Filomat 32:13 (2018), 4627–4638
https://doi.org/10.2298/FIL1813627N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Sherman-Steffensen Type Inequalities

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Abstract. In this work, Sherman-Steffensen type inequalities for convex functions with not necessarily non-negative coefficients are established by using Steffensen's conditions. The Brunk, Bellman and Olkin type inequalities are derived as special cases of the Sherman-Steffensen inequality. The superadditivity of the Jensen-Steffensen functional is investigated via Steffensen's condition for the sequence of the total sums of all entries of the involved vectors of coefficients. Some results of Barić et al. [2] and of Krnić et al. [11] on the monotonicity of the functional are recovered. Finally, a Sherman-Steffensen type inequality is shown for a row graded matrix.

1. Preliminaries and motivation

The celebrated *weighted Jensen's inequality* [5, 14, 18] claims that if f is a real convex function defined on an interval $I \subset \mathbb{R}$, and real coefficients $p_1, p_2, \dots, p_n$ are such that

$$p_i \geq 0, \quad i = 1, 2, \dots, n, \quad \text{and} \quad P_n = p_1 + p_2 + \dots + p_n > 0 \tag{1}$$

then, for any $x_1, x_2, \dots, x_n \in I$,

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{2}$$

Theorem A. (Steffensen [21].) *Assume that f is a real convex function defined on an interval $I \subset \mathbb{R}$. Let $w_1, w_2, \dots, w_n \in \mathbb{R}$. Denote $W_i = w_1 + w_2 + \dots + w_i$ for $i = 1, 2, \dots, n$.*

If

$$W_n \geq W_i \geq 0, \quad i = 1, 2, \dots, n, \quad \text{and} \quad W_n > 0 \quad \text{(Steffensen's condition)}, \tag{3}$$

then, for any monotonic n-tuple $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$,

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{4}$$

Inequality (4) admitting (not necessarily non-negative) weights w_i and satisfying Steffensen's condition (3) is called *Jensen-Steffensen inequality* [1].

2010 *Mathematics Subject Classification*. Primary 26A51, 52A40 ; Secondary 15B51, 26A48
Keywords. convex function, Jensen-Steffensen's inequality, Jensen-Steffensen's functional, Sherman-Steffensen's inequality, column stochastic matrix, doubly stochastic matrix, row graded matrix
Received: 04 January 2018; Accepted: 01 May 2018
Communicated by Fuad Kittaneh
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Example 1.1. Suppose that $f : I \to \mathbb{R}$ is convex on an interval $I = [0, x_1]$ with $f(0) \le 0$, and $x_1 \ge x_2 \ge \ldots \ge x_n \ge 0$.

Brunk inequality [6] asserts that for

$$1 \ge h_1 \ge h_2 \ge \ldots \ge h_n \ge 0, \tag{5}$$

it holds that

$$f\left(\sum_{i=1}^{n} (-1)^{i-1} h_i x_i\right) \le \sum_{i=1}^{n} (-1)^{i-1} h_i f(x_i). \tag{6}$$

A special case of inequality (6) for $h_1 = h_2 = \ldots = h_n = 1$ is *Bellman's inequality* (see [3, p. 462]).

In fact, by considering the $(n+1)$-vector $(x_1, \ldots, x_n, 0)$ and the $(n+1)$-sequence of weights

$$\left(h_1, -h_2, h_3, -h_4, \ldots (-1)^{n-1} h_n, 1 - \sum_{i=1}^{n} (-1)^{i-1} h_i\right),$$

we see from (3) that Steffensen's condition is fulfilled. So, it now follows from inequality (4) in Theorem A that

$$\begin{aligned} & f\left(\sum_{i=1}^{n} (-1)^{i-1} h_i x_i + \left(1 - \sum_{i=1}^{n} (-1)^{i-1} h_i\right) \cdot 0\right) \\ \le & \sum_{i=1}^{n} (-1)^{i-1} h_i f(x_i) + \left(1 - \sum_{i=1}^{n} (-1)^{i-1} h_i\right) f(0) \\ \le & \sum_{i=1}^{n} (-1)^{i-1} h_i f(x_i), \end{aligned}$$

since $f(0) \le 0$ and $1 - \sum\limits_{i=1}^{n} (-1)^{i-1} h_i \ge 0$ (see (5)).

We finish this example by the remark that resignation from the assumption $f(0) \le 0$ leads to the first above inequality only, which is a result due to Olkin [17].

We return to relevant notation, definitions and theorems.

Let $f : I \to \mathbb{R}$ be a function on an interval $I \subset \mathbb{R}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{p} \in \mathcal{P}_n^0$, where

$$\mathcal{P}_n^0 = \{\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : p_i \ge 0, \ \ P_n > 0\} \quad \text{with} \quad P_n = \sum_{i=1}^{n} p_i.$$

The *Jensen functional* is defined by (see [8])

$$J(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right). \tag{7}$$

Equivalently,

$$J(f, \mathbf{x}, \mathbf{p}) = \langle \mathbf{p}, f(\mathbf{x}) \rangle - \langle \mathbf{p}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle}\right), \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, $\mathbf{e} = (1, 1, \ldots, 1) \in \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $f(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_n))$.

