Filomat 32:13 (2018), 4591–4609
https://doi.org/10.2298/FIL1813591Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Commuting Solutions to the Yang-Baxter-like Matrix Equation for Identity Matrix Minus Special Rank-two Matrices☆

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Abstract. Let $A = I - PQ^{\mathrm{T}}$, where P and Q are two $n \times 2$ complex matrices of full column rank such that $\det(Q^{\mathrm{T}}P) = 0$. We find all the commuting solutions of the quadratic matrix equation $AXA = XAX$.

1. Introduction

We consider the solutions of the following quadratic matrix equation

$$AXA = XAX, \tag{1}$$

where both the given A and the unknown X are $n \times n$ complex matrices. The above equation has been called the Yang-Baxter-like matrix equation[4, 6–8]. The equation (1) has its origin in the classical Yang-Baxter equation obtained from Yang [19], which is used to study the many-body problem in 1967, and then by Baxter [2] independently for a lattice model in 1972, which is related to the quantum Yang-Baxter equation. In the past decades, the Yang-Baxter equation has been extensively investigated by mathematicians in knot theory, braid group theory and quantum group theory as well as physicists (see, e.g., [1, 10–12, 18, 25–27, 29] and the references therein). The quadratic matrix equation (1) has been studied using linear algebra techniques in the past few years; see, e.g., [3, 4, 6–8, 28] for more details.

The Yang-Baxter-like matrix equation has two trivial solutions $X = 0$ and $X = A$, but finding nontrivial solutions of (1) is not easy for an arbitrary matrix A. Since solving this equation is equivalent to solving a polynomial system of n^2 quadratic equations with n^2 variables. We limit the task to only finding the solutions that commute with A. Some solutions can be obtained in [6] when the matrix A is a special class of Jordan forms, and a more general result was proved in [9] when the matrix A is a class of diagonalizable matrices, but the general solution has still never been obtained for arbitrary matrices A. In a recent paper [20], the author have found all the solutions of (1), where the given $n \times n$ complex matrix $A = PQ^{\mathrm{T}}$, with two $n \times 2$ matrices P and Q, with the assumption that $Q^{\mathrm{T}}P$ is singular. However, in this paper we intend to

2010 *Mathematics Subject Classification*. 65F10, 65F50
Keywords. Rank-two matrix; Commuting solutions; Jordan form.
Received: 03 January 2018; Accepted: 05 August 2018
Communicated by Dijana Mosić
This work is supported by NNSF of China No. 11461046, NSF of Jiangxi Province with Nos. 20181ACB20001 and 20161ACB21005, and the Fundamental Research Funds for the Central Universities in UIBE (CXTD9-07).
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find all the commuting solutions of (1), where $A = I - PQ^T$ and Q^TP is singular. We would like to find all the solutions X of (1) satisfying $AX = XA$, and such solutions are called the *commuting solutions*.

In the paper [6], it was proved that solving the Yang-Baxter-like matrix equation for any given matrix A is equivalent to solving the same equation, where the matrix A in the equation is replaced by a matrix similar to A, and all the solutions of the two equations are similar with the same similarity matrix. Because any matrix is similar to its Jordan form matrix, solving equation (1) for the given A can be reduced to solving the same equation with the Jordan form of A. We denote the Jordan form of the matrix as J. We shall solve the following simpler Yang-Baxter-like matrix equation

$$JYJ = YJY. \tag{2}$$

From [20] we can get it in a similar way

$$J = diag(I, \Lambda), \tag{3}$$

such that Λ is one of the following three matrices

$$\Lambda_1 \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \Lambda_2 \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}, \lambda \neq 0; \Lambda_3 \equiv \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

and the diagonal block I in (3) is either $(n-3) \times (n-3)$ or $(n-4) \times (n-4)$ accordingly. For each of the three cases of J, there is a corresponding nonsingular matrix $W = [w_1, \cdots, w_n]$, which makes

$$A = I - PQ^T = WJW^{-1}.$$

This paper is organized as follows. Our main result will be presented in the next three sections. We present three examples of our solution result in Section 5.

2. Solutions of equation in the case of $\Lambda = \Lambda_1$

In this and subsequent sections we assume that the known matrix A in equation (1) is $I - PQ^T$, where P and Q are two $n \times 2$ complex matrices of full column rank such that $\det(Q^TP) = 0$, and $P = [p_1, p_2]$, $Q = [q_1, q_2]$. Let J be the Jordan form of A given by (3), where the diagonal block I is either $(n-3) \times (n-3)$ and $\Lambda = \Lambda_1$ or Λ_2 defined by (4), or the diagonal block I is $(n-4) \times (n-4)$ and $\Lambda = \Lambda_3$ in (4). In the current section, we will research all the commuting solutions of (1) when $J = diag(I, \Lambda_1)$, and the other two cases that $J = diag(I, \Lambda_2)$ and $J = diag(I, \Lambda_3)$ will be investigated in section 3 and 4, respectively.

As indicated in section 1, it is well known that solving the quadratic matrix equation (1) is equivalent to solving the equation (2) with J the Jordan form of A, so we just focus on solving (2). To solve the corresponding simplified Yang-Baxter-like matrix equation

$$JYJ = YJY.$$

Let Y be partitioned into the 2×2 block matrix in the same way as J

$$Y = \begin{pmatrix} M & Z \\ W^T & T \end{pmatrix}, \tag{5}$$

where M is $(n-3) \times (n-3)$, $Z = [z_1, z_2, z_3]$ and $W = [w_1, w_2, w_3]$ are $(n-3) \times 3$, and T is 3×3. Let

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

Then (5) becomes

$$Y=\begin{pmatrix} M & z_1 & z_2 & z_3 \\ w_1^{\mathrm{T}} & t_{11} & t_{12} & t_{13} \\ w_2^{\mathrm{T}} & t_{21} & t_{22} & t_{23} \\ w_3^{\mathrm{T}} & t_{31} & t_{32} & t_{33} \end{pmatrix}, \tag{6}$$

and J can be written as

$$J=\begin{pmatrix} I_{n-3} & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

Theorem 2.1 Suppose $A = I - PQ^{\mathrm{T}}$ is such that $\Lambda = \Lambda_1$ in its Jordan form (3) with $det(Q^{\mathrm{T}}P) = 0$. Then all the commuting solutions of (1) are $X = WYW^{-1}$ with Y partition as (6), such that M is an arbitrary $(n-3)\times(n-3)$ projection matrix.

(i)When $t_{11} = 0$, then $t_{12} = 0$, the vectors z_3 and w_1^{T} belong to the range space of M, and $t_{13} = w_1^{\mathrm{T}} z_3$.

(ii)When $t_{11} = 1$, then $t_{12} = 1$, the vectors z_3 and w_1^{T} belong to the null space of M, and $t_{13} = -w_1^{\mathrm{T}} z_3$.

Proof: All the commuting solutions of (2) must satisfy

$$JY = YJ.$$

Then the above matrix equation equivalent to the system

$$\begin{cases} z_2 = z_1 + z_2, \\ z_3 = z_2 + z_3, \\ w_1^{\mathrm{T}} + w_2^{\mathrm{T}} = w_1^{\mathrm{T}}, \\ w_2^{\mathrm{T}} + w_3^{\mathrm{T}} = w_2^{\mathrm{T}}, \\ t_{11} + t_{21} = t_{11}, \\ t_{12} + t_{22} = t_{11} + t_{12}, \\ t_{23} + t_{13} = t_{12} + t_{13}, \\ t_{21} + t_{31} = t_{21}, \\ t_{22} + t_{32} = t_{21} + t_{22}, \\ t_{23} + t_{33} = t_{22} + t_{23}, \\ t_{32} = t_{31} + t_{32}, \\ t_{33} = t_{32} + t_{33}. \end{cases}$$

From the first four equations above, we can solve $z_1 = 0$, $z_2 = 0$, $w_2^{\mathrm{T}} = 0$, $w_3^{\mathrm{T}} = 0$, respectively. We can obtain $t_{21} = t_{32} = 0$ from the fifth, the twelfth and the ninth equations. We can solve $t_{31} = 0$ from $t_{21} + t_{31} = t_{21}$. We can get $t_{11} = t_{22} = t_{33}$ from the two equations of the sixth and the tenth. Then $t_{12} = t_{23}$ from $t_{23} + t_{13} = t_{12} + t_{13}$. Substituting them into Y, then

$$Y=\begin{pmatrix} M & 0 & 0 & z_3 \\ w_1^{\mathrm{T}} & t_{11} & t_{12} & t_{13} \\ 0 & 0 & t_{11} & t_{12} \\ 0 & 0 & 0 & t_{11} \end{pmatrix}.$$

Thus, the matrix equation (2) is equivalent to

$$\begin{cases} M^2 = M, \\ Mz_3 = (1 - t_{11})z_3, \\ w_1^{\mathrm{T}} M = (1 - t_{11})w_1^{\mathrm{T}}, \\ t_{11}^2 = t_{11}, \\ t_{11}^2 + 2t_{11}t_{12} = 2t_{11} + t_{12}, \\ w_1^{\mathrm{T}} z_3 = t_{11} + 2t_{12} + t_{13} - 2t_{11}t_{13} - 2t_{11}t_{12} - t_{12}^2, \\ t_{11}^2 + 2t_{11}t_{12} = t_{12} + 2t_{11}. \end{cases} \tag{7}$$

From the above fourth equation we can know either $t_{11} = 0$ or $t_{11} = 1$. Therefore we have the following result.

