +L_2\delta^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}\left\|\mathcal{D}^{\alpha}u-\mathcal{D}^{\alpha}\tilde{u}\right\|_{C_{1-\alpha}[0,\delta]}+L_2\delta^{\alpha}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left\|\mathcal{D}^{\alpha}u-\mathcal{D}^{\alpha}\tilde{u}\right\|_{C_{1-\alpha}[0,\delta]}\\
&\leq \max\{L_1,L_2\}\left(\delta^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}+\delta^{\alpha}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\right)\left\|u-\tilde{u}\right\|_{C^{\alpha}_{1-\alpha}[0,\delta]}\\
&= L\left\|u-\tilde{u}\right\|_{C^{\alpha}_{1-\alpha}[0,\delta]}
\end{aligned}
$$

Take a monotone non-decreasing sequence $\{u_n\}\subseteq C^{\alpha}_{1-\alpha}[0,\delta]$ converging to u. Therefore, the sequences $x^{1-\alpha}u_n$ and $x^{1-\alpha}\mathcal{D}^{\alpha}u_n$ converge uniformly to functions $x^{1-\alpha}u$ and $x^{1-\alpha}\mathcal{D}^{\alpha}u$ on $[0,\delta]$, respectively. Then, for every $x\in[0,\delta]$, we get

$$x^{1-\alpha}u_1(x)\leq x^{1-\alpha}u_2(x)\leq\cdots\leq x^{1-\alpha}u_n(x)\leq\cdots,$$

and

$$x^{1-\alpha}\mathcal{D}^{\alpha}u_1(x)\leq x^{1-\alpha}\mathcal{D}^{\alpha}u_2(x)\leq\cdots\leq x^{1-\alpha}\mathcal{D}^{\alpha}u_n(x)\leq\cdots,$$

and the convergence of these sequences of real numbers to $x^{1-\alpha}u(x)$ and $x^{1-\alpha}\mathcal{D}^{\alpha}u(x)$, respectively, implies

$$x^{1-\alpha}u_n(x)\leq x^{1-\alpha}u(x),\quad \textit{for all } x\in[0,\delta],\ n\in\mathbb{N},$$

and

$$x^{1-\alpha}\mathcal{D}^{\alpha}u_n(x)\leq x^{1-\alpha}\mathcal{D}^{\alpha}u(x),\quad \textit{for all } x\in[0,\delta],\ n\in\mathbb{N},$$

therefore, the limit is an upper bound for all the terms u_n in the sequence, i.e. $u_n\preceq u$ for $n\in\mathbb{N}$.

Let $\underline{u}$ be a lower solution for (1)-(2) in $C^{\alpha}_{1-\alpha}[0,\delta]$. Indeed, we have $\mathcal{D}^{2\alpha}\underline{u}(x)\leq f(\cdot,\underline{u}(\cdot),\mathcal{D}^{\alpha}\underline{u}(\cdot))(x)$ for every $x\in(0,T]$ and

$$\lim_{x\to0}x^{1-\alpha}\underline{u}(x)\leq u_0,\quad \lim_{x\to0}x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x)\leq u_1.$$

Therefore, by Lemma 3.5 and the monotonicity of Riemann-Liouville fractional integral operator, we deduce

$$
\begin{aligned}
x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x) &= x^{1-\alpha}\Big(\Big[\lim_{x\to0}x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x)\Big]x^{\alpha-1}+I^{\alpha}\mathcal{D}^{2\alpha}\underline{u}(x)\Big)\\
&\leq u_1+x^{1-\alpha}I^{\alpha}\mathcal{D}^{2\alpha}\underline{u}(x)\\
&\leq u_1+x^{1-\alpha}I^{2\alpha}f(\cdot,\tilde{u}(\cdot),\mathcal{D}^{\alpha}\tilde{u}(\cdot))(x)\\
&= x^{1-\alpha}\mathcal{D}^{\alpha}A\underline{u}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
x^{1-\alpha}\underline{u}(x) &= x^{1-\alpha}\Big(\Big[\lim_{x\to0}x^{1-\alpha}\underline{u}(x)\Big]x^{\alpha-1}+I^{\alpha}\mathcal{D}^{\alpha}\underline{u}(x)\Big)\\
&\leq u_0+x^{1-\alpha}I^{\alpha}\mathcal{D}^{\alpha}\underline{u}(x)\\
&= u_0+x^{1-\alpha}\Big(I^{\alpha}\Big(\Big[\lim_{x\to0}x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x)\Big]x^{\alpha-1}+I^{\alpha}\mathcal{D}^{2\alpha}\underline{u}(x)\Big)\Big)\\
&= u_0+\Big[\lim_{x\to0}x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x)\Big]\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}x^{\alpha}+x^{1-\alpha}I^{2\alpha}\mathcal{D}^{2\alpha}\underline{u}(x)\\
&= u_0+u_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}x^{\alpha}+x^{1-\alpha}I^{2\alpha}\mathcal{D}^{2\alpha}\underline{u}(x)\\
&= x^{1-\alpha}A\underline{u}(x).
\end{aligned}
$$