By Jensen's inequality,

$$J(f, \mathbf{x}, \mathbf{p}) \geq 0 \quad \text{for a convex function } f.$$

Theorem B. [8] *If $f : I \to \mathbb{R}$ is a convex function then the function $\mathbf{p} \to J(f, \mathbf{x}, \mathbf{p})$, $\mathbf{p} \in \mathcal{P}_n^0$, is superadditive for any $\mathbf{x} \in I^n$, i.e.,*

$$J(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq J(f, \mathbf{x}, \mathbf{p}) + J(f, \mathbf{x}, \mathbf{q}) \quad \textit{for} \quad \mathbf{p}, \mathbf{q} \in \mathcal{P}_n^0. \tag{9}$$

In consequence, the function $\mathbf{p} \to J(f, \mathbf{x}, \mathbf{p})$, $\mathbf{p} \in \mathcal{P}_n^0$, is monotone for any $\mathbf{x} \in I^n$, i.e.,

$$\mathbf{q} \leq \mathbf{p} \quad \textit{implies} \quad J(f, \mathbf{x}, \mathbf{q}) \leq J(f, \mathbf{x}, \mathbf{p}) \quad \textit{for} \quad \mathbf{q}, \mathbf{p} - \mathbf{q} \in \mathcal{P}_n^0. \tag{10}$$

See [12] for some refinements and converses of the Jensen inequality obtained with the help of Theorem B [8].

We say that a vector $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ is *majorized* by a vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, written as $\mathbf{y} \prec \mathbf{x}$, if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]} \quad \text{for } k = 1, 2, \dots, n$$

with equality for $k = n$, where the symbols $x_{[i]}$ and $y_{[i]}$ denote the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively (see [13, p. 8]).

An $n \times m$ real matrix $\mathbf{S} = (s_{ij})$ is called *column stochastic* if $s_{ij} \geq 0$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and all column sums of $\mathbf{S}$ are equal to 1, i.e., $\sum\limits_{i=1}^{n} s_{ij} = 1$ for $j = 1, 2, \dots, m$. If in addition $m = n$ and the transpose $\mathbf{S}^T = (s_{ji})$ of $\mathbf{S} = (s_{ij})$ is column stochastic, then $\mathbf{S}$ is called *doubly stochastic*.

An important relationship between majorization and double stochasticity is the following (see [13, p. 33]): for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{y} \prec \mathbf{x} \text{ if and only if } \mathbf{y} = \mathbf{x}\mathbf{S} \quad \text{for some doubly stochastic } n \times n \text{ matrix } \mathbf{S}. \tag{11}$$

The next result is called *Majorization Theorem* (see [13, pp. 92-93]).

Theorem C. (Schur [20], Hardy-Littlewood-Pólya [9] and Karamata [10].) *Assume that f is a real convex function defined on an interval $I \subset \mathbb{R}$.*

Then, for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in I^n$,

$$\mathbf{y} \prec \mathbf{x} \quad \textit{implies} \quad \sum_{i=1}^{n} f(y_i) \leq \sum_{i=1}^{n} f(x_i). \tag{12}$$

In [15] the author showed Sherman - Steffensen inequality (15) with non-negative coefficients b_j and with not necessarily non-negative matrix $\mathbf{S}$.

Theorem D. (Niezgoda [15].) *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$, $\mathbf{y} = (y_1, y_2, \dots, y_m) \in I^m$, $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \dots, b_m) \in \mathbb{R}_+^m$. Assume that $\mathbf{x}$ is monotonic.*

If

$$\mathbf{y} = \mathbf{x}\mathbf{S} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{S}^T \tag{13}$$

for some $n \times m$ matrix $\mathbf{S} = (s_{ij})$ such that for each $j = 1, \dots, m$,

$$0 \leq S_{ij} \leq S_{nj} = 1 \quad \textit{for } i = 1, \dots, n, \tag{14}$$

where $S_{ij} = \sum_{k=1}^{i} s_{kj}$, *then*

$$\sum_{j=1}^{m} b_j f(y_j) \le \sum_{i=1}^{n} a_i f(x_i). \tag{15}$$

In the case of non-negative entries of $\mathbf{S}$ with total sums $S_{nj} = 1$, $j = 1, \dots, m$, condition (14) can be removed and then Theorem D becomes Sherman's Theorem [19].

Some applications of Sherman's Theorem can be found in [4, 7, 15, 16].

Remark 1.2. It is not hard to check that Jensen - Steffensen's inequality (4) is a special form of Sherman - Steffensen's inequality (15) with $m = 1$ and $b_1 = 1$.

For $\mathbf{b} = (1, 1, \dots, 1) \in \mathbb{R}_+^m$, inequality (15) yields

$$\sum_{j=1}^{m} f(y_j) \le \sum_{i=1}^{n} a_i f(x_i),$$

where $a_i = \sum_{j=1}^{m} s_{ij}$ is the ith row sum of $\mathbf{S}$, $i = 1, \dots, n$.