(i) When $t_{11} = 0$, from the fifth equation above we can solve $t_{12} = 0$, then we obtain the following system from (7)

$$\begin{cases} M^2 = M, \\ Mz_3 = z_3, \\ w_1^{\mathrm{T}} M = w_1^{\mathrm{T}}, \\ w_1^{\mathrm{T}} z_3 = t_{13}, \end{cases}$$

which is the first statement of Theorem 2.1.

(ii) When $t_{11} = 1$, then $t_{12} = 1$, and (7) is reduced to

$$\begin{cases} M^2 = M, \\ Mz_3 = 0, \\ w_1^{\mathrm{T}} M = 0, \\ w_1^{\mathrm{T}} z_3 = -t_{13}, \end{cases}$$

where w_1 and z_3 are two $(n-3)-$dimensional complex vectors and the second case of Theorem 2.1 is proved.

3. Solutions of equation in the case of $\Lambda = \Lambda_2$

We now consider the second case that the Jordan form of the matrix A is $J = diag(I, \Lambda_2)$. That is to say, J can be written as

$$J = \begin{pmatrix} I_{n-3} & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1-\lambda \end{pmatrix},$$

and Y is partitioned as (6). We solve the equation (2).

Theorem 3.1 Suppose $A = I - PQ^{\mathrm{T}}$ is such that $\Lambda = \Lambda_2$ in its Jordan form (3) with $det(Q^{\mathrm{T}}P) = 0$. Then all the commuting solutions of (1) are $X = WYW^{-1}$ with Y partition as (6), such that M is an arbitrary $(n-3) \times (n-3)$ projection matrix, t_{33} is either 0 or $1 - \lambda$.

(i) When $t_{11} = 0$, then the vectors z_2 and w_1^{T} belong to the range space of M, and $t_{12} = w_1^{\mathrm{T}} z_2$.

(ii) When $t_{11} = 1$, then the vectors z_2 and w_1^{T} belong to the null space of M, and $t_{12} = 1 - w_1^{\mathrm{T}} z_2$.

Proof: All the commuting solutions of (2) must satisfy

$$JY = YJ.$$

Then the above matrix equation is equivalent to the following system

$$\begin{cases} z_1 + z_2 = z_2, \\ (1-\lambda)z_3 = z_3, \\ w_1^{\mathrm{T}} + w_2^{\mathrm{T}} = w_1^{\mathrm{T}}, \\ (1-\lambda)w_3^{\mathrm{T}} = w_3^{\mathrm{T}}, \\ t_{11} + t_{21} = t_{11}, \\ t_{11} + t_{12} = t_{12} + t_{22}, \\ (1-\lambda)t_{13} = t_{13} + t_{23}, \\ t_{21} + t_{22} = t_{22}, \\ (1-\lambda)t_{23} = t_{23}, \\ (1-\lambda)t_{31} = t_{31}, \\ (1-\lambda)t_{32} = t_{31} + t_{32}. \end{cases}$$

From the first equation above, $z_1 = 0$. Since $\lambda \neq 0$, from the second equation, $z_3 = 0$. Similarly, from the two equations of the third and the fourth, we can obtain $w_2^T = 0$, $w_3^T = 0$, respectively. The two equations $(1-\lambda)t_{13} = t_{13} + t_{23}$ and $(1-\lambda)t_{23} = t_{23}$ imply $t_{23} = t_{13} = 0$. The two equations $(1-\lambda)t_{31} = t_{31}$ and $(1-\lambda)t_{32} = t_{31} + t_{32}$ imply $t_{31} = t_{32} = 0$. So $t_{21} = 0$ from $t_{11} + t_{21} = t_{11}$ and $t_{11} = t_{22}$ since $t_{11} + t_{12} = t_{12} + t_{22}$. Substituting them into Y, then

$$Y = \begin{pmatrix} M & 0 & z_2 & 0 \\ w_1^T & t_{11} & t_{12} & 0 \\ 0 & 0 & t_{11} & 0 \\ 0 & 0 & 0 & t_{33} \end{pmatrix}.$$

Thus, the matrix equation $JYJ = YJY$ becomes

$$\begin{cases} M^2 = M, \\ Mz_2 = (1-t_{11})z_2, \\ w_1^T M = (1-t_{11})w_1^T, \\ t_{11}^2 = t_{11}, \\ w_1^T z_2 = t_{12} + 2t_{11} - 2t_{11}t_{12} - t_{11}^2, \\ (1-\lambda)t_{33}^2 = (1-\lambda)^2 t_{33}. \end{cases} \tag{8}$$

The fourth equation of (8) indicates that either t_{11} =0 or t_{11} =1. The equation of $(1-\lambda)t_{33}^2 = (1-\lambda)^2 t_{33}$ implies either t_{33} =0 or $t_{33} = 1-\lambda$.

(i) When $t_{11} = 0$, then (8) is simplified to

$$\begin{cases} M^2 = M, \\ Mz_2 = z_2, \\ w_1^T M = w_1^T, \\ w_1^T z_2 = t_{12}, \end{cases}$$

which proves the first statement of Theorem 3.1.

(ii) When $t_{11} = 1$, and the system (8) is reduced to

$$\begin{cases} M^2 = M, \\ Mz_2 = 0, \\ w_1^T M = 0, \\ w_1^T z_2 = 1 - t_{12}, \end{cases}$$

where either t_{33} =0 or $t_{33} = 1-\lambda$, w_1 and z_2 are two $(n-3)$−dimensional complex vectors and the second statement of Theorem 3.1 is proved.

4. Solutions of equation in the case of $\Lambda = \Lambda_3$

Unlike the previous two cases, both Λ_1 and Λ_2 are 3×3 matrices. In this section we will study the third case Λ_3 is a 4×4 matrix. To solve the equation (2), We define the rank of matrix M as $r(M)$.

The splitting of Y is the same as (5), but M is $(n-4)\times(n-4)$, $Z = [z_1, z_2, z_3, z_4]$ and $W = [w_1, w_2, w_3, w_4]$ are $(n-4)\times 4$, T is 4×4.
Let

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}.$$

The (5) can be written as

$$Y = \begin{pmatrix} M & z_1 & z_2 & z_3 & z_4 \\ w_1^{\mathrm{T}} & t_{11} & t_{12} & t_{13} & t_{14} \\ w_2^{\mathrm{T}} & t_{21} & t_{22} & t_{23} & t_{24} \\ w_3^{\mathrm{T}} & t_{31} & t_{32} & t_{33} & t_{34} \\ w_4^{\mathrm{T}} & t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}, \tag{9}$$

and similarly, J can be written as

$$J = \begin{pmatrix} I_{n-4} & & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

All the commuting solutions of (2) must satisfy

$$JY = YJ.$$

Then the above matrix equation becomes

$$\begin{pmatrix} M & z_1 & z_1+z_2 & z_3 & z_3+z_4 \\ w_1^{\mathrm{T}} & t_{11} & t_{11}+t_{12} & t_{13} & t_{13}+t_{14} \\ w_2^{\mathrm{T}} & t_{21} & t_{21}+t_{22} & t_{23} & t_{23}+t_{24} \\ w_3^{\mathrm{T}} & t_{31} & t_{31}+t_{32} & t_{33} & t_{33}+t_{34} \\ w_4^{\mathrm{T}} & t_{41} & t_{41}+t_{42} & t_{43} & t_{43}+t_{44} \end{pmatrix} = \begin{pmatrix} M & z_1 & z_2 & z_3 & z_4 \\ w_1^{\mathrm{T}}+w_2^{\mathrm{T}} & t_{11}+t_{21} & t_{12}+t_{22} & t_{13}+t_{23} & t_{14}+t_{24} \\ w_2^{\mathrm{T}} & t_{21} & t_{22} & t_{23} & t_{24} \\ w_3^{\mathrm{T}}+w_4^{\mathrm{T}} & t_{31}+t_{41} & t_{32}+t_{42} & t_{33}+t_{43} & t_{34}+t_{44} \\ w_4^{\mathrm{T}} & t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}.$$

