Filomat 32:13 (2018), 4611–4618
https://doi.org/10.2298/FIL1813611A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence Results for Hadamard and Riemann-Liouville Functional Fractional Neutral Integrodifferential Equations with Finite Delay

Mohamed I. Abbas[a]

[a]*Taibah University, Faculty of Science and Arts, Al Ula Branch, Department of Mathematics, Al Ula, Saudi Arabia*
Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt

Abstract. By using Leray-Schauder's alternative, we study the existence and uniqueness of solutions for some Hadamard and Riemann-Liouville fractional neutral functional integrodifferential equations with finite delay, whereas the uniqueness of the solution is established by Banach's contraction principle. An illustrative example is also included.

1. Introduction

In this paper, we establish existence, uniqueness results for the Hadamard and Riemann-Liouville fractional neutral functional integrodifferential equations with finite delay described by

$$ {}_H D^{\alpha}\left[u(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,u_t)\right] = f(t,u_t), \quad t\in J:=[1,T], \tag{1}$$

$$ u(t) = \varphi(t), \quad t\in[1-r,1],\ r>0, \tag{2}$$

where ${}_H D^{\alpha}$ denotes the Hadamard fractional derivative of order $\alpha,\ 0<\alpha\leq 1$, I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i>0,\ i=1,2,\cdots,m,\ f,h_i : J\times C([-r,0],\mathbb{R})\to\mathbb{R}$ are given continuous functions satisfying some assumptions that will be specified later and $\varphi\in C([1-r,1],\mathbb{R})$ with $\varphi(1)=0$. For any function u defined on $[1-r,T]$ and any $t\in J$, we denote by u_t the element of $C([-r,0],\mathbb{R})$ and is defined by $u_t(\theta)=u(t+\theta),\ \theta\in[-r,0]$. Here $u_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

In the last years, there is a strong development of the study of fractional differential equations and inclusions involving Riemann-Liouville and Caputo type fractional derivatives, see [1–3], and the references therein. Besides these derivatives, there is an other fractional derivative introduced by Hadamard in 1892 [12], which is known as Hadamard derivative and differs from aforementioned derivatives in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [8–10] and references cited therein. Recently, several papers were devoted to fractional differential equations and inclusions defined

2010 *Mathematics Subject Classification*. Primary 34A08; Secondary 34K37, 34K40
Keywords. Hadamard fractional derivative, Riemann-Liouville fractional integral, neutral fractional differential equations, fixed point theorems
Received: 03 January 2018; Accepted: 22 January 2018
Communicated by Maria Alessandra RAGUSA
Email address: miabbas77@gmail.com (Mohamed I. Abbas)

by Hadamard fractional derivative [4, 5, 15, 17] etc.
On the other hand, functional and neutral functional differential equations arise in the mathematical modelling of biological, physical, and engineering problems, see, for example, the texts [6, 7, 14, 16] and the references cited therein.

The rest of this paper is organized as follows. In Section 2, we give some notations for Hadamard fractional calculus. In Section 3, we present two existence and uniqueness results by using the Banach contraction principle and Leray-Schauder nonlinear alternative. Finally, an examples is given to illustrate our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that we need in the sequel. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_\infty := \sup\{|u(t)| : t \in J\}.$$

Also $C([-r, 0], \mathbb{R})$ is endowed with the norm

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

The following definitions are devoted to the basic concepts of Hadamard and Riemann-Liuoville types fractional calculus. For more details, see A. A. Kilbas et al. [13].

Definition 2.1. *The Hadamard derivative of fractional order q for a function $g : (1, \infty) \to \mathbb{R}$ is defined as*

$${}_H D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1 < q < n, \ n = [q] + 1,$$

where $[q]$ denotes the integrer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *The Hadamard fractional integral of order q for a function $g : (1, \infty) \to \mathbb{R}$ is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3. *The Riemann-Liuoville fractional integral of order $p > 0$ of a continuous function $h : (1, \infty) \to \mathbb{R}$ is defined by*

$$I^p h(t) = \frac{1}{\Gamma(p)} \int_1^t \frac{h(s)}{(t-s)^{1-p}} ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

3. Existence and uniqueness results

Let us defining what we mean by a solution of problem (1.1) − (1.2).

Definition 3.1. *A function $u \in C([1-r, T], \mathbb{R})$, is said to be a solution of (1.1) − (1.2) if u satisfies the equation ${}_H D^\alpha \left[u(t) - \sum_{i=1}^m I^{\beta_i} h_i(t, u_t)\right] = f(t, u_t)$ on J, and the condition $u(t) = \varphi(t)$ on $[1-r, 1]$.*

To prove the existence of solutions to (1.1) − (1.2), we need the following auxiliary Lemma.

