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Hereditary Properties of Semi-Separation Axioms and Their Applications

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Abstract. The paper studies the open-hereditary property of semi-separation axioms and applies it to the study of digital topological spaces such as an n-dimensional Khalimsky topological space, a Marcus-Wyse topological space and so on. More precisely, we study various properties of digital topological spaces related to low-level and semi-separation axioms such as $T_{\frac{1}{2}}$, semi-$T_{\frac{1}{2}}$, semi-T_1, semi-T_2, *etc*. Besides, using the finite or the infinite product property of the semi-T_i-separation axiom, $i \in \{1, 2\}$, we prove that the n-dimensional Khalimsky topological space is a semi-T_2-space. After showing that not every subspace of the digital topological spaces satisfies the semi-T_i-separation axiom, $i \in \{1, 2\}$, we prove that the semi-T_i-separation property is open-hereditary, $i \in \{1, 2\}$. All spaces in the paper are assumed to be nonempty and connected.

1. Introduction

In relation to the study of semi-separation axioms, many concepts were established such as a regular open set [37], a semi-open set [25], an α-set [32], a preopen set [30], an s-regular set [29] and so on. Furthermore, based on these notions, various types of mappings were developed such as an irresolute map [6], a semi-continuous mapping, a semi-homeomorphism (a bijection such that the images of semi-open sets are semi-open and inverses of semi-open sets are semi-open) [6] and so forth.

The paper [26] introduced that a subset A of a topological space (X, T) is called *generalized closed* (g-closed for short) if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in T$. The paper [25] developed the notion of a $T_{\frac{1}{2}}$-space with the property that every g-closed set is closed. Thus it is obvious that a $T_{\frac{1}{2}}$-space places between a T_0- and a T_1-space. The paper [10] proves that a topological space X is $T_{\frac{1}{2}}$ if and only if each singleton of X is open or closed. Hence a space X satisfies the separation axiom semi-$T_{\frac{1}{2}}$ if for each point p of X at least one of the sets $\{p\}$, $X - \{p\}$ is semi-open, *i.e.* for each point p of X the set $\{p\}$ is either semi-open or semi-closed [7, 26].

The separation axioms semi-T_i, where $i = 0, \frac{1}{2}, 1, 2$, etc (see [3, 5, 25, 27]), are obtained from the definitions of the usual separation axioms T_i after replacing open sets by semi-open ones. Hence the axiom T_i obviously implies the axiom semi-T_i [7] but the converse does not hold. Moreover, in case $i \le j$, the axiom semi-T_j

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implies the axiom semi-T_i, and the converse does not hold [6]. As usual, a property is called a *semi-topological property* if the property is preserved by a semi-homeomorphism. Then the axioms semi-$T_i, i \in \{0, \frac{1}{2}, 1, 2\}$ are proved to have the semi-topological property [6]. Moreover, a property is hereditary if the property passes from a topological space to every subspace with respect to the relative topology [31]. Besides, a property is called *open-hereditary* (*resp. closed-hereditary*) if the property passes from a topological space (X, T) to *every open set* (*resp. every closed set*) of (X, T) with respect to the relative topology.

Since the low-level separation axioms or the semi-separation axioms play important roles in applied topology including digital topology, computational topology and so on, the paper studies their properties on digital topological spaces such as Khalimsky, Marcus-Wyse topological space, axiomatic locally finite space [16, 24], space set topology [18], etc.

Especially, we will study the following topics:

- A study of the preopen or the nowhere dense property of subsets of digital topological spaces.
- Are the semi-T_i-separation axioms, $i \in \{1, 2\}$ hereditary properties ?
- A study of the finite or the infinite product property of the semi-T_i-separation axiom, $i \in \{1, 2\}$.

Since we will often use the terminology " Khalimsky (*resp.* Marcus-Wyse)" in this paper, hereafter we use the notation "*K*-(*resp.* *M*-)" instead of "Khalimsky (*resp.* Marcus-Wyse)", if there is no danger of ambiguity.

This paper is organized as follows. Section 2 provides some basic notions on K-topology. Section 3 studies some topological properties of the n-dimensional K-topological space and of the M-topological space related to the dense and the nowhere dense property of subsets of digital topological spaces. Section 4 investigates some properties of the n-dimensional K-topological space associated with the semi-$T_{\frac{1}{2}}$ separation axiom, semi-open subsets, semi-closed subsets and further, we develop the infinite product property of the semi-T_1-separation axiom. Section 5 proves the open-hereditary property of a semi-T_1- and a semi-T_2-space. Section 6 concludes the paper with a summary and questions for further work.

2. Preliminaries

To study low-level separation axioms and semi-separation axioms of digital topological spaces such as K-topological spaces and M-topological spaces, let us recall basic notions related to this work. A topological space (X, T) is called an Alexandroff space if every point $x \in X$ has the smallest open neighborhood in (X, T) [2]. Motivated by the Alexandroff topological structure [1, 2], several kinds of digital topological spaces and locally finite spaces were developed such as an n-dimensional K-topological space [21], an M-topological space, an axiomatic locally finite space [24], a space set topological space [18] and so on [15, 21, 24, 28]. Furthermore, a study of their properties is included in the papers [11, 20–22, 24, 29].

In digital topology we often take the following: A graph theoretical approach with digital connectivity on $\mathbf{Z}^n$ [34], a K- topological approach [21, 22], an M-topological approach [28], a locally finite topological approach [24] and so on.