For instance, if $a_i = \sum_{j=1}^{m} s_{ij} = 1$ for $i = 1, \dots, n$, then $m = n$ and the last inequality takes the form of HLPK inequality (12).

Example 1.3. Suppose that $f : I \to \mathbb{R}$ is convex on an interval $I = [0, x_1]$ with $f(0) \le 0$, and $\mathbf{x} = (x_1, x_2, \dots, x_n)$ with $x_1 \ge x_2 \ge \dots \ge x_n \ge 0$, and $b_1, \dots, b_m$ are non-negative coefficients.

Let

$$\mathbf{H} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ h_{21} & h_{22} & \cdots & h_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nm} \end{pmatrix}$$

be an $n \times m$ real matrix such that

$$1 \ge h_{1j} \ge h_{2j} \ge \dots \ge h_{nj} \ge 0 \quad \text{for } j = 1, \dots, m. \tag{16}$$

Consider the $(n+1) \times m$ matrix

$$\mathbf{S} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ -h_{21} & -h_{22} & \cdots & -h_{2m} \\ h_{31} & h_{32} & \cdots & h_{3m} \\ -h_{41} & -h_{42} & \cdots & -h_{4m} \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{n-1} h_{n1} & (-1)^{n-1} h_{n2} & \cdots & (-1)^{n-1} h_{nm} \\ 1 - \sum_{i=1}^{n} (-1)^{i-1} h_{i1} & 1 - \sum_{i=1}^{n} (-1)^{i-1} h_{i2} & \cdots & 1 - \sum_{i=1}^{n} (-1)^{i-1} h_{im} \end{pmatrix},$$

It is not hard to verify by (16) that Steffensen's condition (14) is satisfied for each column of $\mathbf{S}$.

It is obvious that

$$\mathbf{S}^T = \begin{pmatrix} h_{11} & -h_{21} & h_{31} & -h_{41} & \cdots & (-1)^{n-1}h_{n1} & 1-\sum\limits_{i=1}^{n}(-1)^{i-1}h_{i1} \\ h_{12} & -h_{22} & h_{32} & -h_{42} & \cdots & (-1)^{n-1}h_{n2} & 1-\sum\limits_{i=1}^{n}(-1)^{i-1}h_{i2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ h_{1m} & -h_{2m} & h_{3m} & -h_{4m} & \cdots & (-1)^{n-1}h_{nm} & 1-\sum\limits_{i=1}^{n}(-1)^{i-1}h_{im} \end{pmatrix}.$$

It now follows from Theorem D applied to the $1 \times (n+1)$ monotonic vector $(x_1, \ldots, x_n, 0)$ that

$$\sum_{j=1}^{m} b_j f\left(\sum_{i=1}^{n}(-1)^{i-1}h_{ij}x_i\right)$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{m} b_j(-1)^{i-1}h_{ij}f(x_i) + \sum_{j=1}^{m} b_j\left(1-\sum_{i=1}^{n}(-1)^{i-1}h_{ij}\right)f(0) \tag{17}$$

with

$$a_i = \sum_{j=1}^{m} b_j(-1)^{i-1}h_{ij} \quad \text{for } i = 1, \ldots, n, \text{ and} \quad a_{n+1} = \sum_{j=1}^{m} b_j\left(1-\sum_{i=1}^{n}(-1)^{i-1}h_{ij}\right).$$

Since $b_j \geq 0$ for $j = 1, \ldots, m$, $f(0) \leq 0$ and $1 - \sum\limits_{i=1}^{n}(-1)^{i-1}h_{ij} \geq 0$ (see (5)), we get

$$\sum_{j=1}^{m} b_j\left(1-\sum_{i=1}^{n}(-1)^{i-1}h_{ij}\right)f(0) \leq 0.$$

Therefore the right hand side of inequality (17) is estimated by upper bound

$$\sum_{i=1}^{n}\sum_{j=1}^{m} b_j(-1)^{i-1}h_{ij}f(x_i).$$

Thus we obtain a generalization of Brunk inequality (cf. Example 1.1):

$$\sum_{j=1}^{m} b_j f\left(\sum_{i=1}^{n}(-1)^{i-1}h_{ij}x_i\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} b_j(-1)^{i-1}h_{ij}f(x_i). \tag{18}$$

In particular, when $\mathbf{H}$ is the matrix of ones, that is, $h_{ij} = 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then we get a Bellman like inequality:

$$\sum_{j=1}^{m} b_j f\left(\sum_{i=1}^{n}(-1)^{i-1}x_i\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} b_j(-1)^{i-1}f(x_i) \tag{19}$$

with

$$a_i = \sum_{j=1}^{m} b_j(-1)^{i-1} \quad \text{for } i = 1, \ldots, n.$$

We also conclude that the resignation from the assumption $f(0) \leq 0$ leads to inequality (17), only, which corresponds to an Olkin's result [17] (see Example 1.1).