Lemma 3.2. *Let $0 < \alpha \leq 1$ and $\sigma : J \rightarrow \mathbb{R}$ be a continuous function. The linear problem*

$$\begin{aligned} {}_{H}D^{\alpha}[u(t) - \gamma(t)] &= \sigma(t), \;\; t \in J & (3)\\ u(t) &= \varphi(t), \;\; t \in [1-r, 1], & (4) \end{aligned}$$

has a unique solution which is given by:

$$u(t) = \begin{cases} \gamma(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\sigma(s)}{s} ds, & \text{if } t \in J \\ \varphi(t), & \text{if } t \in [1-r, 1]. \end{cases} \qquad (5)$$

For the proof of Lemma 3.2, it is useful to refer to [5, 13].

In the sequel, we need the following assumptions.

(H1) The functions $f, h_i : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ are continuous and there exist positive functions μ, ψ_i, $i = 1, 2, \cdots, m$, with bounds $\|\mu\|$ and $\|\psi_i\|$, $i = 1, 2, \cdots, m$, respectively such that:

$$|f(t, x) - f(t, y)| \leq \mu(t)\|x - y\|_C,$$

and

$$|h_i(t, x) - h_i(t, y)| \leq \psi_i(t)\|x - y\|_C,$$

for $t \in J$ and $x, y \in C([-r, 0], \mathbb{R})$.

(H2) There exists a function $p \in C([1, T], \mathbb{R}^+)$ and a continuous nondecreasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ such that

$$|f(t, x)| \leq p(t)\Phi(\|x\|_C), \quad \text{for each } (t, x) \in [1, T] \times C([-r, 0], \mathbb{R}).$$

(H3) There exists a constant $k > 0$ such that

$$|h_i(t, x)| \leq k, \quad \text{for each } (t, x) \in [1, T] \times C([-r, 0], \mathbb{R}), \; i = 1, 2, \cdots, m.$$

(H4) There exist constants $0 < \alpha \leq 1$ and $M, k > 0$ such that:

$$\frac{M}{\frac{\|p\|_\infty \Phi(M)(\log T)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^m \frac{k\, T^{\beta_i}}{\Gamma(\beta_i+1)}} > 1.$$

Our first existence result for (1.1) – (1.2) is based on the Banach contraction principle.

Theorem 3.3. *Assume that assumption* (*H*1) *hold. If*

$$\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\|\mu\| + \sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i+1)}\|\psi_i\| < 1,$$

then there exists a unique solution for (1.1) – (1.2) *on the interval* $[1-r, T]$.

Proof Transform the problem (1.1) – (1.2) into a fixed point problem. Consider the operator $\mathcal{N} : C([1-r, T], \mathbb{R}) \rightarrow C([1-r, T], \mathbb{R})$ defined by

$$\mathcal{N}u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u_s)}{s} ds + \sum_{i=1}^m I^{\beta_i} h_i(s, u_s), & t \in J \\ \varphi(t), & t \in [1-r, 1]. \end{cases} \qquad (6)$$

Let $u, v \in C([1-r, T], \mathbb{R})$. Then, for $t \in J$,

$$\begin{aligned}
|\mathcal{N}(u)(t) - \mathcal{N}(v)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, u_s) - f(s, v_s)| \frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i - 1} |h_i(s, u_s) - h_i(s, v_s)| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \|\mu\| \|u - v\|_{[1-r,T]} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \|\psi_i\| \|u - v\|_{[1-r,T]} \int_0^t (t-s)^{\beta_i - 1} ds \\
&= \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \|\mu\| \|u - v\|_{[1-r,T]} + \sum_{i=1}^m \frac{t^{\beta_i}}{\Gamma(\beta_i + 1)} \|\psi_i\| \|u - v\|_{[1-r,T]}.
\end{aligned}$$

Consequently,

$$\|\mathcal{N}(u)(t) - \mathcal{N}(v)(t)\|_{[1-r,T]} \leq \left(\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|\mu\| + \sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i + 1)} \|\psi_i\| \right) \|u - v\|_{[1-r,T]},$$

which implies that $\mathcal{N}$ is a contraction, and hence $\mathcal{N}$ has a unique fixed point by Banach's contraction principle.

Our second existence result for (1.1) – (1.2) is based on the following nonlinear alternative of Leray-Schauder.