From the two equations of $z_1 + z_2 = z_2$ and $z_3 + z_4 = z_4$, we can obtain $z_1 = 0$, $z_3 = 0$, respectively. We can solve $w_2^{\mathrm{T}} = 0$ from $w_1^{\mathrm{T}} = w_1^{\mathrm{T}} + w_2^{\mathrm{T}}$, so $w_4^{\mathrm{T}} = 0$ from $w_3^{\mathrm{T}} = w_3^{\mathrm{T}} + w_4^{\mathrm{T}}$ and $t_{21} = t_{23} = t_{41} = t_{43} = 0$ from equating entries (2,2), (2,4), (4,2), (4,4) of both sides in above system. Then $t_{11} = t_{22}$ from $t_{11} + t_{12} = t_{12} + t_{22}$. $t_{13} = t_{24}$ from $t_{13} + t_{14} = t_{14} + t_{24}$, so $t_{31} = t_{42}$ from $t_{31} + t_{32} = t_{32} + t_{42}$ and $t_{33} = t_{44}$ since $t_{33} + t_{34} = t_{34} + t_{44}$.
Then (9) becomes

$$Y = \begin{pmatrix} M & 0 & z_2 & 0 & z_4 \\ w_1^{\mathrm{T}} & t_{11} & t_{12} & t_{13} & t_{14} \\ 0 & 0 & t_{11} & 0 & t_{13} \\ w_3^{\mathrm{T}} & t_{31} & t_{32} & t_{33} & t_{34} \\ 0 & 0 & t_{31} & 0 & t_{33} \end{pmatrix}.$$

Thus, the equation (2) is equivalent to the system

$$\begin{cases} M^2 = M, \\ Mz_2 + t_{11}z_2 + t_{31}z_4 = z_2, \\ Mz_4 + t_{13}z_2 + t_{33}z_4 = z_4, \\ w_1^{\mathrm{T}}M + t_{11}w_1^{\mathrm{T}} + t_{13}w_3^{\mathrm{T}} = w_1^{\mathrm{T}}, \\ w_3^{\mathrm{T}}M + t_{31}w_1^{\mathrm{T}} + t_{33}w_3^{\mathrm{T}} = w_3^{\mathrm{T}}, \\ t_{11}^2 + t_{13}t_{31} = t_{11}, \\ t_{11}t_{13} + t_{13}t_{33} = t_{13}, \\ w_1^{\mathrm{T}}z_4 + t_{11}t_{14} + t_{11}t_{13} + t_{12}t_{13} + t_{13}t_{34} + t_{13}t_{33} + t_{14}t_{33} = 2t_{13} + t_{14}, \\ t_{11}t_{31} + t_{31}t_{33} = t_{31}, \\ w_3^{\mathrm{T}}z_2 + t_{31}t_{12} + t_{11}t_{31} + t_{11}t_{32} + t_{32}t_{33} + t_{31}t_{33} + t_{31}t_{34} = 2t_{31} + t_{32}, \\ t_{33}^2 + t_{13}t_{31} = t_{33}, \\ w_1^{\mathrm{T}}z_2 + t_{11}^2 + 2t_{11}t_{12} + t_{13}t_{32} + t_{13}t_{31} + t_{14}t_{31} = 2t_{11} + t_{12}, \\ w_3^{\mathrm{T}}z_4 + t_{31}t_{14} + t_{13}t_{31} + t_{13}t_{32} + 2t_{34}t_{33} + t_{33}^2 = 2t_{33} + t_{34}. \end{cases} \tag{10}$$

By observing the above equations, the equation $t_{11}t_{13} + t_{13}t_{33} = t_{13}$ implies either $t_{13} = 0$ or $t_{11} + t_{33} = 1$ and the equation $t_{11}t_{31} + t_{31}t_{33} = t_{31}$ implies either $t_{31} = 0$ or $t_{11} + t_{33} = 1$. We consider the first case $t_{11} + t_{33} = 1$, which leads to the following proposition.

Proposition 4.1. *Suppose $A = I - PQ^{\mathrm{T}}$ is such that $\Lambda = \Lambda_3$ in its Jordan form (3) with $det(Q^{\mathrm{T}}P) = 0$. Then all the commuting solutions of (1) are $X = WYW^{-1}$ with Y partition as (9), such that M is an arbitrary $(n-4)\times(n-4)$ projection matrix, the vectors S, S^{T} belong to the null space of $I - M$ and the vectors T, T^{T} belong to the null space of M, when $t_{11} + t_{33} = 1$,*

(i) $t_{13} = 0$, $t_{31} \neq 0$, then $t_{11} = 0$ or 1, $t_{33} = 1 - t_{11}$ and t_{31} is an arbitrary non-zero complex number, t_{12} and t_{32} are two arbitrary complex numbers.

If $t_{11} = 0$, then $t_{33} = 1$, z_4 belongs to the null space of M and w_1^{T} belongs to the range space of M, the vectors of z_4 and w_1^{T} satisfy $w_1^{\mathrm{T}}z_4 = 0$, $z_2 = t_{31}z_4 + S$, $w_3^{\mathrm{T}} = -t_{31}w_1^{\mathrm{T}} + T^{\mathrm{T}}$, $t_{14} = \frac{t_{12}-w_1^{\mathrm{T}}z_2}{t_{31}}$ and $t_{34} = 1 - t_{31}t_{14} - w_3^{\mathrm{T}}z_4$.

If $t_{11} = 1$, then $t_{33} = 0$, z_4 belongs to the range space of M and w_1^{T} belongs to the null space of M, the vectors of z_4 and w_1^{T} satisfy $w_1^{\mathrm{T}}z_4 = 0$, $z_2 = -t_{31}z_4 + T$, $w_3^{\mathrm{T}} = t_{31}w_1^{\mathrm{T}} + S^{\mathrm{T}}$, $t_{14} = \frac{1-t_{12}-w_1^{\mathrm{T}}z_2}{t_{31}}$ and $t_{34} = t_{31}t_{14} + w_3^{\mathrm{T}}z_4$.

(ii) $t_{13} = t_{31} = 0$, then $t_{11} = 0$ or 1, $t_{33} = 1 - t_{11}$, t_{14} and t_{32} are two arbitrary complex numbers.

If $t_{11} = 0$, then $t_{33} = 1$, z_2, w_1^{T} belong to the range space of M and z_4, w_3^{T} belong to the null space of M, the vectors of z_4, w_1^{T}, z_2 and w_3^{T}, which satisfy the two equations of $w_1^{\mathrm{T}}z_4 = 0$ and $w_3^{\mathrm{T}}z_2 = 0$, $t_{12} = w_1^{\mathrm{T}}z_2$ and $t_{34} = 1 - w_3^{\mathrm{T}}z_4$.

If $t_{11} = 1$, then $t_{33} = 0$, z_2, w_1^{T} belong to the null space of M and z_4, w_3^{T} belong to the range space of M, the vectors of z_4, w_1^{T}, z_2 and w_3^{T}, which satisfy the two equations of $w_1^{\mathrm{T}}z_4 = 0$ and $w_3^{\mathrm{T}}z_2 = 0$, $t_{12} = 1 - w_1^{\mathrm{T}}z_2$ and $t_{34} = w_3^{\mathrm{T}}z_4$.

(iii)$t_{13} \neq 0$, $t_{31} = 0$, then $t_{11} = 0$ or 1, $t_{33} = 1 - t_{11}$ and t_{13} is an arbitrary non-zero complex number, t_{14} and t_{32} are two arbitrary complex numbers.

If $t_{11} = 0$, then $t_{33} = 1$, z_2 belongs to the range space of M and w_3^{T} belongs to the null space of M, the vectors of z_2 and w_3^{T} satisfy $w_3^{\mathrm{T}}z_2 = 0$, $z_4 = -t_{13}z_2 + T$, $w_1^{\mathrm{T}} = t_{13}w_3^{\mathrm{T}} + S^{\mathrm{T}}$, $t_{12} = t_{13}t_{32} + w_1^{\mathrm{T}}z_2$ and $t_{34} = 1 - t_{13}t_{32} - w_3^{\mathrm{T}}z_4$.

If $t_{11} = 1$, then $t_{33} = 0$, z_2 belongs to the null space of M and w_3^{T} belongs to the range space of M, the vectors of z_2 and w_3^{T} satisfy $w_3^{\mathrm{T}}z_2 = 0$, $z_4 = t_{13}z_2 + S$, $w_1^{\mathrm{T}} = -t_{13}w_3^{\mathrm{T}} + T^{\mathrm{T}}$, $t_{12} = 1 - t_{13}t_{32} - w_1^{\mathrm{T}}z_2$ and $t_{34} = t_{13}t_{32} + w_3^{\mathrm{T}}z_4$.

(iv)$t_{13} \neq 0$, $t_{31} \neq 0$, t_{11} and t_{13} are two arbitrary non-zero complex numbers, t_{12} and t_{14} are two arbitrary complex numbers, w_1 and z_2 are two $(n-4)$–dimensional complex vectors, then $t_{33} = 1 - t_{11}$, $t_{31} = \frac{t_{11}(1-t_{11})}{t_{13}}$, $z_4 = \frac{(t_{33}I-M)z_2}{t_{31}}$, $w_3^{\mathrm{T}} = \frac{w_1^{\mathrm{T}}(t_{33}I-M)}{t_{13}}$, $t_{34} = 1 - t_{12} - \frac{w_1^{\mathrm{T}}z_4}{t_{13}}$ and $t_{32} = \frac{t_{11}+t_{12}-2t_{11}t_{12}-t_{14}t_{31}-w_1^{\mathrm{T}}z_2}{t_{13}}$.