Therefore, $\underline{u}\preceq A\underline{u}$. Thus an application of the Theorem 2.10, together with Lemma 3.8, yields the existence and uniqueness of the solution of $u\in C^{\alpha}_{1-\alpha}[0,\delta]$ of the problem (1)-(2). Moreover, the unique solution of

(1)-(2) can be obtained as $\lim_{n\to\infty} A^n u$ for every $u \in C^{\alpha}_{1-\alpha}[0,\delta]$. In particular, the unique solution $u \in C^{\alpha}_{1-\alpha}[0,\delta]$ of (1)-(2) can be obtained as $\lim_{n\to\infty} u_n(x)$ where

$$u_n(x) = u_0 x^{\alpha-1} + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + I^{2\alpha} f(\cdot, u_{n-1}(\cdot), \mathcal{D}^{\alpha} u_{n-1}(\cdot))(x),$$

where $u_0(x) = \underline{u}(x)$. □

Example 4.3. *Let us consider the following linear initial value problem*

$$\begin{cases} \mathcal{D}^{2\alpha} u(x) - \kappa_1 x^{\alpha} \mathcal{D}^{\alpha} u(x) - \kappa_2 u(x) = x^{-\gamma}, \quad x \in (0,T], \\ \lim_{x\to 0} x^{1-\alpha} u(x) = a, \qquad \lim_{x\to 0} x^{1-\alpha} \mathcal{D}^{\alpha} u(x) = b, \end{cases} \tag{6}$$

where $0 < \alpha \le 1$, $T \ge 1$, $a, b, \kappa_1, \kappa_2 \ge 0$, $\max\{\kappa_1, \kappa_2\} < \left(\frac{\Gamma(\alpha)}{\Gamma(3\alpha)} + \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\right)^{-1}$ *and* $\gamma < 1$*. In this case, it is easy to see that the conditions* (H1)-(H2) *hold. On the other hand,* $u_0(x) = a x^{\alpha-1} + b\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1}$ *is a lower solution of* (6)*. Therefore, using Theorem 4.2, the initial value problem* (6) *has a unique solution in* $C^{\alpha}_{1-\alpha}[0,1]$*. Furthermore, to solve* (6)*, we can apply the method of successive approximations by setting*

$$u_n(x) = u_0(x) + \kappa_1 I^{2\alpha}\Big(x^{\alpha} \mathcal{D}^{\alpha} u_{n-1}(x)\Big) + \kappa_2\Big(I^{2\alpha} u_{n-1}(x)\Big) + I^{2\alpha} x^{-\gamma}, \quad (n \in \mathbb{N}).$$

We can now form the first few successive approximations as follows

$$\begin{aligned} u_1(x) \;=\;& a x^{\alpha-1} + b \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} \\ &+ b\kappa_1 \frac{\Gamma(2\alpha)}{\Gamma(4\alpha)} x^{4\alpha-1} + a\kappa_2 \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} x^{3\alpha-1} + b\kappa_2 \frac{\Gamma(\alpha)}{\Gamma(4\alpha)} x^{4\alpha-1} + \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+2\alpha)} x^{2\alpha-\gamma}. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} u_2(x) \;=\;& a x^{\alpha-1} + b \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} \\ &+ \kappa_1 \Bigg(b \frac{\Gamma(2\alpha)}{\Gamma(4\alpha)} x^{4\alpha-1} + b\kappa_1 \frac{\Gamma(2\alpha)}{\Gamma(3\alpha)} \frac{\Gamma(4\alpha)}{\Gamma(6\alpha)} x^{6\alpha-1} + a\kappa_2 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)} x^{5\alpha-1} \\ &\qquad + b\kappa_2 \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \frac{\Gamma(4\alpha)}{\Gamma(6\alpha)} x^{6\alpha-1} + \frac{\Gamma(1-\gamma+\alpha)}{\Gamma(1-\gamma+2\alpha)} \frac{\Gamma(2\alpha)}{\Gamma(4\alpha)} x^{4\alpha-\gamma} \Bigg) \\ &+ \kappa_2 \Bigg(a \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} x^{3\alpha-1} + b \frac{\Gamma(\alpha)}{\Gamma(4\alpha)} x^{4\alpha-1} + b\kappa_1 \frac{\Gamma(2\alpha)}{\Gamma(6\alpha)} x^{6\alpha-1} + a\kappa_2 \frac{\Gamma(\alpha)}{\Gamma(5\alpha)} x^{5\alpha-1} \\ &\qquad + b\kappa_2 \frac{\Gamma(\alpha)}{\Gamma(6\alpha)} x^{6\alpha-1} + \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+4\alpha)} x^{4\alpha-\gamma} \Bigg) + \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+2\alpha)} x^{2\alpha-\gamma}. \end{aligned} \tag{8}$$

It is interesting to point out that u_n, $n = 1, 2$ *of* (7) *and* (8) *serve as approximate solutions of increasing accuracy as* $n \to \infty$.