Lemma 3.4. ***(Nonlinear alternative [11])***
Let $\mathbb{E}$ be a Banach space, C a closed, convex subset of $\mathbb{E}$, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(U)$ is a relatively compact subset of C) map. Then either

(i) F has a fixed point in U, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

Theorem 3.5. *Assume that assumptions* $(H2) - (H4)$ *hold. Then* (1.1) – (1.2) *has at least one solution on* $[1-r, T]$.

Proof We consider the operator $\mathcal{N} : C([1-r, T], \mathbb{R}) \to C([1-r, T], \mathbb{R})$ defined by (6).
We shall show that the operator $\mathcal{N}$ is continuous and completely continuous.
Step 1: $\mathcal{N}$ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([1-r,T],\mathbb{R})$. Let $\eta > 0$ such that $\|u_n\|_\infty \leq \eta$. Then

$$\begin{aligned}
|\mathcal{N}(u_n)(t) - \mathcal{N}(u)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,u_{ns}) - f(s,u_s)|\frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1}|h_i(s,u_{ns}) - h_i(s,u_s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{t}{s}\right)^{\alpha-1} \sup_{s\in[1,T]} |f(s,u_{ns}) - f(s,u_s)|\frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_0^T (t-s)^{\beta_i-1} \sup_{s\in[1,T]} |h_i(s,u_{ns}) - h_i(s,u_s)|ds \\
&\leq \frac{\|f(\cdot,u_{n\cdot}) - f(\cdot,u_\cdot)\|_\infty}{\Gamma(\alpha)}\int_1^T \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{\|h_i(\cdot,u_{n\cdot}) - h_i(\cdot,u_\cdot)\|_\infty}{\Gamma(\beta_i)}\int_0^T (t-s)^{\beta_i-1}ds \\
&\leq \frac{(\log T)^\alpha \|f(\cdot,u_{n\cdot}) - f(\cdot,u_\cdot)\|_\infty}{\Gamma(\alpha+1)} + \sum_{i=1}^m \frac{T^{\beta_i}\|h_i(\cdot,u_{n\cdot}) - h_i(\cdot,u_\cdot)\|_\infty}{\Gamma(\beta_i+1)}.
\end{aligned}$$

Since f and h_i are continuous functions, we have

$$\|\mathcal{N}(u_n)(t) - \mathcal{N}(u)(t)\|_\infty \leq \frac{(\log T)^\alpha \|f(\cdot,u_{n\cdot}) - f(\cdot,u_\cdot)\|_\infty}{\Gamma(\alpha+1)} + \sum_{i=1}^m \frac{T^{\beta_i}\|h_i(\cdot,u_{n\cdot}) - h_i(\cdot,u_\cdot)\|_\infty}{\Gamma(\beta_i+1)} \to 0,$$

as $n \to \infty$. Consequently, $\mathcal{N}$ is continuous.

Step 2: $\mathcal{N}$ maps bounded sets into bounded sets in $C([1-r,T],\mathbb{R})$.
Indeed, it is sufficient to show that for any $\eta^* > 0$ there exists a positive constant L such that for each $u \in \mathcal{B}_{\eta^*} := \{u \in C([1-r,T],\mathbb{R}) : \|u\|_\infty \leq r^*\}$, we have $\|\mathcal{N}(u)\|_\infty \leq L$.

$$\begin{aligned}
|\mathcal{N}(u)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,u_s)|\frac{ds}{s} + \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1}|h_i(s,u_s)|ds \\
&\leq \frac{\|p\|_\infty \Phi(\|u\|_{[1-r,T]})}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + \sum_{i=1}^m \frac{k}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1}ds \\
&\leq \frac{(\log T)^\alpha \|p\|_\infty \Phi(\eta^*)}{\Gamma(\alpha+1)} + \sum_{i=1}^m \frac{k\,T^{\beta_i}}{\Gamma(\beta_i+1)} := L.
\end{aligned}$$

Step 3: $\mathcal{N}$ maps bounded sets into equicontinuous sets of $C([1-r,T],\mathbb{R})$.
Let $t_1, t_2 \in [1,T],\ t_1 < t_2$, $\mathcal{B}_{\eta^*}$ be a bounded set of $C([1-r,T],\mathbb{R})$ as in **Step 2**, and let $u \in \mathcal{B}_{\eta^*}$. Then