Proof: When $t_{11} + t_{33} = 1$, the system (10) becomes the following form after some simplifications.

$$\begin{cases} t_{11} + t_{33} = 1, \\ t_{11}^2 + t_{13}t_{31} = t_{11}, \\ t_{33}^2 + t_{13}t_{31} = t_{33}, \\ M^2 = M, \\ Mz_2 = (1-t_{11})z_2 - t_{31}z_4, \\ Mz_4 = (1-t_{33})z_4 - t_{13}z_2, \\ w_1^{\mathrm{T}}M = (1-t_{11})w_1^{\mathrm{T}} - t_{13}w_3^{\mathrm{T}}, \\ w_3^{\mathrm{T}}M = (1-t_{33})w_3^{\mathrm{T}} - t_{31}w_1^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_2 + 2t_{11}t_{12} + t_{13}t_{32} + t_{14}t_{31} = t_{11} + t_{12}, \\ w_3^{\mathrm{T}}z_4 + t_{31}t_{14} + t_{13}t_{32} + 2t_{34}t_{33} = t_{33} + t_{34}, \\ w_1^{\mathrm{T}}z_4 = t_{13}(1 - t_{12} - t_{34}), \\ w_3^{\mathrm{T}}z_2 = t_{31}(1 - t_{12} - t_{34}). \end{cases} \tag{11}$$

From the first three equations of (11) indicate that $t_{11}t_{33} = t_{13}t_{31}$.

(i) When $t_{13} = 0$, $t_{31} \neq 0$, we can get either $t_{11} = 0$ or $t_{11} = 1$ from the second equation in above system.

If $t_{11} = 0$, then $t_{33} = 1$. The (11) can be written as

$$\begin{cases} M^2 = M, \\ Mz_4 = 0, \\ w_1^{\mathrm{T}}M = w_1^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_4 = 0, \\ Mz_2 = z_2 - t_{31}z_4, \\ w_3^{\mathrm{T}}M = -t_{31}w_1^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_2 + t_{14}t_{31} = t_{12}, \\ w_3^{\mathrm{T}}z_4 + t_{31}t_{14} = 1 - t_{34}, \\ w_3^{\mathrm{T}}z_2 = t_{31}(1 - t_{12} - t_{34}). \end{cases} \tag{12}$$

From the above system, we can know that for any the vectors of z_4 and w_1^{T}, which satisfy z_4 belongs to the null space of M, w_1^{T} belongs to the range space of M and $w_1^{\mathrm{T}}z_4 = 0$. From the second equation of (12), we can obtain $(I - M)z_4 = z_4$, then $r(I - M) = r(I - M, z_4)$. The fifth equation of the system above is simplified to $(I - M)z_2 = t_{31}z_4$, and such the solution vector z_2 must exist. $z_2 = z_0 + S$, where z_0 is a special solution to the equation $(I - M)z_2 = t_{31}z_4$ and S is a fundamental system of solutions to the equation $(I - M)z_2 = 0$. Obviously z_0 can be taken as $t_{31}z_4$ and S satisfies the equation of $(I - M)S = 0$, thus

$$z_2 = t_{31}z_4 + S.$$

Similarly to the idea above, $r(M) = r(M, w_1^{\mathrm{T}})$ can be obtained from the equation of $w_1^{\mathrm{T}}M = w_1^{\mathrm{T}}$. Then the solution vector w_3^{T} of the equation $w_3^{\mathrm{T}}M = -t_{31}w_1^{\mathrm{T}}$ can be written as

$$w_3^{\mathrm{T}} = -t_{31}w_1^{\mathrm{T}} + T^{\mathrm{T}},$$

where $-t_{31}w_1^{\mathrm{T}}$ is a special solution to the equation $w_3^{\mathrm{T}}M = -t_{31}w_1^{\mathrm{T}}$ and T^{T} is a fundamental system of solutions to the equation $w_3^{\mathrm{T}}M = 0$, T^{T} satisfies the equation of $T^{\mathrm{T}}M = 0$. Substituting z_2 and w_3^{T} into the last equation of (12), we can get

$$-t_{31}^2 w_1^{\mathrm{T}}z_4 - t_{31}w_1^{\mathrm{T}}S + t_{31}T^{\mathrm{T}}z_4 + T^{\mathrm{T}}S = t_{31}(1 - t_{12} - t_{34}).$$

The above equation is simplified to $-t_{31}w_1^{\mathrm{T}}S + t_{31}T^{\mathrm{T}}z_4 = t_{31}(1 - t_{12} - t_{34})$. Since $t_{31} \neq 0$, t_{31} can be an arbitrary non-zero complex number. For any t_{12} we can solve $t_{14} = \frac{t_{12} - w_1^{\mathrm{T}}z_2}{t_{31}}$ from the seventh equation of (12). The equation $w_3^{\mathrm{T}}z_4 + t_{31}t_{14} = 1 - t_{34}$ implies $t_{34} = 1 - t_{31}t_{14} - w_3^{\mathrm{T}}z_4$. Since t_{32} does not appear at all in the case, it is a free variable in all the commuting solutions.

When $t_{13} = 0$, $t_{31} \neq 0$, $t_{11} = 1$, then $t_{33} = 0$. The (11) becomes

$$\begin{cases} M^2 = M, \\ Mz_4 = z_4, \\ w_1^{\mathrm{T}}M = 0, \\ w_1^{\mathrm{T}}z_4 = 0, \\ Mz_2 = -t_{31}z_4, \\ w_3^{\mathrm{T}}M = w_3^{\mathrm{T}} - t_{31}w_1^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_2 + t_{14}t_{31} = 1 - t_{12}, \\ w_3^{\mathrm{T}}z_4 + t_{31}t_{14} = t_{34}, \\ w_3^{\mathrm{T}}z_2 = t_{31}(1 - t_{12} - t_{34}). \end{cases}$$

Through observation, we can easily find that it is similar to the above situation, so I will just give some brief explanations. For any the vectors of z_4 and w_1^{T}, which satisfy z_4 belongs to the range space of M, w_1^{T} belongs to the null space of M and $w_1^{\mathrm{T}}z_4 = 0$.

$$z_2 = -t_{31}z_4 + T,$$

where $-t_{31}z_4$ is a special solution to the equation $Mz_2 = -t_{31}z_4$ and T is a fundamental system of solutions to the equation $Mz_2 = 0$.

$$w_3^{\mathrm{T}} = t_{31}w_1^{\mathrm{T}} + S^{\mathrm{T}},$$

where $t_{31}w_1^{\mathrm{T}}$ is a special solution to the equation $w_3^{\mathrm{T}}(I - M) = t_{31}w_1^{\mathrm{T}}$ and S^{T} is a fundamental system of solutions to the equation $w_3^{\mathrm{T}}(I - M) = 0$. From the last three equations in above, we can solve t_{31}, which is an arbitrary non-zero complex number, $t_{14} = \frac{1-t_{12}-w_1^{\mathrm{T}}z_2}{t_{31}}$ and $t_{34} = w_3^{\mathrm{T}}z_4 + t_{14}t_{31}$, where t_{32} and t_{12} are two arbitrary complex numbers, which is the proof of the first case of Proposition 4.1.

(ii) When $t_{13} = t_{31} = 0$, we can solve $t_{11} = 0$ or $t_{11} = 1$ from the equation of $t_{11}^2 + t_{13}t_{31} = t_{11}$. If $t_{11} = 0$, then $t_{33} = 1$. The system of (11) becomes

$$\begin{cases} M^2 = M, \\ Mz_2 = z_2, \\ Mz_4 = 0, \\ w_1^{\mathrm{T}}M = w_1^{\mathrm{T}}, \\ w_3^{\mathrm{T}}M = 0, \\ w_1^{\mathrm{T}}z_4 = 0, \\ w_3^{\mathrm{T}}z_2 = 0, \\ w_1^{\mathrm{T}}z_2 = t_{12}, \\ w_3^{\mathrm{T}}z_4 = 1 - t_{34}. \end{cases}$$

Through the above system, For any the vectors of z_2, z_4, w_1^{T} and w_3^{T}, which satisfy z_2, w_1^{T} belong to the range space of M, z_4, w_3^{T} belong to the null space of M, $w_1^{\mathrm{T}}z_4 = 0$ and $w_3^{\mathrm{T}}z_2 = 0$. we can solve $t_{34} = 1 - w_3^{\mathrm{T}}z_4$ and $t_{12} = w_1^{\mathrm{T}}z_2$ from the last two equations in above.