Remark 4.4. *In a similar way, we can deal with the following initial value problem of the nonlinear sequential fractional differential equation, more general than those in* (1)-(2)*, are defined by*

$$\mathcal{D}^{n\alpha} u(x) = f(x, u(x), \mathcal{D}^{\alpha} u(x), \mathcal{D}^{2\alpha} u(x), \cdots, \mathcal{D}^{(n-1)\alpha} u(x)), \quad x \in (0,T], \tag{9}$$

$$\lim_{x\to 0} x^{1-\alpha} \mathcal{D}^{k\alpha} u(x) = u_k, \quad (k = 0, \cdots, n-1), \tag{10}$$

where $0 < \alpha \le 1$.

We could carry out a similar argument to prove the existence and uniqueness results for problem (9)- (10). We do not try to give here an account of the extremely wide details on this topic, we only confine ourself to introduce the necessary spaces of appropriate order on them, and omit the full details of the processes.

Definition 4.5. *For $0 < \alpha \leq 1$, we define the following space*

$$C^{n\alpha}_{1-\alpha}[0,T] = \{u \in C_{1-\alpha}[0,T] : \mathcal{D}^{k\alpha} u \in C_{1-\alpha}[0,T],\ k = 1,2,\cdots,n-1\},$$

equipped with the norm

$$\|u\|_{C^{n\alpha}_{1-\alpha}[0,T]} = \sum_{k=0}^{n-1} \|\mathcal{D}^{k\alpha} u\|_{C_{1-\alpha}[0,T]}.$$

Definition 4.6. *We define the following order relation for $C^{n\alpha}_{1-\alpha}[0,T]$,*

$$u \leq v \quad \Longleftrightarrow \qquad x^{1-\alpha}\mathcal{D}^{k\alpha}u(x) \leq x^{1-\alpha}\mathcal{D}^{k\alpha}v(x), \quad x \in [0,T],\ k = 0,1,\cdots,n-1.$$

Theorem 4.7. *Let $0 < \alpha \leq 1$. Then the space $C^{n\alpha}_{1-\alpha}[0,T]$ is a Banach space and $(C^{n\alpha}_{1-\alpha}[0,T], \leq)$ is a partially ordered set and every pair of elements has a lower bound and an upper bound.*

Definition 4.8. *A function $\underline{u} \in C^{n\alpha}_{1-\alpha}[0,T]$ is called a lower solution of the initial value problem* (9)-(10), *if $\mathcal{D}^{n\alpha}\underline{u}(x) \leq f(\cdot, \underline{u}(\cdot), \mathcal{D}^{\alpha}\underline{u}(\cdot), \mathcal{D}^{2\alpha}\underline{u}(\cdot), \cdots, \mathcal{D}^{(n-1)\alpha}\underline{u}(\cdot))(x)$ for every $x \in (0,T]$ and*

$$\lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}\underline{u}(x) \leq u_k, \quad k = 0,1,\cdots,n-1.$$

To prove the main results, we need the following assumptions:

(H3) $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for every $u \in C^{n\alpha}_{1-\alpha}[0,T]$,

$$f(\cdot, u(\cdot), \mathcal{D}^{\alpha}\ u(\cdot), \mathcal{D}^{2\alpha}u(\cdot), \cdots, \mathcal{D}^{(n-1)\alpha}u(\cdot))(x) \in C_{\gamma}[0,T],$$

for some $0 \leq \gamma < 1$.

(H4) f is non-decreasing in all its arguments except for the first argument and there exist $L > 0$ such that

$$f(x, u_1, \cdots, u_n) - f(x, \tilde{u}_1, \cdots, \tilde{u}_n) \leq L \sum_{i=1}^{n} (u_i - \tilde{u}_i) \quad u_i \geq \tilde{u}_i,\ i = 1,2,\cdots,n.$$

Theorem 4.9. *Assume that* (H3)-(H4) *hold. Then there exists $0 < \delta \leq T$ such that the existence of a lower solution for* (9)-(10) *in $C^{n\alpha}_{1-\alpha}[0,\delta]$ provides the existence of a unique solution $u \in C^{n\alpha}_{1-\alpha}[0,\delta]$ for* (9)-(10).

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