Note that one of the main assumptions of Theorem D is the *non-negativity* of the weights b_j. The purpose of the present paper is to extend this theorem to some other classes of weights by employing the above-mentioned Steffensen's condition (3).

The paper is arranged in the folowing way. In Section 1 we collect some needed terminology, definitions and facts. In Example 1.1, Brunk, Olkin and Bellman's inequalities are also presented. In Theorem D we demonstrate a Sherman-Steffensen type inequality with non-negative coefficients b_j, $j = 1, \dots, m$. As consequence, we show a multivariate Brunk like inequality (see Example 1.3).

In Section 2 we establish the property of superadditivity of the Jensen-Steffensen functional (see Theorem 2.1). Here the main assumption is Steffensen's condition for some real l-tuple prepared by total sums of all entries of the involved coefficient vectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_l$ in $\mathbb{R}^n$. The case $l = 2$ leads to some resuls due to Barić et al. [2] and to Krnić et al. [11], and gives the monotonicity of the functional (see Corollary 2.4).

In Section 3 we investigate the Sherman-Steffensen type inequality for possibly negative coefficients b_js. To do so, we employ the property of monotonicity of the Jensen-Steffensen functional. Again, the basic assumptions in Theorem 3.1 are Steffensen's conditions:
(i) for the last column of the matrix,
(ii) for the pairs of the total sums of entries of any column and of its difference with the next column of the matrix,
(iii) for the difference of any two successive columns of the matrix.
Finally, we also apply $n \times m$ row graded matrices in order to simplify conditions ensuring the validity of the inequality.

2. Superadditivity of the Jensen-Steffensen functional

For a function $f : I \to \mathbb{R}$ on an interval $I \subset \mathbb{R}$, $\mathbf{x} \in I^n$ and $\mathbf{p} \in \mathbb{R}^n$ with $P_n = \langle \mathbf{p}, \mathbf{e} \rangle \neq 0$, we define the Jensen-Steffensen functional by

$$J(f, \mathbf{x}, \mathbf{p}) = \langle \mathbf{p}, f(\mathbf{x}) \rangle - \langle \mathbf{p}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}, \mathbf{x} \rangle}{\langle \mathbf{p}, \mathbf{e} \rangle} \right). \tag{20}$$

Here we *do not assume* that $\mathbf{p}$ is non-negative n-tuple.

We begin this section with a complement to Theorem B, which also extends a result of Krnić et al. [11, Theorem 2.1]. A consequence of Theorem 2.1 is Corollary 2.4 that will be used in the proofs of Theorems 3.1-3.2.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. Let $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^n$ and $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_l \in \mathbb{R}^n$ be such that $\langle \mathbf{p}_j, \mathbf{e} \rangle \neq 0$ for $j = 1, \dots, l$, and*

$$\sum_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e} \rangle > 0 \quad \text{and} \quad \sum_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e} \rangle \geq \sum_{j=1}^{i} \langle \mathbf{p}_j, \mathbf{e} \rangle \geq 0 \quad \text{for } i = 1, \dots, l. \tag{21}$$

Then for any monotonic n-tuple $\mathbf{x} \in I^n$,

$$\langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{x} \rangle}{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{e} \rangle} \right) \leq \sum_{i=1}^{l} \langle \mathbf{p}_i, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x} \rangle}{\langle \mathbf{p}_i, \mathbf{e} \rangle} \right). \tag{22}$$

Moreover, the Jensen-Steffensen functional, under Steffensen's condition (21), is superadditive, i.e., (22) takes the equivalent form

$$J\left(f, \mathbf{x}, \sum_{i=1}^{l} \mathbf{p}_i \right) \geq \sum_{i=1}^{l} J(f, \mathbf{x}, \mathbf{p}_i). \tag{23}$$

Proof. The following identity holds

$$\frac{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{x}\rangle}{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{e}\rangle} = \sum_{i=1}^{l} w_i \frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}, \tag{24}$$

where

$$w_i = \frac{\langle \mathbf{p}_i, \mathbf{e}\rangle}{\sum\limits_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e}\rangle} \quad \text{for } i = 1, \ldots, l. \tag{25}$$

Since $f : I \to \mathbb{R}$ is convex on I, and $1 = W_l \geq W_i \geq 0$ for $i = 1, \ldots, l$ (see (21)), where

$$W_i = \sum_{j=1}^{i} w_j = \sum_{j=1}^{i} \frac{\langle \mathbf{p}_j, \mathbf{e}\rangle}{\sum\limits_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e}\rangle} \quad \text{for } i = 1, \ldots, l,$$

by Jensen-Steffensen's inequality (see Theorem A) one has

$$f\left(\frac{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{x}\rangle}{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{e}\rangle}\right) = f\left(\sum_{i=1}^{l} w_i \frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}\right) \leq \sum_{i=1}^{l} w_i f\left(\frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}\right)$$

$$= \sum_{i=1}^{l} \frac{\langle \mathbf{p}_i, \mathbf{e}\rangle}{\sum\limits_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e}\rangle} f\left(\frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}\right) = \frac{1}{\sum\limits_{j=1}^{l} \langle \mathbf{p}_j, \mathbf{e}\rangle} \sum_{i=1}^{l} \langle \mathbf{p}_i, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}\right). \tag{26}$$

Now, it is not hard to see that (26) implies (22).