When $t_{13} = t_{31} = 0$, $t_{11} = 1$, then $t_{33} = 0$. The system (11) is reduced to

$$\begin{cases} M^2 = M, \\ Mz_2 = 0, \\ Mz_4 = z_4, \\ w_1^{\mathrm{T}}M = 0, \\ w_3^{\mathrm{T}}M = w_3^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_4 = 0, \\ w_3^{\mathrm{T}}z_2 = 0, \\ w_1^{\mathrm{T}}z_2 = 1 - t_{12}, \\ w_3^{\mathrm{T}}z_4 = t_{34}. \end{cases}$$

Similar to the above, For any the vectors of z_2, z_4, w_1^{T} and w_3^{T}, which satisfy z_4, w_3^{T} belong to the range space of M, z_2, w_1^{T} belong to the null space of M, $w_1^{\mathrm{T}}z_4 = 0$ and $w_3^{\mathrm{T}}z_2 = 0$. The last two equations of the system above imply $t_{34} = w_3^{\mathrm{T}}z_4$, $t_{12} = 1 - w_1^{\mathrm{T}}z_2$. Clearly the above two systems does not involve t_{14}, t_{32}, so they are free variables in all commuting solutions, which is the second statement of Proposition 4.1.

(iii) When $t_{13} \neq 0$, $t_{31} = 0$, the second equation of (11) indicates that either $t_{11} = 0$ or $t_{11} = 1$. If $t_{11} = 0$, then $t_{33} = 1$. The (11) now can be written as

$$\begin{cases} M^2 = M, \\ Mz_2 = z_2, \\ w_3^{\mathrm{T}}M = 0, \\ w_3^{\mathrm{T}}z_2 = 0, \\ Mz_4 = -t_{13}z_2, \\ w_1^{\mathrm{T}}M = w_1^{\mathrm{T}} - t_{13}w_3^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_2 + t_{13}t_{32} = t_{12}, \\ w_3^{\mathrm{T}}z_4 + t_{13}t_{32} = 1 - t_{34}, \\ w_1^{\mathrm{T}}z_4 = t_{13}(1 - t_{12} - t_{34}). \end{cases} \tag{13}$$

From the above system, we can know that for any the vectors of z_2 and w_3^{T}, which satisfy z_2 belongs to the range space of M, w_3^{T} belongs to the null space of M and $w_3^{\mathrm{T}} z_2 = 0$. The equation of $Mz_2 = z_2$ indicates that $r(M) = r(M, z_2)$, so the solution vector z_4 of equation $Mz_4 = -t_{13}z_2$ must exist and can be expressed as

$$z_4 = -t_{13}z_2 + T,$$

where $-t_{13}z_2$ is a special solution to the equation $Mz_4 = -t_{13}z_2$, T is a fundamental system of solutions to the equation $Mz_4 = 0$ and T satisfies the equation of $MT = 0$. From the third equation in above, we can obtain $w_3^{\mathrm{T}}(I - M) = w_3^{\mathrm{T}}$, so $r(I - M) = r(I - M, w_3^{\mathrm{T}})$. Then the solution vector w_1^{T} of the equation $w_1^{\mathrm{T}}(I - M) = t_{13}w_3^{\mathrm{T}}$ can be written as

$$w_1^{\mathrm{T}} = t_{13}w_3^{\mathrm{T}} + S^{\mathrm{T}},$$

where $t_{13}w_3^{\mathrm{T}}$ is a special solution to the equation $w_1^{\mathrm{T}}(I - M) = t_{13}w_3^{\mathrm{T}}$ and S^{T} is a fundamental system of solutions to the equation $w_1^{\mathrm{T}}(I - M) = 0$. S^{T} satisfies the equation of $S^{\mathrm{T}}(I - M) = 0$. Substituting z_4 and w_1^{T} into the last equation in above, since $t_{13} \neq 0$, t_{13} can be an arbitrary non-zero complex number. From the seventh and the eighth equations of (13), we can solve $t_{12} = w_1^{\mathrm{T}} z_2 + t_{13}t_{32}$, $t_{34} = 1 - t_{13}t_{32} - w_3^{\mathrm{T}} z_4$, respectively, where t_{32} and t_{14} are two arbitrary complex numbers.

When $t_{13} \neq 0$, $t_{31} = 0$, $t_{11} = 1$, then $t_{33} = 0$. The system (11) are simplified to

$$\begin{cases} M^2 = M, \\ Mz_2 = 0, \\ w_3^{\mathrm{T}} M = w_3^{\mathrm{T}}, \\ w_3^{\mathrm{T}} z_2 = 0, \\ Mz_4 = z_4 - t_{13}z_2, \\ w_1^{\mathrm{T}} M = -t_{13}w_3^{\mathrm{T}}, \\ w_1^{\mathrm{T}} z_2 + t_{13}t_{32} = 1 - t_{12}, \\ w_3^{\mathrm{T}} z_4 + t_{13}t_{32} = t_{34}, \\ w_1^{\mathrm{T}} z_4 = t_{13}(1 - t_{12} - t_{34}). \end{cases}$$

The observation show that when $t_{11} = 1$, it is very similar to the above case. So I don't give the detailed proof process, and I only give some conclusions. For any the vectors of z_2, w_3^{T}, which satisfy z_2 belongs to the null space of M, w_3^{T} belongs to the range space of M and $w_3^{\mathrm{T}} z_2 = 0$.

$$z_4 = t_{13}z_2 + S,$$

where $t_{13}z_2$ is a special solution to the equation $(I - M)z_4 = t_{13}z_2$ and S is a fundamental system of solutions to the equation $(I - M)z_4 = 0$, S satisfies the equation of $(I - M)S = 0$.

$$w_1^{\mathrm{T}} = -t_{13}w_3^{\mathrm{T}} + T^{\mathrm{T}},$$

where $-t_{13}w_3^{\mathrm{T}}$ is a special solution to the equation $w_1^{\mathrm{T}} M = -t_{13}w_3^{\mathrm{T}}$ and T^{T} is a fundamental system of solutions to the equation $w_1^{\mathrm{T}} M = 0$, T^{T} satisfies the equation of $T^{\mathrm{T}} M = 0$. From the last three equations in above, we can obtain t_{13} is an arbitrary non-zero complex number, $t_{34} = w_3^{\mathrm{T}} z_4 + t_{13}t_{32}$ and $t_{12} = 1 - t_{13}t_{32} - w_1^{\mathrm{T}} z_2$, where t_{14} and t_{32} are three arbitrary complex numbers, which is the proof of the third case of Proposition 4.1.

(iv) When $t_{13} \neq 0$, $t_{31} \neq 0$, we can get either $t_{11} \neq 0$ or 1. The (11) becomes

$$\begin{cases} t_{11} + t_{33} = 1, \\ t_{11}^2 + t_{13}t_{31} = t_{11}, \\ t_{33}^2 + t_{13}t_{31} = t_{33}, \\ M^2 = M, \\ (t_{33}I - M)z_2 = t_{31}z_4, \\ (t_{11}I - M)z_4 = t_{13}z_2, \\ w_1^{\mathrm{T}}(t_{33}I - M) = t_{13}w_3^{\mathrm{T}}, \\ w_3^{\mathrm{T}}(t_{11}I - M) = t_{31}w_1^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_2 + 2t_{11}t_{12} + t_{13}t_{32} + t_{14}t_{31} = t_{11} + t_{12}, \\ w_3^{\mathrm{T}}z_4 + t_{31}t_{14} + t_{13}t_{32} + 2t_{34}t_{33} = t_{33} + t_{34}, \\ w_1^{\mathrm{T}}z_4 = t_{13}(1 - t_{12} - t_{34}), \\ w_3^{\mathrm{T}}z_2 = t_{31}(1 - t_{12} - t_{34}). \end{cases}$$

For any $t_{11} \neq 0$, 1, $t_{13} \neq 0$, the first three equations of the above system indicate that $t_{33} = 1 - t_{11}$ and $t_{31} = \frac{t_{11}(1-t_{11})}{t_{13}}$. From the fifth equation in above, we can solve $z_4 = \frac{(t_{33}I-M)z_2}{t_{31}}$, where z_2 is any complex $(n-3)$−dimensional complex vector. For any w_1^{T}, the seventh equation of the above system implies $w_3^{\mathrm{T}} = \frac{w_1^{\mathrm{T}}(t_{33}I-M)}{t_{13}}$. Let's substitute the values of t_{31}, z_4 and w_3^{T} into the two equations of the sixth and the eighth, so we can verify that both of these equations are satisfied. From the eleventh equation in above system, we can obtain $t_{34} = 1 - t_{12} - \frac{w_1^{\mathrm{T}}z_4}{t_{13}}$. substituting t_{34}, z_4 and w_3^{T} into the last equation, which is always satisfied. By substituting t_{33}, t_{34}, w_3^{T} and z_4 into the ninth equation and the tenth equation, we will find that these two equations are the same. So we can solve $t_{32} = \frac{t_{11}+t_{12}-2t_{11}t_{12}-t_{14}t_{31}-w_1^{\mathrm{T}}z_2}{t_{13}}$, where t_{12} and t_{14} are two arbitrary complex numbers and the last statement of Proposition 4.1 is proved.

We now consider the next case $t_{13} = 0$ and $t_{31} = 0$, which leads to the next proposition.