First of all, let us recall some basic notions from digital graph theory [34, 35]. Motivated by the study of low-dimensional digital images $X \subset \mathbf{Z}^n, n \in \{1, 2, 3\}$ [34], we have studied nD digital images $X \subset \mathbf{Z}^n, n \in \mathbf{N}$ with the k-adjacency relations of $\mathbf{Z}^n, n \in \mathbf{N}$ [12] (see also [14]), as follows:

For a natural number m, $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$-adjacent if

$$\text{at most } m \text{ of their coordinates differ by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

In terms of the operator of (2.1), the $k(m, n)$-adjacency relations of $\mathbf{Z}^n, n \in \mathbf{N}$, are obtained [12] (see also [14]) as follows:

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.2}$$

A. Rosenfeld [34] called a set $X(\subset \mathbf{Z}^n)$ with a k-adjacency a *digital image*, denoted by (X, k). Indeed, to study digital images on $\mathbf{Z}^n$ in the graph-theoretical approach [34, 35], using the k-adjacency relations of $\mathbf{Z}^n$ of (2.2), we say that a digital k-neighborhood of p in $\mathbf{Z}^n$ is the set [34]

$$N_k(p) := \{q \in \mathbf{Z}^n \,|\, p \text{ is } k\text{-adjacent to } q\} \cup \{p\}. \tag{2.3}$$

For instance, for a point $p := (x, y) \in \mathbf{Z}^2$ we have

$$\begin{Bmatrix} N_4(p) = \{(x \pm 1, y), p, (x, y \pm 1)\} \\ N_8(p) = \{(x \pm 1, y), p, (x, y \pm 1), (x \pm 1, y \pm 1).\} \end{Bmatrix}$$

Second, let us now recall basic notions of the n-dimensional K-topological space, $n \geq 1$. *Khalimsky line topology* κ on $\mathbf{Z}$, denoted by $(\mathbf{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} \,|\, n \in \mathbf{Z}\}$ as a subbase [2] (see also [21]), where $a, b \in \mathbf{Z}$, we often use the notation $[a, b]_{\mathbf{Z}} := \{x \in \mathbf{Z} \,|\, a \leq x \leq b\}$. In the present paper we call $([a, b]_{\mathbf{Z}}, \kappa_{[a,b]_{\mathbf{Z}}})$(or for short $[a, b]_{\mathbf{Z}}$ if there is no danger of ambiguity) a Khalimsky interval. Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, \kappa)$ is called the *Khalimsky topology* on $\mathbf{Z}^n$ (or the *n-dimensional Khalimsky topological space*), denoted by $(\mathbf{Z}^n, \kappa^n)$. Besides, for a subset $X \subset \mathbf{Z}^n$, the subspace induced by $(\mathbf{Z}^n, \kappa^n)$ is obtained, denoted by (X, κ^n_X) and called a K-topological space.

Let us now investigate the structure of $(\mathbf{Z}^n, \kappa^n)$. A point $x = (x_i)_{i \in [1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd, and *pure closed* if each of the coordinates is even and the other points in $\mathbf{Z}^n$ are called *mixed* [22]. These points are shown like the following symbols: The symbols ■(*resp.* •) means a pure closed point(*resp.* a mixed point) (see Figure 1) and further, a black jumbo dot represents a pure open point. In addition, in the present paper we denote by $(\mathbf{Z}^n)_o$ (*resp.* $(\mathbf{Z}^n)_e$) the set of all pure open (*resp.* pure closed) points of $(\mathbf{Z}^n, \kappa^n)$. Besides, we use the notation $(\mathbf{Z}^n)_m$ for the set of all mixed points of $(\mathbf{Z}^n, \kappa^n)$.

In relation to the further statement of a mixed point in $(\mathbf{Z}^2, \kappa^2)$, for the point $p = (2m, 2n+1)$(*resp.* $p = (2m+1, 2n)$), we call the point p *closed-open* (*resp. open-closed*) [36]. In terms of this perspective, we clearly observe that the *smallest (open) neighborhood* of the point $p := (p_1, p_2)$ of $\mathbf{Z}^2$, denoted by $SN_K(p) \subset \mathbf{Z}^2$, is the following:

$$SN_K(p) := \begin{Bmatrix} \{p\} \text{ if } p \text{ is pure open,} \\ \{(p_1, p_2 \pm 1), p\} \text{ if } p \text{ is open-closed,} \\ \{(p_1 \pm 1, p_2), p\} \text{ if } p \text{ is closed-open,} \\ N_8(p) \text{ if } p \text{ is pure closed.} \end{Bmatrix} \tag{2.4}$$

Hereafter, in (X, κ^n_X), for a point $p \in X$ we use the notation $SN_K(p) \cap X := SN_X(p)$ for short.

Third, let us now recall basic concepts on Marcus-Wyse topology as another digital space. The *M-topology* on $\mathbf{Z}^2$, denoted by $(\mathbf{Z}^2, \gamma)$, is induced by the set $\{U_p\}$ in (2.5) below as a subbase [28], where for point $p = (x, y) \in \mathbf{Z}^2$

$$U_p := SN_M(p) := N_4(p) \text{ if } x + y \text{ is even.} \tag{2.5}$$

In relation to the further statement of a point in $\mathbf{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [28].

In all subspaces of $(\mathbf{Z}^2, \gamma)$ of Figure 1-3, the symbol ◊ means a *double even point* or a *even point*, and the symbol • means an *odd point*. In view of (2.5), we can obviously obtain the following: under $(\mathbf{Z}^2, \gamma)$ the singleton with either a double even point or an even point is the closure containing the given point. In addition, the singleton with an odd point is clearly the smallest open neighborhood of the given point. For a set $X \subset \mathbf{Z}^2$ we can take the *subspace*, denoted by (X, γ_X), induced by $(\mathbf{Z}^2, \gamma)$. As usual, for a subset $X \subset \mathbf{Z}^2$ we will consider (X, γ_X) [28] as a subspace of $(\mathbf{Z}^2, \gamma)$, and it is called an M-topological space.

A locally finite space is a topological space in which every point has a finite neighborhood [31]. It is clear that whereas every locally finite space is Alexandroff, not every Alexandroff space is locally finite. Indeed, an Alexandroff space with T_0 axiom need not be locally finite. But under $(\mathbf{Z}^n, \kappa^n)$ these notions of local finiteness and Alexandroff are equivalent.

3. Some Topological Properties of Digital Topological Spaces

In this section, we investigate various topological properties of $(\mathbf{Z}^n, \kappa^n)$ and $(\mathbf{Z}^2, \gamma)$ such as a dense subset, a nowhere dense subset and so forth, which will be substantially used in Sections 4 and 5. Indeed, for a topological space (X, T) we say that a subset A of X is *nowhere dense* if $Int(Cl(A)) = \emptyset$ [31], where "Cl(*resp.* Int)" means the closure (*resp.* the interior) operator of the given set. Also, a subset A of X is called dense if $Cl(A) = X$. To do this work, we now recall some properties of an open and a closed set of $(\mathbf{Z}^2, \kappa^2)$.