By using the notation (20) for the Jensen-Steffensen functional, we have

$$\sum_{i=1}^{l} J(f, \mathbf{x}, \mathbf{p}_i) = \sum_{i=1}^{l} \langle \mathbf{p}_i, f(\mathbf{x})\rangle - \sum_{i=1}^{l} \langle \mathbf{p}_i, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}_i, \mathbf{x}\rangle}{\langle \mathbf{p}_i, \mathbf{e}\rangle}\right)$$

and

$$J\left(f, \mathbf{x}, \sum_{i=1}^{l} \mathbf{p}_i\right) = \langle \sum_{i=1}^{l} \mathbf{p}_i, f(\mathbf{x})\rangle - \langle \sum_{i=1}^{l} \mathbf{p}_i, \mathbf{e}\rangle f\left(\frac{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{x}\rangle}{\langle \sum\limits_{i=1}^{l} \mathbf{p}_i, \mathbf{e}\rangle}\right).$$

Therefore (22) can be restated as

$$J\left(f, \mathbf{x}, \sum_{i=1}^{l} \mathbf{p}_i\right) \geq \sum_{i=1}^{l} J(f, \mathbf{x}, \mathbf{p}_i),$$

as claimed. ∎

Remark 2.2. In context of Theorem 2.1, if the coefficient vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_l \in \mathbb{R}^n$ have non-negative entries with positive sums, then the requirement (21) holds valid. Thus Theorem 2.1 is a generalization of Theorem B.

Remark 2.3. Observe that the requirement (21) can be viewed as the Steffensen's condition related to the l-tuple

$$(\langle \mathbf{p}_1, \mathbf{e}\rangle, \langle \mathbf{p}_2, \mathbf{e}\rangle, \dots, \langle \mathbf{p}_l, \mathbf{e}\rangle).$$

The next discussion concerns the monotonicity property (30) of the Jensen-Steffensen functional $\mathbf{p} \to J(f, \mathbf{x}, \mathbf{p})$. To this end, a sufficient condition is (27), which relaxes the standard requirement $\mathbf{p} \geq \mathbf{q}$ with $\langle \mathbf{p}, \mathbf{e}\rangle > 0$, $\langle \mathbf{q}, \mathbf{e}\rangle > 0$ and $p_i, q_i \geq 0$ (cf. [2, 11]).

Corollary 2.4 (Cf. Barić et al. [2], Krnić et al. [11, Theorem 2.1]). *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. Let $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^n$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ be such that*

$$P_n = \langle \mathbf{p}, \mathbf{e}\rangle > 0, \quad Q_n = \langle \mathbf{q}, \mathbf{e}\rangle > 0 \quad \text{and} \quad P_n - Q_n = \langle \mathbf{p} - \mathbf{q}, \mathbf{e}\rangle > 0. \tag{27}$$

Then for any monotonic n-tuple $\mathbf{x} \in I^n$,

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) + J(f, \mathbf{x}, \mathbf{q}). \tag{28}$$

If in addition the entries of $\mathbf{p} - \mathbf{q}$ fulfill the Steffensen's condition, that is,

$$P_n - Q_n > 0 \quad \text{and} \quad P_n - Q_n \geq P_i - Q_i \geq 0 \quad \text{for } i = 1, \dots, n, \tag{29}$$

then for any monotonic n-tuple $\mathbf{x} \in I^n$,

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{q}). \tag{30}$$

Proof. We denote

$$\mathbf{p}_1 = \mathbf{p} - \mathbf{q} \quad \text{and} \quad \mathbf{p}_2 = \mathbf{q}.$$

Then via (27) we have

$$\langle \mathbf{p}_1 + \mathbf{p}_2, \mathbf{e}\rangle > 0 \quad \text{and} \quad \langle \mathbf{p}_1 + \mathbf{p}_2, \mathbf{e}\rangle \geq \langle \mathbf{p}_1, \mathbf{e}\rangle \geq 0.$$

It now follows from Theorem 2.1 for $l = 2$ that

$$J(f, \mathbf{x}, \mathbf{p}_1 + \mathbf{p}_2) \geq J(f, \mathbf{x}, \mathbf{p}_1) + J(f, \mathbf{x}, \mathbf{p}_2), \tag{31}$$

i.e.,

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) + J(f, \mathbf{x}, \mathbf{q}),$$

as was to be proved.