Proposition 4.2. *Suppose $A = I - PQ^{\mathrm{T}}$ is such that $\Lambda = \Lambda_3$ in its Jordan form (3) with det$(Q^{\mathrm{T}}P) = 0$. Then all the commuting solutions of (1) are $X = WYW^{-1}$ with Y partition as (9), such that M is an arbitrary $(n-4) \times (n-4)$ projection matrix, when $t_{13} = t_{31} = 0$, and*

(i)$t_{11} = t_{33} = 0$, then the vectors $z_2, z_4, w_1^{\mathrm{T}}$, w_3^{T} belong to the range space of M, and $t_{12} = w_1^{\mathrm{T}}z_2$, $t_{14} = w_1^{\mathrm{T}}z_4$, $t_{32} = w_3^{\mathrm{T}}z_2$, $t_{34} = w_3^{\mathrm{T}}z_4$.

(ii)$t_{11} = t_{33} = 1$, then the vectors $z_2, z_4, w_1^{\mathrm{T}}$, w_3^{T} belong to the null space of M, and $t_{12} = 1 - w_1^{\mathrm{T}}z_2$, $t_{14} = -w_1^{\mathrm{T}}z_4$, $t_{32} = -w_3^{\mathrm{T}}z_2$, $t_{34} = 1 - w_3^{\mathrm{T}}z_4$.

Proof: When $t_{13} = 0$ and $t_{31} = 0$. Substituting them into (10) and after simple simplification, it becomes the following form

$$\begin{cases} M^2 = M, \\ Mz_2 = (1 - t_{11})z_2, \\ Mz_4 = (1 - t_{33})z_4, \\ w_1^{\mathrm{T}}M = (1 - t_{11})w_1^{\mathrm{T}}, \\ w_3^{\mathrm{T}}M = (1 - t_{33})w_3^{\mathrm{T}}, \\ w_1^{\mathrm{T}}z_2 = 2t_{11} + t_{12} - 2t_{11}t_{12} - t_{11}^2, \\ w_1^{\mathrm{T}}z_4 = t_{14} - t_{11}t_{14} - t_{33}t_{14}, \\ w_3^{\mathrm{T}}z_2 = t_{32} - t_{11}t_{32} - t_{32}t_{33}, \\ w_3^{\mathrm{T}}z_4 = 2t_{33} + t_{34} - 2t_{33}t_{34} - t_{33}^2, \\ t_{11}^2 = t_{11}, \\ t_{33}^2 = t_{33}. \end{cases} \tag{14}$$

The equation $t_{11}^2 = t_{11}$ indicates that either $t_{11} = 0$ or $t_{11} = 1$, and the last equation above indicates that either $t_{33} = 0$ or $t_{33} = 1$. Therefore we have the following result.

When $t_{11} = t_{33} = 0$, then the above system is reduced to

$$\begin{cases} M^2 = M, \\ Mz_2 = z_2, \\ Mz_4 = z_4, \\ w_1^{\mathrm{T}} M = w_1^{\mathrm{T}}, \\ w_3^{\mathrm{T}} M = w_3^{\mathrm{T}}, \\ w_1^{\mathrm{T}} z_2 = t_{12}, \\ w_1^{\mathrm{T}} z_4 = t_{14}, \\ w_3^{\mathrm{T}} z_2 = t_{32}, \\ w_3^{\mathrm{T}} z_4 = t_{34}, \end{cases}$$

which is the first statement of Proposition 4.2.

When $t_{11} = t_{33} = 1$, then the system of (14) becomes

$$\begin{cases} M^2 = M, \\ Mz_2 = 0, \\ Mz_4 = 0, \\ w_1^{\mathrm{T}} M = 0, \\ w_3^{\mathrm{T}} M = 0, \\ w_1^{\mathrm{T}} z_2 = 1 - t_{12}, \\ w_1^{\mathrm{T}} z_4 = -t_{14}, \\ w_3^{\mathrm{T}} z_2 = -t_{32}, \\ w_3^{\mathrm{T}} z_4 = 1 - t_{34}, \end{cases}$$

where z_2, z_4, w_1 and w_3 are four complex $(n-4)$−dimensional complex vectors and the second statement of Proposition 4.2 is proved.

When $t_{11} = 0$ and $t_{33} = 1$ or when $t_{11} = 1$ and $t_{33} = 0$, which belong to the case of *proposition 4.1*(ii), so we don't discuss them here.

When $t_{13} = 0$, $t_{11} + t_{33} = 1$ or when $t_{31} = 0$, $t_{11} + t_{33} = 1$, which belong to the cases of *proposition 4.1*(i), (iii), respectively, so we don't discuss them here.

Summarizing all of the above cases gives the main result.

Theorem 4.1 Suppose $A = I - PQ^{\mathrm{T}}$ is such that $\Lambda = \Lambda_3$ in its Jordan form (3) with $det(Q^{\mathrm{T}}P) = 0$. Then all the commuting solutions of (1) are $X = WYW^{-1}$ with Y partition as (9), such that M is an arbitrary $(n-4) \times (n-4)$ projection matrix. Then the following are true:

(i)If $t_{11} + t_{33} = 1$, all the commuting solutions of (1) are given by *proposition 4.1*.

(ii)If $t_{13} = t_{31} = 0$, all the commuting solutions of (1) are given by *proposition 4.2*.

5. Illustrating examples

In this section we give a few examples to illustrate our results.

Example 1. Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$, $Q^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and

$$A = I - PQ^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

then $A = WJW^{-1}$, where

$$W = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \mathit{and}\ J = \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

By Theorem 2.1, we have the following conclusions:

All the commuting solutions of (1) are $X = WYW^{-1}$, in which $Y = 0$,

$$\begin{pmatrix} 1 & 0 & 0 & z_3 \\ w_1^{\mathrm{T}} & 0 & 0 & w_1^{\mathrm{T}} z_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & z_3 \\ w_1^{\mathrm{T}} & 1 & 1 & -w_1^{\mathrm{T}} z_3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where z_3 and w_1^{T} are two arbitrary complex numbers.

Example 2. Let $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$, and

$$A = I - PQ^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & -2 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then $A = WJW^{-1}$, where

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \mathit{and}\ J = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & 3 \end{pmatrix}.$$

Form the first equation of (8) we can know that M is any 2×2 projection matrix. By solving the equation $M^2 = M$, we see that $M = 0, I$, and

$$\begin{pmatrix} \frac{1 \pm \sqrt{1-4bc}}{2} & b \\ c & \frac{1 \mp \sqrt{1-4bc}}{2} \end{pmatrix}, \forall\, b, c,$$

where z_2, w_1^{T} are two vectors, $z_2 = [z_{21}, z_{22}]^{\mathrm{T}}$, $w_1 = [w_{11}^{\mathrm{T}}, w_{12}^{\mathrm{T}}]^{\mathrm{T}}$.

By Theorem 3.1, all the commuting solutions of (1) are as follows:

When it's the first case of theorem 3.1, then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & z_{21} & 0 \\ 0 & 1 & 0 & z_{22} & 0 \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & w_{11}^{\mathrm{T}} z_{21} + w_{12}^{\mathrm{T}} z_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{33} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1 \pm \sqrt{1-4bc}}{2} & b & 0 & \frac{1 \pm \sqrt{1-4bc}}{2c} z_{22} & 0 \\ c & \frac{1 \mp \sqrt{1-4bc}}{2} & 0 & z_{22} & 0 \\ \frac{1 \pm \sqrt{1-4bc}}{2b} w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & \frac{1 \pm \sqrt{1-4bc}}{2bc} w_{12}^{\mathrm{T}} z_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{33} \end{pmatrix}.$$

When it's the second case of theorem 3.1, then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} 0 & 0 & 0 & z_{21} & 0 \\ 0 & 0 & 0 & z_{22} & 0 \\ w_{11}^T & w_{12}^T & 1 & 1 - w_{11}^T z_{21} - w_{12}^T z_{22} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t_{33} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t_{33} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & -\frac{1\mp\sqrt{1-4bc}}{2c} z_{22} & 0 \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & z_{22} & 0 \\ -\frac{1\mp\sqrt{1-4bc}}{2b} w_{12}^T & w_{12}^T & 1 & 1 - \frac{1\mp\sqrt{1-4bc}}{2bc} w_{12}^T z_{22} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t_{33} \end{pmatrix},$$

where b and c are two arbitrary complex numbers, t_{33} is either 0 or 3 in all the commuting solutions above.

Example 3. Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $Q^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$, and

$$A = I - PQ^T = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then $A = WJW^{-1}$, where

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \textit{ and } J = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

Form the first equation of (10) we can know that M is any 2×2 projection matrix. Similar to example two in above, we can get $M = 0, I$, and

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b \\ c & \frac{1\mp\sqrt{1-4bc}}{2} \end{pmatrix}, \forall\, b, c,$$

where z_2, z_4, w_1^T and w_3^T are four vectors, $z_2 = [z_{21}, z_{22}]^T$, $z_4 = [z_{41}, z_{42}]^T$ $w_1 = [w_{11}^T, w_{12}^T]^T$, $w_3 = [w_{31}^T, w_{32}^T]^T$.