According to the properties (2.3) and (2.4), by using some properties of the closure and the interior operator, we obtain the following:

Lemma 3.1. *A subset B of $(\mathbf{Z}^2, \kappa^2)$ is open if and only if*

$$\begin{Bmatrix} N_8(p) \subset B \text{ whenever } p := (2m, 2n) \in B, \\ \{2m+1\} \times [2n-1, 2n+1]_{\mathbf{Z}} \subset B \text{ whenever } (2m+1, 2n) \in B, \text{and} \\ [2m-1, 2m+1]_{\mathbf{Z}} \times \{2n+1\} \subset B \text{ whenever } (2m, 2n+1) \in B. \end{Bmatrix} \tag{3.1}$$

Lemma 3.2. *A subset C of $(\mathbf{Z}^2, \kappa^2)$ is closed if and only if*

$$\begin{Bmatrix} N_8(q) \subset C \text{ whenever } q := (2m+1, 2n+1) \in C, \\ [2m, 2m+2]_{\mathbf{Z}} \times \{2n\} \subset C \text{ whenever } (2m+1, 2n) \in C, \text{and} \\ \{2m\} \times [2n, 2n+2]_{\mathbf{Z}} \subset C \text{ whenever } (2m, 2n+1) \in C. \end{Bmatrix} \tag{3.2}$$

As a generalization of Lemmas 3.1 and 3.2, we obtain the following:

Remark 3.3. If a subset B of $(\mathbf{Z}^n, \kappa^n)$ is open then

$$\begin{Bmatrix} N_{3^n-1}(p) \subset B \text{ whenever } p := (2m_1, 2m_2, \cdots, 2m_n) \in B, \\ \{2m_1+1\} \times \cdots \times \{2m_{i-1}+1\} \times N_2(2m_i) \times \{2m_{i+1}+1\} \times \cdots \times \{2m_n+1\} \subset B \\ \text{whenever } p_i := (2m_1+1, \cdots, 2m_{i-1}+1, 2m_i, 2m_{i+1}+1, \cdots, 2m_n+1) \in B, i \in [1, n]_{\mathbf{Z}}. \end{Bmatrix}$$

Remark 3.4. If a subset C of $(\mathbf{Z}^n, \kappa^n)$ is closed then

$$\begin{Bmatrix} N_{3^n-1}(q) \subset C \text{ whenever } q := (2m_1+1, 2m_2+1, \cdots, 2m_n+1) \in C, \text{and} \\ \{2m_1\} \times \cdots \times \{2m_{i-1}\} \times N_2(2m_i+1) \times \{2m_{i+1}\} \times \cdots \times \{2m_n\} \subset C \\ \text{whenever } q_i := (2m_1, \cdots, 2m_{i-1}, 2m_i+1, 2m_{i+1}, \cdots, 2m_n) \in C, i \in [1, n]_{\mathbf{Z}}. \end{Bmatrix}$$

In view of the property (3.2), under $(\mathbf{Z}^n, \kappa^n)$, for the point $q := (2m_1+1, 2m_2+1, \cdots, 2m_n+1)$, $m_i \in \mathbf{Z}$ the closure of the singleton $\{q\}$ is the set

$$\overline{\{q\}} := N_{3^n-1}(q) \tag{3.3}$$

By using the above properties, we now investigate dense and nowhere dense subsets of $(\mathbf{Z}^n, \kappa^n)$ and $(\mathbf{Z}^2, \gamma)$.

Theorem 3.5. *(1) In $(\mathbf{Z}^n, \kappa^n)$, $(\mathbf{Z}^n)_o$ is a dense subset of $(\mathbf{Z}^n, \kappa^n)$.*
(2) Under $(\mathbf{Z}^n, \kappa^n)$, $(\mathbf{Z}^n)_e$ is a nowhere dense subset of $(\mathbf{Z}^n, \kappa^n)$.
(3) Under $(\mathbf{Z}^n, \kappa^n)$, $(\mathbf{Z}^n)_m$ is a nowhere dense subset of $(\mathbf{Z}^n, \kappa^n)$.

Proof. (1) Owing to Remarks 3.3 and 3.4, for any $p \in \mathbf{Z}^n \setminus (\mathbf{Z}^n)_o$ we obtain $(SN_K(p) \setminus \{p\}) \cap (\mathbf{Z}^n)_o \neq \emptyset$ so that $Cl((\mathbf{Z}^n)_o) = \mathbf{Z}^n$ because the derived set of $(\mathbf{Z}^n)_o$ is equal to $\mathbf{Z}^n \setminus (\mathbf{Z}^n)_o$, which completes the proof.

(2) Since $Cl((\mathbf{Z}^n)_e) = (\mathbf{Z}^n)_e$ and any nonempty subset of $(\mathbf{Z}^n)_e$ is not an open subset of $(\mathbf{Z}^n, \kappa^n)$(see Remark 3.3), the interior of $Cl((\mathbf{Z}^n)_e)$ is an empty set. Namely, $Int(Cl((\mathbf{Z}^n)_e) = \emptyset$, which completes the proof.

(3) Since $Cl((\mathbf{Z}^n)_m) = \mathbf{Z}^n \setminus (\mathbf{Z}^n)_o$ and any nonempty subset of $(\mathbf{Z}^n)_m$ is not an open subset of $(\mathbf{Z}^n, \kappa^n)$(see Remark 3.3), we conclude that $Int(Cl((\mathbf{Z}^n)_m))$ is an empty set, which completes the proof. □

In view of Remark 3.3, we obtain the following:

Remark 3.6. Under $(\mathbf{Z}^n, \kappa^n)$, every nonempty open set O contains at least one point in $(\mathbf{Z}^n)_o$.

Let us move onto the study of the above properties from the viewpoint of M-topology, as follows.