Assume in addition that (29) holds, that is, the entries of $\mathbf{p}_1 = \mathbf{p} - \mathbf{q}$ fulfill the Steffensen's condition. Then we get $J(f, \mathbf{x}, \mathbf{p}_1) \geq 0$ (see Theorem A). For this reason, (31) gives

$$J(f, \mathbf{x}, \mathbf{p}_1 + \mathbf{p}_2) \geq J(f, \mathbf{x}, \mathbf{p}_2).$$

In other words,

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{q}),$$

completing the proof. ∎

Remark 2.5. Likewise in Remark 2.3, the restriction (27) is the Steffensen's condition related to the pair $(\langle \mathbf{p} - \mathbf{q}, \mathbf{e}\rangle, \langle \mathbf{q}, \mathbf{e}\rangle)$.

3. Sherman - Steffensen like inequalities with possibly negative coefficients b_j

We now give another result of Sherman-Steffensen type, where the non-negativity of the coefficients b_i is relaxed at the expense of restricting the matrix $\mathbf{G}$.

For an $n \times m$ matrix $\mathbf{G} = (g_{ij})$ we denote the ith sum of the jth column by $G_{ij} = \sum_{k=1}^{i} g_{kj}$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. The jth column sum of $\mathbf{G}$ is denoted by $G_{nj} = \sum_{k=1}^{n} g_{kj}$. The symbol $\mathbf{g}_j$ stands for the jth column of $\mathbf{G}$ for $j = 1, 2, \ldots, m$.

Theorem 3.1. *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$, $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in I^m$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$. Denote $B_j = b_1 + b_2 + \ldots + b_j$ for $j = 1, 2, \ldots, m$.*

If

$$B_j \geq 0, \quad j = 1, 2, \ldots, m, \tag{32}$$

$$\mathbf{x} \textit{ is monotonic}, \tag{33}$$

and

$$\mathbf{y} = \mathbf{xG} \quad \textit{and} \quad \mathbf{a} = \mathbf{bG}^T \tag{34}$$

for some $n \times m$ matrix $\mathbf{G} = (g_{ij})$ satisfying

$$G_{nm} > 0 \quad \textit{and} \quad G_{nm} \geq G_{im} \geq 0, \quad i = 1, \ldots, n, \tag{35}$$

$$G_{nj} > 0 \quad \textit{and} \quad G_{n,j+1} > 0, \quad j = 1, \ldots, m-1, \tag{36}$$

$$G_{nj} - G_{n,j+1} > 0 \quad \textit{and} \quad G_{nj} - G_{n,j+1} \geq G_{ij} - G_{i,j+1} \geq 0, \quad i = 1, \ldots, n, \; j = 1, \ldots, m-1, \tag{37}$$

then

$$\sum_{j=1}^{m} b_j G_{nj} f\left(\frac{y_j}{G_{nj}}\right) \leq \sum_{i=1}^{n} a_i f(x_i). \tag{38}$$

Proof. Remind that the Jensen-Steffensen functional corresponding to the jth column $\mathbf{g}_j$ of $\mathbf{G}$ is given by

$$J(f, \mathbf{x}, \mathbf{g}_j) = \sum_{i=1}^{n} g_{ij} f(x_i) - G_{nj} f\left(\frac{1}{G_{nj}} \sum_{i=1}^{n} g_{ij} x_i\right) \quad \text{for } j = 1, 2, \ldots, m. \tag{39}$$

We shall prove with the aid of condition (32) that

$$\sum_{j=1}^{m} b_j J(f, \mathbf{x}, \mathbf{g}_j) \geq 0. \tag{40}$$

To see this we invoke to Abel's identity

$$\sum_{j=1}^{m} b_j J(f, \mathbf{x}, \mathbf{g}_j) = \sum_{j=1}^{m-1} \Big(J(f, \mathbf{x}, \mathbf{g}_j) - J(f, \mathbf{x}, \mathbf{g}_{j+1}) \Big) B_j + J(f, \mathbf{x}, \mathbf{g}_m) B_m. \tag{41}$$

By using (35) and Jensen-Steffensen's inequality (see Theorem A), we obtain

$$G_{nm} f\left(\sum_{i=1}^{n} \frac{g_{im}}{G_{nm}} x_i\right) \leq \sum_{i=1}^{n} g_{im} f(x_i).$$

In other words, we have

$$J(f, \mathbf{x}, \mathbf{g}_m) \geq 0. \tag{42}$$

Furthermore, by (32) and (42) we have

$$J(f, \mathbf{x}, \mathbf{g}_m) B_m \geq 0. \tag{43}$$

By (36)-(37) we have

$$G_{nj} > 0, \quad G_{n,j+1} > 0 \quad \text{and} \quad G_{nj} - G_{n,j+1} > 0, \quad j = 1, \ldots, m-1. \tag{44}$$

In light of (37), (44) and Corollary 2.4 applied to $\mathbf{g}_j$ and $\mathbf{g}_{j+1}$, we see that

$$J(f, \mathbf{x}, \mathbf{g}_j) \geq J(f, \mathbf{x}, \mathbf{g}_{j+1}) \quad \text{for } j = 1, 2, \ldots, m-1. \tag{45}$$

By combining (32), (41), (43) and (45), we deduce that (40) is satisfied, as claimed.