$$\begin{aligned}
&|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right] f(s,u_s)\frac{ds}{s}\right. \\
&+ \left.\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1} f(s,u_s)\frac{ds}{s}\right| + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\left|\int_{t_1}^{t_2}(t_2-s)^{\beta_i-1}h_i(s,u_s)ds\right. \\
&+ \left.\int_0^{t_1}[(t_2-s)^{\beta_i-1} - (t_1-s)^{\beta_i-1}]h_i(s,u_s)ds\right| \\
&\leq \frac{\|p\|_\infty\Phi(\eta^*)}{\Gamma(\alpha)}\left(\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s} + \int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right) \\
&+ \sum_{i=1}^{m}\frac{k}{\Gamma(\beta_i)}\left(\int_{t_1}^{t_2}(t_2-s)^{\beta_i-1}ds + \int_0^{t_1}[(t_2-s)^{\beta_i-1} - (t_1-s)^{\beta_i-1}]ds\right).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)| &\leq \frac{\|p\|_\infty\Phi(\eta^*)}{\Gamma(\alpha)}\left(\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\right. \\
&+ \left.\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right) + \sum_{i=1}^{m}\frac{k}{\Gamma(\beta_i+1)}\left(t_2^{\beta_i} - t_1^{\beta_i}\right).
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

As a consequence of **Steps 1** to **3** it follows by the Arzelá-Ascoli theorem that $\mathcal{N} : C([1-r,T],\mathbb{R}) \to C([1-r,T],\mathbb{R})$ is continuous and completely continuous.

Step 4: We show that there exists an open set $U \subseteq C([1-r,T],\mathbb{R})$ with $u \neq \lambda\mathcal{N}(u)$ for $\lambda \in (0,1)$ and $u \in \partial U$.
Let $u \in C([1-r,T],\mathbb{R})$ and $u = \lambda\mathcal{N}(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,T]$, we have

$$u(t) = \lambda\left(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,u_s)\frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i}h_i(s,u_s)ds\right).$$

$$\begin{aligned}
|u(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} p(s)\Phi(\|u_s\|_C)\frac{ds}{s} \\
&+ \sum_{i=1}^{m}\frac{k}{\Gamma(\beta_i)}\int_0^t(t-s)^{\beta_i-1}ds \\
&\leq \frac{\|p\|_\infty\Phi(\|u\|_{[1-r,T]})(\log T)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^{m}\frac{k\,T^{\beta_i}}{\Gamma(\beta_i+1)},
\end{aligned}$$

which can be expressed as

$$\frac{\|u\|_{[1-r,T]}}{\frac{\|p\|_\infty\Phi(\|u\|_{[1-r,T]})(\log T)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^{m}\frac{k\,T^{\beta_i}}{\Gamma(\beta_i+1)}} \leq 1.$$

In view of $(H4)$, there exists M such that $\|u\|_{[1-r,T]} \neq M$. Let us set

$$U = \{u \in C([1-r,T],\mathbb{R}) : \|u\|_{[1-r,T]} < M\}.$$

Note that the operator $\mathcal{N} : \overline{U} \to C([1-r,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda\mathcal{N}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathcal{N}$ has a fixed point $u \in \overline{U}$ which is a solution of (1.1) − (1.2). This completes the proof.

4. An illustrative example

In this section, we illustrate the existence results obtained in Section 3 with the aid of the following example. Consider the following Hadamard and Riemann-Liouville fractional neutral functional integrodifferential equation:

$$\begin{cases} {}_HD^{\frac{1}{2}}\left[u(t) - \sum_{i=1}^{3} I^{\frac{3i+2}{3}} \frac{|u|_t}{(i+4\log t)(1+|u|_t)}\right] = \frac{1}{4+e-t^2}\left(\frac{|u|_t}{2(1+|u|_t)} + \frac{1}{4}\right),\ t \in [1,\ \sqrt{e}\,], \\ u(t) = \varphi(t),\ t \in [1-r,1]. \end{cases} \tag{7}$$

Here, $\alpha = \frac{1}{2}$, $m = 3$, $\beta_1 = \frac{5}{3}$, $\beta_2 = \frac{8}{3}$, $\beta_3 = \frac{11}{3}$, $T = \sqrt{e}$, and
$h_i(t,x) = \frac{|x|}{(i+4\log t)(1+|x|)}$, $i = 1,2,3$, $\quad f(t,x) = \frac{1}{4+e-t^2}\left(\frac{|x|}{2(1+|x|)} + \frac{1}{4}\right)$.
Clearly,

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left|\frac{1}{2(4+e-t^2)}\left(\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right)\right| \\ &\leq \frac{1}{2(4+e-t^2)}\frac{\|x-y\|}{(1+|x|)(1+|y|)} \\ &\leq \frac{1}{2(4+e-t^2)}\|x-y\|, \end{aligned}$$

and

$$\begin{aligned} |h_i(t,x) - h_i(t,y)| &= \left|\frac{1}{(i+4\log t)}\left(\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right)\right| \\ &\leq \frac{1}{(i+4\log t)}\frac{\|x-y\|}{(1+|x|)(1+|y|)} \\ &\leq \frac{1}{(i+4\log t)}\|x-y\|, \quad \text{for } x,y \in \mathbb{R},\ i = 1,2,3. \end{aligned}$$