Lemma 3.7. *A subset B of $(\mathbf{Z}^2, \gamma)$ is open if and only if*

$$N_4(p) \subset B \text{ whenever } p \in \{(2m, 2n), (2m+1, 2n+1)\}, p \in B \tag{3.4}$$

Lemma 3.8. *A subset C of $(\mathbf{Z}^2, \gamma)$ is closed if and only if*

$$N_4(q) \subset C \text{ whenever } q \in \{(2m+1, 2n), (2m, 2n+1)\}, q \in C \tag{3.5}$$

Based on Lemma 3.7 and 3.8, we obtain the following:

Theorem 3.9. *(1) In $(\mathbf{Z}^2, \gamma)$, $(\mathbf{Z}^2)_o$ is a dense subset of $(\mathbf{Z}^2, \gamma)$.*
(2) Under $(\mathbf{Z}^2, \gamma)$, $(\mathbf{Z}^2)_e$ is a nowhere dense subset of $(\mathbf{Z}^2, \gamma)$.

Proof. (1) Owing to Lemmas 3.7 and 3.8, for any $p \in \mathbf{Z}^2 \setminus (\mathbf{Z}^2)_o$ we obtain $(SN_M(p) \setminus \{p\}) \cap (\mathbf{Z}^2)_o \neq \emptyset$ so that $Cl((\mathbf{Z}^2)_o) = \mathbf{Z}^2$ because the derived set of $(\mathbf{Z}^2)_o$ is equal to $\mathbf{Z}^2 \setminus (\mathbf{Z}^2)_o$, which completes the proof.

(2) Since $Cl((\mathbf{Z}^2)_e) = (\mathbf{Z}^2)_e$ and any nonempty subset of $(\mathbf{Z}^2)_e$ is not an open subset of $(\mathbf{Z}^2, \gamma)$ (see Lemmas 3.7 and 3.8), we conclude that the interior of $Cl((\mathbf{Z}^2)_e)$ is an empty set. Namely, $Int(Cl((\mathbf{Z}^2)_e) = \emptyset$, which completes the proof. □

4. Some Properties of the Semi-T_1-Structure Related to Digital Topological Spaces

This section proves the infinite product property of the semi-T_1-separation axiom (see Theorem 4.4 and Remark 4.5). Owing to this property, for example $(\mathbf{Z}^n, \kappa^n), n \in \mathbf{N}$ is a semi-T_1-space and further, $(\mathbf{Z}^2, \gamma)$ is also a semi-T_1-space (see Example 4.6 and Theorem 4.7). Recall that a set A of a topological space (X, T) is called *semi-open* if there is an open set O such that $O \subset A \subset Cl(O)$ (or equivalently, $A \subset Cl(Int(A))$) [25]. In addition, a set B of a topological space (X, T) is called a *semi-closed* if there is a closed set F such that $Int(F) \subset B \subset F$ (or equivalently, $Int(Cl(B)) \subset B$) [25]. We recall that a topological space (X, T) is a semi-$T_{\frac{1}{2}}$ space if each singleton of (X, T) is semi-open or semi-closed. For instance, $(\mathbf{Z}, \kappa)$ is a semi-$T_{\frac{1}{2}}$-space [4].

It is well known that each of $(\mathbf{Z}^n, \kappa^n)$ and $(\mathbf{Z}^2, \gamma)$ is an Alexandroff space with the axiom T_0, $n \in \mathbf{N}$ [22]. As a generalization of this approach, we obtain the following:

Theorem 4.1. ([4]) *An Alexandroff space with the separation axiom T_0 is a semi-$T_{\frac{1}{2}}$ space.*

By Theorem 4.1, for example, $(\mathbf{Z}^n, \kappa^n)$ is also a semi-$T_{\frac{1}{2}}$-space [15]. In view of Theorem 4.1, we can observe that a semi-$T_{\frac{1}{2}}$-separation axiom does not place between the T_0- and the $T_{\frac{1}{2}}$-separation axiom.

As a semi-separation axiom which is stronger than the semi-$T_{\frac{1}{2}}$-separation axiom, we introduce the following:

Definition 4.2. ([27]) We say that a topological space (X, T) is a semi-T_1-space if any two distinct points $p, q \in X$ have their own semi-open sets $SO(p)$ and $SO(q)$ such that $q \notin SO(p)$ and $p \notin SO(q)$, where $SO(x)$ means a semi-open set containing the given point x.

Besides, it turns out that a topological space (X, T) is a semi-T_1-space if and only if every singleton is semi-closed [27].

Example 4.3. The Khalimsky line $(\mathbf{Z}, \kappa)$ is a semi-T_1 space [32].

More precisely, since every singleton $\{2n\}, n \in \mathbf{Z}$ is a closed set in $(\mathbf{Z}, \kappa)$, it is semi-closed. Next, we need to only prove that every singleton $\{2n+1\}, n \in \mathbf{Z}$ is semi-closed in $(\mathbf{Z}, \kappa)$. Since we have the following property with $\mathbf{Z} \setminus Cl(\{2n+1\}) := O \in (\mathbf{Z}, \kappa)$,

$$O \subset \mathbf{Z} \setminus \{2n+1\} \subset Cl(O) = \mathbf{Z} \setminus \{2n+1\}, \tag{4.1}$$

because $Cl(\{2n+1\}) = \{2n, 2n+1, 2n+2\}$, which guarantees the assertion.

Unlike the product property of the separation axioms T_0 and T_1, it is well known that the separation axiom $T_{\frac{1}{2}}$ does not have the product property. For instance, whereas $(\mathbf{Z}, \kappa)$ is a $T_{\frac{1}{2}}$-space, the product space $(\mathbf{Z}^2, \kappa^2)$ is not a $T_{\frac{1}{2}}$-space.