We now deduce from (39) and (40) that

$$\sum_{j=1}^{m} b_j G_{nj} f\left(\frac{1}{G_{nj}} \sum_{i=1}^{n} g_{ij} x_i\right) \leq \sum_{j=1}^{m} b_j \sum_{i=1}^{n} g_{ij} f(x_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j g_{ij} f(x_i). \tag{46}$$

It follows from the first equality of (34) that $(y_1, y_2, \ldots, y_m) = (x_1, x_2, \ldots, x_n)\mathbf{G}$. That is, $y_j = \sum\limits_{i=1}^{n} g_{ij} x_i$, $j = 1, 2, \ldots, m$. The second equality of (34) gives $a_i = \sum\limits_{j=1}^{m} b_j g_{ij}$, $i = 1, 2, \ldots, n$. So, we find from (46) that

$$\sum_{j=1}^{m} b_j G_{nj} f\left(\frac{y_j}{G_{nj}}\right) \leq \sum_{i=1}^{n} a_i f(x_i),$$

which completes the proof of inequality (38). ■

We now discuss some simplifications of the assumptions in Theorem 3.1.

Theorem 3.2. *Under the assumptions of Theorem 3.1 with the condition (37) replaced by*

$$G_{nj} - G_{n,j+1} > 0, \quad j = 1, \ldots, m-1, \tag{47}$$

then

$$\sum_{j=1}^{m-1} B_j J(f, \mathbf{x}, \mathbf{g}_j - \mathbf{g}_{j+1}) + \sum_{j=1}^{m} b_j G_{nj} f\left(\frac{y_j}{G_{nj}}\right) \leq \sum_{i=1}^{n} a_i f(x_i). \tag{48}$$

Proof. We proceed as in the proof of Theorem 3.1. We apply the Abel's identity

$$\sum_{j=1}^{m} b_j J(f, \mathbf{x}, \mathbf{g}_j) = \sum_{j=1}^{m-1} \Big(J(f, \mathbf{x}, \mathbf{g}_j) - J(f, \mathbf{x}, \mathbf{g}_{j+1})\Big) B_j + J(f, \mathbf{x}, \mathbf{g}_m) B_m. \tag{49}$$

By virtue of (35) and of Jensen-Steffensen inequality (see Theorem A), we obtain

$$J(f, \mathbf{x}, \mathbf{g}_m) = \sum_{i=1}^{n} g_{im} f(x_i) - G_{nm} f\left(\frac{1}{G_{nm}} \sum_{i=1}^{n} g_{im} x_i\right) \geq 0. \tag{50}$$

By (32) and (50),

$$J(f,\mathbf{x},\mathbf{g}_m)B_m \geq 0. \tag{51}$$

From (35) and (47) we have

$$G_{nj} > 0, \quad G_{n,j+1} > 0 \quad \text{and} \quad G_{nj} - G_{n,j+1} > 0, \quad j = 1,\dots,m-1. \tag{52}$$

It now follows from (52) and Corollary 2.4 that

$$J(f,\mathbf{x},\mathbf{g}_j) \geq J(f,\mathbf{x},\mathbf{g}_{j+1}) + J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}) \quad \text{for } j = 1,2,\dots,m-1. \tag{53}$$

By making use of (32), (49), (51) and (53) we infer that

$$\sum_{j=1}^{m} b_j J(f,\mathbf{x},\mathbf{g}_j) \geq \sum_{j=1}^{m-1} \Big(J(f,\mathbf{x},\mathbf{g}_j) - J(f,\mathbf{x},\mathbf{g}_{j+1}) \Big) B_j \geq \sum_{j=1}^{m-1} J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}) B_j. \tag{54}$$

Therefore,

$$\sum_{j=1}^{m-1} J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}) B_j + \sum_{j=1}^{m} b_j G_{nj} f\left(\frac{1}{G_{nj}} \sum_{i=1}^{n} g_{ij} x_i \right)$$

$$\leq \sum_{j=1}^{m} b_j \sum_{i=1}^{n} g_{ij} f(x_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j g_{ij} f(x_i). \tag{55}$$

Moreover, by (34) we have $y_j = \sum\limits_{i=1}^{n} g_{ij} x_i$, $j = 1,\dots,m$, and $a_i = \sum\limits_{j=1}^{m} b_j g_{ij}$, $i = 1,\dots,n$. Finally, from (55) we get

$$\sum_{j=1}^{m-1} J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}) B_j + \sum_{j=1}^{m} b_j G_{nj} f\left(\frac{y_j}{G_{nj}} \right) \leq \sum_{i=1}^{n} a_i f(x_i),$$

completing the proof of (48). ∎

Remark 3.3. Under the full condition (37), the Jensen-Steffensen functionals

$$J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}), \quad j = 1,\dots,m-1$$

are non-negative (see Theorem A). Then (48) in Theorem 3.2 is a refinement of inequality (38) in Theorem 3.1.
Conversely, with (47) in place of (37), the Jensen-Steffensen functionals can be negative in (48).