Hence, assumption (H1) hold with $\mu(t) = \frac{1}{2(4+e-t^2)}$, $\psi_i(t) = \frac{1}{(i+4\log t)}$, $\|\mu\| = \frac{1}{8}$ and $\|\psi_i\| = \frac{1}{i+2}$. Since,

$$\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\|\mu\| + \sum_{i=1}^{m}\frac{T^{\beta_i}}{\Gamma(\beta_i+1)}\|\psi_i\| \approx 0.9309 < 1,$$

therefore, by Theorem 3.3, there exists a unique solution for (7) on the interval $[1-r,\ \sqrt{e}\,]$.

Also, we have $|f(t,x)| \leq \frac{1}{4}(\frac{3}{4})$ and $|h_i(t,x)| \leq \frac{1}{3+4\log\sqrt{e}} = \frac{1}{5}$. Thus we get $p(t) = \frac{1}{4}$, $\Phi(\|x\|) = \frac{3}{4}$ and $k = \frac{1}{5}$. Further, using the assumption (H4),

$$\frac{M}{\frac{\|p\|_\infty\Phi(M)(\log T)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^{m}\frac{k\ T^{\beta_i}}{\Gamma(\beta_i+1)}} > 1.$$

We find that $M > 0.729604$. Therefore, all the conditions of Theorem 3.5 are satisfied. Hence, problem (7) has at least one solution on $[1-r,\ \sqrt{e}\,]$.

References

[1] M. I. Abbas, *Existence and uniqueness of solution for a boundary value problem of fractional order involving two Caputos fractional derivatives*, Advances in Difference Equations (2015), 2015:252.

[2] M. I. Abbas, *Ulam Stability of Fractional Impulsive Differential Equations with Riemann-Liouville Integral Boundary Conditions*, Journal of Contemporary Mathematical Analysis, Vol. 50, No. 5 (2015) 209-219.

[3] R. Agarwal, Y. Zhou, and Y. He, *Existence of fractional neutral functional differential equations*, Comput. Math. Appl., vol. 59 (2010) 1095-1100.

[4] B. Ahmad and S. K. Ntouyas, *Eexistence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations*, EJDE, Vol. 2017, No. 36 (2017), 1-11.

[5] B. Ahmad, S. K. Ntouyas, and J. Tariboonc, *A study of mixed Hadamard and RiemannLiouville fractional integro-differential inclusions via endpoint theory*, Appl. Math. Letters 52 (2016) 9-14.

[6] C. Bianca , M. Pennisi, S. Motta , M.A. Ragusa, *Immune system network and cancer vaccine*, AIP Conference Proceedings, vol. 1389 (2011), 945-948, doi: 10.1063/1.3637764.

[7] C. Bianca , F. Pappalardo, S. Motta, M.A. Ragusa, *Persistence analysis in a Kolmogorov-type model for cancer-immune system competition*, AIP Conference Proceedings, vol. 1558 (2013), 1797-1800, doi: 10.1063/1.4825874.

[8] P. L. Butzer, A. A. Kilbas, J. J. Trujillo, *Compositions of Hadamard-type fractional integration operators and the semigroup property*, J. Math. Anal. Appl. 269 (2002), 387-400.

[9] P. L. Butzer, A. A. Kilbas, J. J. Trujillo, *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl. 269 (2002), 1-27.

[10] P. L. Butzer, A. A. Kilbas, J. J. Trujillo, *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals*, J. Math. Anal. Appl. 270 (2002), 1-15.

[11] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[12] J. Hadamard, *Essai sur l'etude des fonctions donnees par leur developpment de Taylor*, J. Mat. Pure Appl. Ser. 8 (1892) 101-186.

[13] A.A. Kilbas, Hari M. Srivastava, J. Juan Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[14] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers, vol. 463, 1999.

[15] Q. Ma, R. Wang, J. Wang, Y. Ma, *Qualitative analysis for solutions of a certain more generalized two-dimensional fractional differential system with Hadamard derivative*, Appl. Math. Comput. (2015), 257:436-45.

[16] J. Hale and S. Verduyn Lunel, *Introduction to Functional Differential Equations*, ser. Applied Mathematical Sciences. New York: Springer, vol. 99, 1993.

[17] J. Wang and Y. Zhang, *On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives*, Appl. Math. Letters 39 (2015) 85-90.