The paper [4] proves the finite product property of the semi-$T_{\frac{1}{2}}$-separation axiom. Motivated by this fact and by using the finite product property of the interior operator and the product property of the closure operator, we study the finite product property of a semi-T_1-space. Indeed, the paper [8] referred to the finite product property of a semi-T_1-space (see Corollary 3.2 of [8]) by using some properties of a semi-R_0-space. However, the present paper proves this product property by using the definition of a semi-T_1-space in a simpler way and further, proves the infinite product property a semi-T_1-space (see Remark 4.5), as follows:

Theorem 4.4. *The semi-T_1-separation axiom has the finite product property.*

Proof. Let (X_α, T_α) be semi-T_1-spaces, $\alpha \in M := [1, n]_{\mathbf{Z}}$. Then we prove that the product space $(\prod_{\alpha\in M} X_\alpha, \prod_{\alpha\in M} T_\alpha)$ is also a semi-T_1-space. More precisely, under $(\prod_{\alpha\in M} X_\alpha, \prod_{\alpha\in M} T_\alpha)$, take any point $p := (p_\alpha)_{\alpha\in M} \in \mathbf{Z}^n$. Then we need to prove that each singleton $\{p\}$ is semi-closed in the product space. Owing to the the property "semi-T_1" of (X_α, T_α), each singleton $\{p_\alpha\}$ consisting of one of the coordinates of the given point p has a closed set in (X_α, T_α), denoted by C_α, such that

$$Int(C_\alpha) \subset \{p_\alpha\} \subset C_\alpha. \tag{4.2}$$

Let us recall both the product property of the interior operator, *i.e.* in case M is finite, for any subset $A_\alpha \subset X_\alpha$, $Int(\prod_{\alpha\in M} A_\alpha) = \prod_{\alpha\in M} Int(A_\alpha)$ and in case M is infinite, $Int(\prod_{\alpha\in M} A_\alpha) \subset \prod_{\alpha\in M} Int(A_\alpha)$; and the product property of the closure operator, *i.e.* $Cl(\prod_{\alpha\in M} A_\alpha) = \prod_{\alpha\in M} Cl(A_\alpha)$. According to the property (4.2), for any singleton $\{p\}$ in $(\prod_{\alpha\in M} X_\alpha, \prod_{\alpha\in M} T_\alpha)$ we obtain

$$Int(\prod_{\alpha\in M} C_\alpha) = \prod_{\alpha\in M} Int(C_\alpha) \subset \{p\} \subset \prod_{\alpha\in M} C_\alpha. \tag{4.3}$$

Indeed, owing to the property (4.4) below, $\prod_{\alpha\in M} C_\alpha$ of (4.3) is closed.

$$\prod_{\alpha\in M} C_\alpha = \prod_{\alpha\in M} Cl(C_\alpha) = Cl(\prod_{\alpha\in M} C_\alpha). \tag{4.4}$$

Due to the property (4.3), the singleton $\{p\}$ is semi-closed in $(\prod_{\alpha\in M} X_\alpha, \prod_{\alpha\in M} T_\alpha)$, which completes the proof. □

Remark 4.5. As for the assertion of Theorem 4.4, we have the infinite product property of the semi-T_1-separation axiom because the property (4.3) can be generalized into the following property

$$Int(\prod_{\alpha\in M} C_\alpha) \subset \prod_{\alpha\in M} Int(C_\alpha) \subset \{p\} \subset \prod_{\alpha\in M} C_\alpha,$$

and further, owing to (4.4), the assertion is proved.

By Example 4.3 and Theorem 4.4, we obtain the following:

Example 4.6. Since $(\mathbf{Z}, \kappa)$ is obviously a semi-T_1 space, by Theorem 4.4, $(\mathbf{Z}^n, \kappa^n)$ is a semi-T_1 space, $n \in \mathbf{N}$ [32].

Le us now prove that the M-topological space $(\mathbf{Z}^2, \gamma)$ is also a semi-T_1-space, as follows:

Theorem 4.7. *$(\mathbf{Z}^2, \gamma)$ is a semi-T_1-space.*

Proof. Let us take any two distinct points $p, q \in \mathbf{Z}^2$. Without loss of generality, according to the property (2.5), we may take $p \in \{(2m, 2n), (2m+1, 2n+1) \,|\, m, n \in \mathbf{Z}\}$ and $q \in \{(2m+1, 2n), (2m, 2n+1) \,|\, m, n \in \mathbf{Z}\}$. Then we prove that each singleton $\{p\}$ and $\{q\}$ is semi-closed.

(Case 1) Since the singleton $\{p\}$ is closed in $(\mathbf{Z}^2, \gamma)$, it is obviously semi-closed.

(Case 2) Whereas the singleton $\{q\}$ is open in $(\mathbf{Z}^2, \gamma)$, we have the following property (see Lemmas 3.7 and 3.8)

$$\mathbf{Z}^2 \setminus Cl(\{q\}) := O \subset \mathbf{Z}^2 \setminus \{q\} = Cl(O), \tag{4.5}$$

which implies that the singleton $\{q\}$ is semi-closed. □

5. Open-Hereditary Property of a Semi-T_1 and a Semi-T_2-Space

In this section we study the hereditary property of the semi-T_1-separation axiom. Indeed, not every subspace of the digital topological spaces satisfies the semi-T_1-separation axiom, as the following remark shows:

Remark 5.1. The Khalimsky subspace $([0,5]_{\mathbf{Z}}, \kappa_{[0,5]_{\mathbf{Z}}})$ does not satisfy the semi-T_1-separation axiom because the singleton $\{1\}$ is not semi-closed in $([0,5]_{\mathbf{Z}}, \kappa_{[0,5]_{\mathbf{Z}}})$ (for more details, see the proof of Lemma 5.3(2) below).

To study the hereditary property of semi-separation axioms, let us recall basic notion of digital paths, as follows:

Definition 5.2. ([13]) (1) We say that a finite sequence $P := (x_i)_{i\in[0,m]_{\mathbf{Z}}}$ in $(\mathbf{Z}^n, \kappa^n)$ is *simple K-path* if x_i and x_j in P are K-adjacent if and only if either $j = i+1$ or $i = j+1$, where two distinct points x and y are called *K-adjacent* if $y \in SN_K(x)$ or $x \in SN_K(y)$ [22].