To give further simplifications of Theorem 3.2, we introduce row graded matrices, as follows.
A real $n \times m$ matrix $\mathbf{G} = (g_{ij})$ is said to be *row graded* if

$$g_{ij} \geq g_{i,j+1} \quad \text{for } i = 1,2,\dots n,\ j = 1,2,\dots,m-1. \tag{56}$$

For a row graded matrix $\mathbf{G} = (g_{ij})$, the vector $\mathbf{g}_j - \mathbf{g}_{j+1}$ for $j = 1,\dots,m-1$, has non-negative entries. As a result, condition (47) implies (37), so the Jensen-Steffensen functional $J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}) \geq 0$ is non-negative.
Therefore we obtain

Corollary 3.4. *Under the assumptions of Theorem 3.1 with the condition (37) replaced by*

$$G_{nj} - G_{n,j+1} > 0, \quad j = 1,\dots,m-1, \tag{57}$$

for a row graded matrix $\mathbf{G}$*, then*

$$\sum_{j=1}^{m} b_j G_{nj} f\left(\frac{y_j}{G_{nj}} \right) \leq \sum_{j=1}^{m-1} B_j J(f,\mathbf{x},\mathbf{g}_j - \mathbf{g}_{j+1}) + \sum_{j=1}^{m} b_j G_{nj} f\left(\frac{y_j}{G_{nj}} \right) \leq \sum_{i=1}^{n} a_i f(x_i). \tag{58}$$

References

[1] A. Abramovich, M. Klaričić Bakula, M. Matić, J. Pečarić, A variant of Jensen - Steffensen's inequality and quasi-arithmetic means, Journal of Mathematical Analysis and Applications 307 (2005) 370–386.
[2] J. Barić, M. Matić, J. Pečarić, On the bounds for the normalized Jensen functional and Jensen - Steffensen inequality, Mathematical Inequalities and Applications 12 (2009) 413–432.
[3] M. Bjelica, Refinement and converse of Brunk-Olkin inequality, Journal of Mathematical Analysis and Applications 227 (1998) 462–467.
[4] J. Borcea, Equilibrium points of logarithmic potentials, Transactions of the American Mathematical Society 359 (2007) 3209–3237.
[5] W. W. Breckner, T. Trif, Convex Functions and Related Functional Equations: Selected Topics, Cluj University Press, Cluj, 2008.
[6] H. D. Brunk, On an inequality for convex functions, Proceedings of the American Mathematical Society 7 (1956) 817–824.
[7] A.-M. Burtea, Two examples of weighted majorization, Annals of the University of Craiova, Mathematics and Computer Science Series 32 (2010) 92–99.
[8] S. S. Dragomir, J. E. Pečarić, L. E. Persson, Properties of some functionals related to Jensen's inequality, Acta Mathematica Hungarica 70 (1996) 129–143.
[9] G. M. Hardy, J. E. Littlewood, G. Pólya, Inequalities, (2nd edition), Cambridge University Press, Cambridge, 1952.
[10] J. Karamata, Sur une inégalité rélative aux fonctions convexes, Publications Mathématiques de l'Université de Belgrade 1 (1932) 145-148.
[11] M. Krnić, N. Lovričević, J. Pečarić, On some properties of Jensen - Steffensen's functional, Annals of the University of Craiova, Mathematics and Computer Science Series 38 (2011) 43-54.
[12] M. Krnić, N. Lovričević, J. Pečarić, Jessen's functional, its properties and applications, An. Şt. Univ. Ovidius Constanţa 20 (2012) 225–248.
[13] A. W. Marshall, I. Olkin, B. C. Arnold, Inequalities: Theory of Majorization and Its Applications, (2nd edition), Springer, New York 2011.
[14] C. P. Niculescu, L.-E. Persson, Convex Functions And Their Applications: A contemporary approach, Springer, Berlin/New York, 2004.
[15] M. Niezgoda, Remarks on Sherman like inequalities for (α, β)-convex functions, Mathematical Inequalities and Applications 17 (2014) 1579–1590.
[16] M. Niezgoda, Vector majorization and Schur concavity of some sums generated by the Jensen and Jensen-Mercer functionals, Mathematical Inequalities and Applications 18 (2015) 769–786.
[17] I. Olkin, On inequalities of Szegö and Bellman, Proceedings of the National Academy of Sciences USA 45 (1959) 230-231.
[18] J. E. Pečarić, F. Proschan, Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, Inc, 1992.
[19] S. Sherman, On a theorem of Hardy, Littlewood, Pólya, and Blackwell, Proceedings of the National Academy of Sciences USA 37 (1957) 826–831.
[20] I. Schur, Über eine Klasse von Mittelbildungen mit Anwendungen die Determinanten-Theorie Sitzungsber, Berliner Mathematische Gessellschaft 22 (1923) 9-20.
[21] J. F. Steffensen, On certain inequalities and methods of approximation, Journal of the Institute of Acturies 51 (1919) 274-297.