By Theorem 4.1, all the commuting solutions of (1) are as follows:
When it is a case of proposition 4.1(i), then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} 0 & 0 & 0 & t_{31}z_{41} & 0 & z_{41} \\ 0 & 0 & 0 & t_{31}z_{42} & 0 & z_{42} \\ 0 & 0 & 0 & t_{12} & 0 & t_{12}/t_{31} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1^T & \beta_2^T & t_{31} & t_{32} & 1 & 1 - t_{12}t_{31} - \beta_1^T z_{41} - \beta_2^T z_{42} \\ 0 & 0 & 0 & t_{31} & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 1 & 0 & \alpha_2 & 0 & 0 \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & t_{12} & 0 & \frac{t_{12}-w_{11}^{\mathrm{T}}\alpha_1-w_{12}^{\mathrm{T}}\alpha_2}{t_{31}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -t_{31}w_{11}^{\mathrm{T}} & -t_{31}w_{12}^{\mathrm{T}} & t_{31} & t_{32} & 1 & 1-t_{12}+w_{11}^{\mathrm{T}}\alpha_1+w_{12}^{\mathrm{T}}\alpha_2 \\ 0 & 0 & 0 & t_{31} & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & z_{21} & 0 & -\frac{1\mp\sqrt{1-4bc}}{2b}z_{42} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & t_{31}z_{42}+\alpha_2 & 0 & z_{42} \\ \frac{1\pm\sqrt{1-4bc}}{2b}w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & t_{12} & 0 & t_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w_{31}^{\mathrm{T}} & -t_{31}w_{12}^{\mathrm{T}}+\beta_2^{\mathrm{T}} & t_{31} & t_{32} & 1 & t_{34} \\ 0 & 0 & 0 & t_{31} & 0 & 1 \end{pmatrix};$$

where $w_{31}^{\mathrm{T}} = -\frac{t_{31}(1\pm\sqrt{1-4bc})w_{12}^{\mathrm{T}}+(1\mp\sqrt{1-4bc})\beta_2^{\mathrm{T}}}{2b}$, $z_{21} = \frac{t_{31}(-1\pm\sqrt{1-4bc})z_{42}+(1\pm\sqrt{1-4bc})\alpha_2}{2c}$, $t_{14} = \frac{2bct_{12}-(1\pm\sqrt{1-4bc})w_{12}^{\mathrm{T}}\alpha_2}{2bct_{31}}$ and $t_{34} = 1-t_{12}-\frac{1\mp\sqrt{1-4bc}}{2bc}\beta_2^{\mathrm{T}}z_{42}+\frac{1\pm\sqrt{1-4bc}}{2bc}w_{12}^{\mathrm{T}}\alpha_2$.

$$\begin{pmatrix} 0 & 0 & 0 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 & 0 \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 1 & t_{12} & 0 & \frac{1-t_{12}-w_{11}^{\mathrm{T}}\beta_1-w_{12}^{\mathrm{T}}\beta_2}{t_{31}} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ t_{31}w_{11}^{\mathrm{T}} & t_{31}w_{12}^{\mathrm{T}} & t_{31} & t_{32} & 0 & 1-t_{12}-w_{11}^{\mathrm{T}}\beta_1-w_{12}^{\mathrm{T}}\beta_2 \\ 0 & 0 & 0 & t_{31} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & -t_{31}z_{41} & 0 & z_{41} \\ 0 & 1 & 0 & -t_{31}z_{42} & 0 & z_{42} \\ 0 & 0 & 1 & t_{12} & 0 & \frac{1-t_{12}}{t_{31}} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha_1^{\mathrm{T}} & \alpha_2^{\mathrm{T}} & t_{31} & t_{32} & 0 & 1-t_{13}+\alpha_1^{\mathrm{T}}z_{41}+\alpha_2^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & t_{31} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & z_{21} & 0 & \frac{1\pm\sqrt{1-4bc}}{2c}z_{42} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & -t_{31}z_{42}+\beta_2 & 0 & z_{42} \\ -\frac{1\mp\sqrt{1-4bc}}{2b}w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 1 & t_{12} & 0 & t_{14} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ w_{31}^{\mathrm{T}} & t_{31}w_{12}^{\mathrm{T}}+\alpha_2^{\mathrm{T}} & t_{31} & t_{32} & 0 & t_{34} \\ 0 & 0 & 0 & t_{31} & 0 & 0 \end{pmatrix},$$

where $z_{21} = -\frac{t_{31}(1\pm\sqrt{1-4bc})z_{42}+(1\mp\sqrt{1-4bc})\beta_2}{2c}$, $w_{31}^{\mathrm{T}} = \frac{t_{31}(-1\pm\sqrt{1-4bc})w_{12}^{\mathrm{T}}+(1\pm\sqrt{1-4bc})\alpha_2^{\mathrm{T}}}{2b}$, $t_{14} = \frac{2bc(1-t_{12})-(1\mp\sqrt{1-4bc})w_{12}^{\mathrm{T}}\beta_2}{2bct_{31}}$ and $t_{34} = 1-t_{12}-\frac{1\mp\sqrt{1-4bc}}{2bc}w_{12}^{\mathrm{T}}\beta_2+\frac{1\pm\sqrt{1-4bc}}{2bc}\alpha_2^{\mathrm{T}}z_{42}$.

When it is a case of proposition 4.1(ii), then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & z_{41} \\ 0 & 0 & 0 & 0 & 0 & z_{42} \\ 0 & 0 & 0 & 0 & 0 & t_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w_{31}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 1 & 1-w_{31}^{\mathrm{T}}z_{41}-w_{32}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & z_{21} & 0 & 0 \\ 0 & 1 & 0 & z_{22} & 0 & 0 \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & w_{11}^{\mathrm{T}}z_{21}+w_{12}^{\mathrm{T}}z_{22} & 0 & t_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{32} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & \frac{1\pm\sqrt{1-4bc}}{2c}z_{22} & 0 & -\frac{1\mp\sqrt{1-4bc}}{2c}z_{42} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & z_{22} & 0 & z_{42} \\ \frac{1\pm\sqrt{1-4bc}}{2b}w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & \frac{1\pm\sqrt{1-4bc}}{2bc}w_{12}^{\mathrm{T}}z_{22} & 0 & t_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1\mp\sqrt{1-4bc}}{2b}w_{32}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 1 & 1-\frac{1\mp\sqrt{1-4bc}}{2bc}w_{32}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 0 & 0 & 0 & z_{21} & 0 & 0 \\ 0 & 0 & 0 & z_{22} & 0 & 0 \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 1 & 1-w_{11}^{\mathrm{T}}z_{21}-w_{12}^{\mathrm{T}}z_{22} & 0 & t_{14} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & z_{41} \\ 0 & 1 & 0 & 0 & 0 & z_{42} \\ 0 & 0 & 1 & 1 & 0 & t_{14} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ w_{31}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 0 & w_{31}^{\mathrm{T}}z_{41}+w_{32}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & -\frac{1\mp\sqrt{1-4bc}}{2c}z_{22} & 0 & \frac{1\pm\sqrt{1-4bc}}{2c}z_{42} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & z_{22} & 0 & z_{42} \\ -\frac{1\mp\sqrt{1-4bc}}{2b}w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 1 & 1-\frac{1\mp\sqrt{1-4bc}}{2bc}w_{12}^{\mathrm{T}}z_{22} & 0 & t_{14} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1\pm\sqrt{1-4bc}}{2b}w_{32}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 0 & \frac{1\pm\sqrt{1-4bc}}{2bc}w_{32}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

When it is a case of proposition 4.1(iii), then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 \\ t_{13}w_{31}^{\mathrm{T}} & t_{13}w_{32}^{\mathrm{T}} & 0 & t_{13}t_{32} & t_{13} & t_{14} \\ 0 & 0 & 0 & 0 & 0 & t_{13} \\ w_{31}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 1 & 1-t_{13}t_{32}-w_{31}^{\mathrm{T}}\beta_1-w_{32}^{\mathrm{T}}\beta_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & z_{21} & 0 & -t_{13}z_{21} \\ 0 & 1 & 0 & z_{22} & 0 & -t_{13}z_{22} \\ \alpha_1^{\mathrm{T}} & \alpha_2^{\mathrm{T}} & 0 & t_{13}t_{32}+\alpha_1^{\mathrm{T}}z_{21}+\alpha_2^{\mathrm{T}}z_{22} & t_{13} & t_{14} \\ 0 & 0 & 0 & 0 & 0 & t_{13} \\ 0 & 0 & 0 & t_{32} & 1 & 1-t_{13}t_{32} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & \frac{1\pm\sqrt{1-4bc}}{2c}z_{22} & 0 & z_{41} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & z_{22} & 0 & -t_{13}z_{22}+\beta_2 \\ w_{11}^{\mathrm{T}} & t_{13}w_{32}^{\mathrm{T}}+\alpha_2^{\mathrm{T}} & 0 & t_{12} & t_{13} & t_{14} \\ 0 & 0 & 0 & 0 & 0 & t_{13} \\ -\frac{1\mp\sqrt{1-4bc}}{2b}w_{32}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 1 & t_{34} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $z_{41} = -\frac{t_{13}(1\pm\sqrt{1-4bc})z_{22}+(1\mp\sqrt{1-4bc})\beta_2}{2c}$, $w_{11}^{\mathrm{T}} = \frac{t_{13}(-1\pm\sqrt{1-4bc})w_{32}^{\mathrm{T}}+(1\pm\sqrt{1-4bc})\alpha_2^{\mathrm{T}}}{2b}$, $t_{12} = t_{13}t_{32} + \frac{1\pm\sqrt{1-4bc}}{2bc}\alpha_2^{\mathrm{T}}z_{22}$ and $t_{34} = 1 - t_{13}t_{32} - \frac{1\mp\sqrt{1-4bc}}{2bc}w_{32}^{\mathrm{T}}\beta_2$.