(2) We say that a finite sequence $P := (x_i)_{i\in[0,m]_{\mathbf{Z}}}$ in $(\mathbf{Z}^2, \gamma)$ is *simple M-path* if x_i and x_j in P are M-adjacent if and only if either $j = i+1$ or $i = j+1$, where two distinct points x and y are called *M-adjacent* if $y \in SN_M(x)$ or $x \in SN_M(y)$ [17].

Let us investigate the hereditary property of a semi-T_1-space with the following lemma.

Lemma 5.3. *(1) Any simple K-path $P := (p_1, p_2, \cdots, p_l)$ in $(\mathbf{Z}^n, \kappa^n)$ is a semi-T_1-space, where $p_1, p_l \in (\mathbf{Z}^n)_o$ and $|P| \geq 3$.*

(2) A simple K-path $P := (p_1, p_2, \cdots, p_l)$ in $(\mathbf{Z}^n, \kappa^n)$ is not a semi-T_1-space, where either of p_1 and p_l belongs to $(\mathbf{Z}^n)_e$ and $|P| \geq 3$.

(3) Any simple M-path $Q := (m_1, m_2, \cdots, m_l)$ in $(\mathbf{Z}^2, \gamma)$ is a semi-T_1-space, where both m_1 and m_l are odd points in $(\mathbf{Z}^2, \gamma)$ and $|Q| \geq 3$.

(4) A simple M-path $Q := (m_1, m_2, \cdots, m_l)$ in $(\mathbf{Z}^2, \gamma)$ is not a semi-T_1-space, where either of m_1 and m_l is double even or even in $(\mathbf{Z}^2, \gamma)$ and $|Q| \geq 3$.

Proof. (1) Under the hypothesis, according to Definition 2, the number l should be odd. Since each point $p_{2t} \in P, t \in [1, \frac{l-1}{2}]_{\mathbf{Z}}$ belongs to the set $(\mathbf{Z}^n)_e$, the singleton $\{p_{2t}\}$ is semi-closed because $\{p_{2t}\}$ is closed in (P, κ^n_P). Next, since the singleton $\{p_{2t-1}\}$, $p_{2t-1} \in P, t \in [1, \frac{l-1}{2}]_{\mathbf{Z}}$, belongs to the set $(\mathbf{Z}^n)_o \cup (\mathbf{Z}^n)_m$, the singleton $\{p_{2t-1}\}$ is also semi-closed because for the singleton $\{p_{2t-1}\}$ we have the following property with $P \setminus Cl(\{p_{2t-1}\}) := O \in (P, \kappa^n_P)$

$$O \subset P \setminus \{p_{2t-1}\} \subset Cl(O). \tag{5.1}$$

For instance, consider the simple K-path $(P := (p_i)_{i\in[1,5]_{\mathbf{Z}}}, \kappa_P^2)$ in Figure 1(a). Then (P, κ_P^2) is a semi-T_1-space. To be specific, the singleton $\{p\}, p \in \{p_2, p_4\}$, is closed in (P, κ_P^2) and it is obviously semi-closed. Next, whereas the singleton $\{q\}, q \in \{p_1, p_3, p_5\}$, is open in (P, κ_P^2) and it is obviously semi-closed in (P, κ_P^2).

(2) Under the hypothesis, to prove that the given simple K-path $P := (p_i)_{[1,5]_{\mathbf{Z}}}$ is not a semi-T_1-space, we may assume that p_1 belongs to $(\mathbf{Z}^n)_e$. Let us now take the point $p_2 \in P$. Then the singleton $\{p_2\}$ is not semi-closed so that P is not a semi-T_1-space. For instance, consider the simple K-path $(P := (p_i)_{i\in[1,5]_{\mathbf{Z}}}, \kappa_P^2)$ in Figure 1(b). Then the singleton $\{p_2\}$ is not semi-closed in (P, κ_P^2) because

$$P \setminus Cl(\{p_2\}) := O \subset P \setminus \{p_2\} \nsubseteq Cl(O),$$

and $Cl(O) = \{p_3, p_4, p_5\}$.

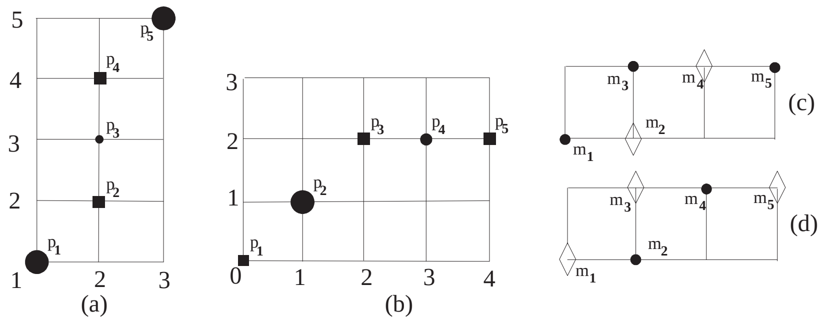

Figure 1: Configuration of the semi-T_1-axiom of both a simple K-path and a simple M-path.

(3) Under the hypothesis, according to Definition 2, the number l should be odd. Since each point $m_{2t} \in Q, t \in [1, \frac{l-1}{2}]_{\mathbf{Z}}$ is double even or even points in $(\mathbf{Z}^2, \gamma)$, the singleton $\{m_{2t}\}$ is semi-closed because $\{m_{2t}\}$ is closed in (Q, γ_Q). Next, since the point $m_{2t-1} \in Q, t \in [1, \frac{l}{2}]_{\mathbf{Z}}$ is odd point, by using the method similar to (5.1), the singleton $\{m_{2t-1}\}$ is semi-closed.

For instance, consider the simple M-path $(Y := (m_i)_{i\in[1,5]_{\mathbf{Z}}}, \gamma_Y)$ in Figure 1(c), where $m_1 := (1, 0), m_2 := (2, 0), m_3 := (2, 1), m_4 := (3, 1), m_5 := (4, 1)$. Then (Y, γ_Y) is a semi-T_1-space. To be specific, since each singleton $\{p\} \subset \{m_2, m_4\}$ is closed in (Y, γ_Y), it is obviously semi-closed. Next, whereas each singleton $\{p\} \subset \{m_1, m_3, m_5\}$ is open in (Y, γ_Y), it is obviously semi-closed in (Y, γ_Y).

(4) Under the hypothesis, consider the M-path $(Z := (m_i)_{i\in[1,l]_{\mathbf{Z}}}, \gamma_Z)$ in Figure 1(d). We may assume that m_1 is a double even or an even point in $(\mathbf{Z}^2, \gamma)$. Then the point $m_2 \in Q$ is odd point in $(\mathbf{Z}^2, \gamma)$ so that the singleton $\{m_2\}$ is not semi-closed. To be specific, take $Z := (m_i)_{i\in[1,5]_{\mathbf{Z}}}$, where $m_1 := (0, 0), m_2 := (1, 0), m_3 := (1, 1), m_4 := (2, 1), m_5 := (3, 1)$ (see Figure 1 (d)). Then the point $\{m_2\}$ is not semi-closed in (Z, γ_Z) because

$$Q \setminus Cl(\{m_2\}) := O \subset P \setminus \{m_2\} \nsubseteq Cl(O),$$

$Cl(O) = \{m_3, m_4, m_5\}$. Hence (Z, γ_Z) is not a semi-T_1-space. □

Let us now ask if a finite K-plane is a semi-T_1 space.

Proposition 5.4. *(1) Let X be the set* $[2m + 1, 2m + k + 1]_{\mathbf{Z}} \times [2n + 1, 2n + k + 1]_{\mathbf{Z}}$, *where* $m, n \in \mathbf{Z}, k \in 2\mathbf{N}$. *Then the subspace* $(X, \kappa_X^2)(\subset (\mathbf{Z}^2, \kappa^2))$ *is a semi-*T_1*-space.*

(2) Let Y be the set $[2m, 2m + k]_{\mathbf{Z}} \times [2n + 1, 2n + k]_{\mathbf{Z}}, m, n \in \mathbf{Z}, k \in 2\mathbf{N}$. *Then* (Y, κ_Y^2) *is not a semi-*T_1*-space.*

(3) Let Z be the set $[2m, 2m + k + 1]_{\mathbf{Z}} \times [2n, 2n + k + 1]_{\mathbf{Z}}, m, n \in \mathbf{Z}, k \in 2\mathbf{N}$. *Then* (Z, κ_Z^2) *is not a semi-*T_1*-space.*

(4) Let W be the set $[2m+1, 2m+k+1]_{\mathbf{Z}} \times [2n, 2n+k+1]_{\mathbf{Z}}, m, n \in \mathbf{Z}, k \in 2\mathbf{N}$. *Then* (W, κ_W^2) *is not a semi-*T_1*-space.*

Proof. (1) By Theorem 4.4, Remark 5.1 and Lemma 5.3, the proof is completed because both $[2m+1, 2m+k+1]_{\mathbf{Z}}$ and $[2n + 1, 2n + k + 1]_{\mathbf{Z}}$ are simple K-paths.

(2) Based on Lemmas 3.1 and 3.2, let us take the singleton $\{p\}$, $p := (1,1)$ in Figure 2(a). Then the singleton $\{p\}$ is not semi-closed in (Y, κ^2_Y).

(3) Based on Lemmas 3.1 and 3.2, consider the singleton $\{q\}$, $q := (1,1)$ in Figure 2(b). Then the singleton $\{q\}$ is not semi-closed in (Z, κ^2_Z).

(4) By using the method similar to the proof of (2), the proof is completed. □

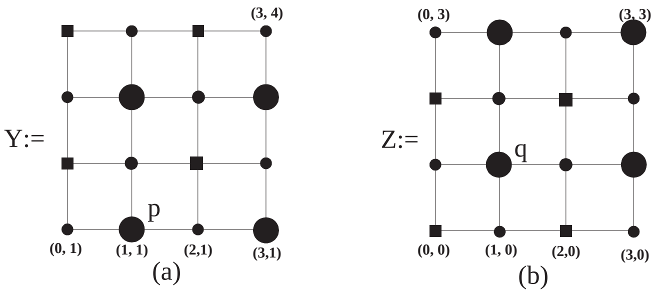

Figure 2: (a)-(b) Configuration of the non-hereditary both the semi-T_1- and the semi-T_2-space, where (Y, κ^2_Y) and (Z, κ^2_Z) are portions of $(\mathbf{Z}^2, \kappa^2)$.

In view of Theorems 4.4 and 4.7, and Example 4.6 and Proposition 5.4(2)-(4), we obtain the following:

Remark 5.5. (1) Let X be an Alexandroff semi-T_1-space and $Y \subset X$. Then Y need not be a semi-T_1-space.

(2) Let X_1, X_2 be an Alexandroff semi-T_1-space and $Z \subset X_1 \times X_2$. Then Z need not be a semi-T_1-space.

In view of Proposition 5.4 (2)-(4), not every finite K-plane is not a semi-T_1-space. Besides, owing to Example 4.6 and Theorem 4.7, we obtain the following:

Proposition 5.6. *A semi-T_1-separation property is not hereditary.*

Motivated by Proposition 5.4, we prove that a semi-T_1 space has the open-hereditary property.