$$\begin{pmatrix} 0 & 0 & 0 & z_{21} & 0 & t_{13}z_{21} \\ 0 & 0 & 0 & z_{22} & 0 & t_{13}z_{22} \\ \beta_1^{\mathrm{T}} & \beta_2^{\mathrm{T}} & 1 & 1-t_{13}t_{32}-\beta_1^{\mathrm{T}}z_{21}-\beta_2^{\mathrm{T}}z_{22} & t_{13} & t_{14} \\ 0 & 0 & 0 & 1 & 0 & t_{13} \\ 0 & 0 & 0 & t_{32} & 0 & t_{13}t_{32} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & 0 & 0 & \alpha_2 \\ -t_{13}w_{31}^{\mathrm{T}} & -t_{13}w_{32}^{\mathrm{T}} & 1 & 1-t_{13}t_{32} & t_{13} & t_{14} \\ 0 & 0 & 0 & 1 & 0 & t_{13} \\ w_{31}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 0 & t_{13}t_{32}+w_{31}^{\mathrm{T}}\alpha_1+w_{32}^{\mathrm{T}}\alpha_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & -\frac{1\mp\sqrt{1-4bc}}{2c}z_{22} & 0 & z_{41} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & z_{22} & 0 & t_{13}z_{22}+\alpha_2 \\ w_{11}^{\mathrm{T}} & -t_{13}w_{32}^{\mathrm{T}}+\beta_2^{\mathrm{T}} & 1 & t_{12} & t_{13} & t_{14} \\ 0 & 0 & 0 & 1 & 0 & t_{13} \\ \frac{1\pm\sqrt{1-4bc}}{2b}w_{32}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & t_{32} & 0 & t_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $z_{41} = \frac{t_{13}(-1\pm\sqrt{1-4bc})z_{22}+(1\mp\sqrt{1-4bc})\alpha_2}{2c}$, $w_{11}^{\mathrm{T}} = -\frac{t_{13}(1\pm\sqrt{1-4bc})w_{32}^{\mathrm{T}}+(1\mp\sqrt{1-4bc})\beta_2^{\mathrm{T}}}{2b}$, $t_{12} = 1-t_{13}t_{32}-\frac{1\mp\sqrt{1-4bc}}{2bc}\beta_2^{\mathrm{T}}z_{22}$ and $t_{34} = t_{13}t_{32}+\frac{1\pm\sqrt{1-4bc}}{2bc}w_{32}^{\mathrm{T}}\alpha_2$.

When it is a case of proposition 4.1(iv), then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} M & 0 & z_2 & 0 & z_4 \\ w_1^{\mathrm{T}} & t_{11} & t_{12} & t_{13} & t_{14} \\ 0 & 0 & t_{11} & 0 & t_{13} \\ w_3^{\mathrm{T}} & \frac{t_{11}(1-t_{11})}{t_{13}} & t_{32} & 1-t_{11} & t_{34} \\ 0 & 0 & \frac{t_{11}(1-t_{11})}{t_{13}} & 0 & 1-t_{11} \end{pmatrix},$$

where $M = 0, I$, and

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b \\ c & \frac{1\mp\sqrt{1-4bc}}{2} \end{pmatrix},$$

$z_4 = \frac{t_{13}((1-t_{11})I-M)z_2}{t_{11}(1-t_{11})}$, $t_{32} = \frac{t_{13}(t_{11}+t_{12}-2t_{11}t_{12})-t_{14}t_{11}(1-t_{11})-t_{13}w_1^{\mathrm{T}}z_2}{t_{13}^2}$, $w_3^{\mathrm{T}} = \frac{w_1^{\mathrm{T}}((1-t_{11})I-M)}{t_{13}}$ and $t_{34} = 1-t_{12}-\frac{w_1^{\mathrm{T}}z_4}{t_{13}}$ in the above case.

When it is a case of proposition 4.2(i), then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y = 0$,

$$\begin{pmatrix} 1 & 0 & 0 & z_{21} & 0 & z_{41} \\ 0 & 1 & 0 & z_{22} & 0 & z_{42} \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & w_{11}^{\mathrm{T}}z_{21}+w_{12}^{\mathrm{T}}z_{22} & 0 & w_{11}^{\mathrm{T}}z_{41}+w_{12}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w_{31}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & w_{31}^{\mathrm{T}}z_{21}+w_{32}^{\mathrm{T}}z_{22} & 0 & w_{31}^{\mathrm{T}}z_{41}+w_{32}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1\pm\sqrt{1-4bc}}{2} & b & 0 & \frac{1\pm\sqrt{1-4bc}}{2c}z_{22} & 0 & \frac{1\pm\sqrt{1-4bc}}{2c}z_{42} \\ c & \frac{1\mp\sqrt{1-4bc}}{2} & 0 & z_{22} & 0 & z_{42} \\ \frac{1\pm\sqrt{1-4bc}}{2b}w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 0 & \frac{1\pm\sqrt{1-4bc}}{2bc}w_{12}^{\mathrm{T}}z_{22} & 0 & \frac{1\pm\sqrt{1-4bc}}{2bc}w_{12}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1\pm\sqrt{1-4bc}}{2b}w_{32}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & \frac{1\pm\sqrt{1-4bc}}{2bc}w_{32}^{\mathrm{T}}z_{22} & 0 & \frac{1\pm\sqrt{1-4bc}}{2bc}w_{32}^{\mathrm{T}}z_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

When it is a case of proposition 4.2(ii), then all the commuting solutions of (1) are $X = WYW^{-1}$ in which $Y =$

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & z_{21} & 0 & z_{41} \\ 0 & 0 & 0 & z_{22} & 0 & z_{42} \\ w_{11}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 1 & 1 - w_{11}^{\mathrm{T}} z_{21} - w_{12}^{\mathrm{T}} z_{22} & 0 & -w_{11}^{\mathrm{T}} z_{41} - w_{12}^{\mathrm{T}} z_{42} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ w_{31}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & -w_{31}^{\mathrm{T}} z_{21} - w_{32}^{\mathrm{T}} z_{22} & 1 & 1 - w_{31}^{\mathrm{T}} z_{41} - w_{32}^{\mathrm{T}} z_{42} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1 \pm \sqrt{1-4bc}}{2} & b & 0 & -\frac{1 \mp \sqrt{1-4bc}}{2c} z_{22} & 0 & -\frac{1 \mp \sqrt{1-4bc}}{2c} z_{42} \\ c & \frac{1 \mp \sqrt{1-4bc}}{2} & 0 & z_{22} & 0 & z_{42} \\ -\frac{1 \mp \sqrt{1-4bc}}{2b} w_{12}^{\mathrm{T}} & w_{12}^{\mathrm{T}} & 1 & 1 - \frac{1 \mp \sqrt{1-4bc}}{2bc} w_{12}^{\mathrm{T}} z_{22} & 0 & -\frac{1 \mp \sqrt{1-4bc}}{2bc} w_{12}^{\mathrm{T}} z_{42} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1 \mp \sqrt{1-4bc}}{2b} w_{32}^{\mathrm{T}} & w_{32}^{\mathrm{T}} & 0 & -\frac{1 \mp \sqrt{1-4bc}}{2bc} w_{32}^{\mathrm{T}} z_{22} & 1 & 1 - \frac{1 \mp \sqrt{1-4bc}}{2bc} w_{32}^{\mathrm{T}} z_{42} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where $S = [\alpha_1, \alpha_2]^{\mathrm{T}}$, $T = [\beta_1, \beta_2]^{\mathrm{T}}$, $S^{\mathrm{T}} = [\alpha_1^{\mathrm{T}}, \alpha_2^{\mathrm{T}}]$ and $T^{\mathrm{T}} = [\beta_1^{\mathrm{T}}, \beta_2^{\mathrm{T}}]$ are four any vectors, b and c are two arbitrary complex numbers in all the commuting solutions above.

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