Theorem 5.7. *A semi-T_1-separation property is open-hereditary.*

Proof. We need to prove that if $(X, T) := X$ is a semi-T_1 space and Y is an open subset of X, then the subspace $(Y, T_Y) := Y$ is a semi-T_1-space. Let $p \in Y \subset X$. We need to prove that there is a closed set F' in (Y, T_Y) such that $Int(F') \subset \{p\} \subset F'$. Owing to the semi-$T_1$ structure of (X, T), for any point $p \in Y \subset X$ there is a closed set F in (X, T) such that

$$Int(F) \subset \{p\} \subset F \text{ or } Int(Cl(\{p\})) \subset \{p\}. \tag{5.3}$$

Let us now consider the following two cases:

(Case 1) In (5.3), assume $Int(Cl(\{p\})) = \{p\}$, *i.e.* we may assume $Int(F) \neq \emptyset$. Then the closed set F contains a unique non-empty open set denoted by $O(p)(\in (X, T))$ such that $O(p) = \{p\} \subset F$. It is obvious that if the set $\{p\}$ is open in X, then it is open in Y and further, since F is a closed set in X, $F \cap Y$ is also closed in (Y, T_Y). Thus we obtain the closed set $F \cap Y$ in (Y, T_Y) such that

$$F \cap Y \subset Y \subset X. \tag{5.4}$$

Let us now adapt the interior operator into (5.4). Then we have

$$\left\{\begin{array}{l} Int_X(F \cap Y) = Int_X(Y) \cap Int_Y(F \cap Y) \\ = Y \cap Int_Y(F \cap Y) = Int_Y(F \cap Y), \end{array}\right\} \tag{5.5}$$

where for a subset $B \subset Y \subset X$ we denote by $Int_X(B)$ (*resp.* $Int_Y(B)$) the interior of B under the topology (X, T) (*resp.* (Y, T_Y)). In (5.5), since $Int_X(F \cap Y) = \{p\}$, we have $Int_Y(F \cap Y) = \{p\}$ so that in (Y, T_Y)

$$Int_Y(F \cap Y) \subset \{p\} \subset F \cap Y.$$

Hence the subspace Y is semi-T_1.

(Case 2) In (5.3), assume $Int(Cl(\{p\})) = \emptyset$. It is obvious that if the set $\{p\}$ is nowhere dense in X, then it is also nowhere dense in Y. Hence the subspace Y is semi-T_1. □

Let us now move onto the study of semi-T_2-structure of digital topological spaces.

Definition 5.8. ([27]) We say that a topological space (X, T) is a semi-T_2-space if any two distinct points $p, q \in X$ have their own semi-open sets $SO(p)$ and $SO(q)$ such that $SO(p) \cap SO(q) = \emptyset$, where $SO(x)$ means a semi-open set containing the given point x.

Theorem 5.9. *$(\mathbf{Z}^2, \gamma)$ is a semi-T_2-space.*

Proof. Consider two distinct points p and q in $(\mathbf{Z}^2, \gamma)$.

(Case 1) In case p and q are not M-adjacent to each other. Namely, $p \notin SN_M(q)$ and $q \notin SN_M(p)$. Take the sets

$$\begin{Bmatrix} X(p) := SN_M(p) \setminus (SN_M(p) \cap SN_M(q)) \text{ and} \\ X(q) := SN_M(q) \setminus (SN_M(q) \cap SN_M(p)). \end{Bmatrix}$$

Then $X(p) = SO(p)$ and $X(q) = SO(q)$ such that $SO(p) \cap SO(q) = \emptyset$.

(Case 2) In case p and q are M-adjacent. Namely, $p \in SN_M(q)$ or $q \in SN_M(p)$ so that we have the following two cases.

(Case 2-1) In case each of p and q is an odd point in $(\mathbf{Z}^2, \gamma)$, we have $SO(p) = \{p\}$ and $SO(q) = \{q\}$. Then $SO(p) \cap SO(q) = \emptyset$.

(Case 2-2) In case each of p is a double even or even point and q is an odd point in $(\mathbf{Z}^2, \gamma)$, we have $SO(p) = SN_M(p) \setminus \{q\}$ and $SO(q) = \{q\}$.

Then $SO(p) \cap SO(q) = \emptyset$. □

The paper [33] developed the product property of a semi-T_2-space.

Theorem 5.10. ([33]) *A semi-T_2-space has the finite product property.*

Remark 5.11. Since $(\mathbf{Z}, \kappa)$ is a semi-T_2-space, by Theorem 5.9, we confirm that $(\mathbf{Z}^n, \kappa^n)$ is a semi-T_2-space [32].

Proposition 5.12. *(1) A semi-T_2-separation property is not hereditary.*
(2) A semi-T_2-separation property is not closed-hereditary.

Proof. (1) As a counterexample, consider the subspace (X, κ_X), where $X := [0, 1]_{\mathbf{Z}}$ (see Figure 3(e)). Whereas the space $(\mathbf{Z}, \kappa)$ is a semi-T_2-space, the subspace (X, κ_X) is not a semi-T_2-space because $SO(0) = X$ and $SO(1) = \{1\}$.

(2) As a counterexample, consider the subspace (Y, κ_Y), where $Y := [0, 2]_{\mathbf{Z}}$ (see Figure 3(c)). Although the space $(\mathbf{Z}, \kappa)$ is a semi-T_2-space, the closed subspace (Y, κ_Y) is not a semi-T_2-space because for the distinct points 0 and 1 there are smallest semi-open sets

$$SO(0) = \{0, 1\} \text{ and } SO(1) = \{1\}$$

so that $SO(0) \cap SO(1) \neq \emptyset$, which implies that (Y, κ_Y) is not a semi-T_2-space. □

Theorem 5.13. *A semi-T_2-separation property is open-hereditary.*

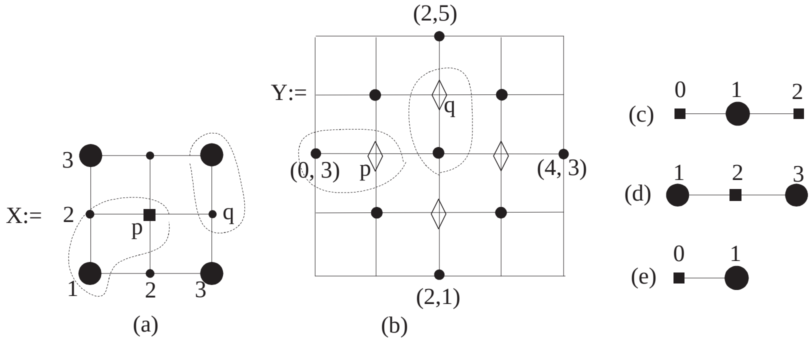

Figure 3: (a)-(b) Configuration of the open-hereditary property of the semi-T_2-structure of both $(\mathbf{Z}^n, \kappa^n)$ and $(\mathbf{Z}^2, \gamma)$; (c) Non-closed-hereditary property of a semi-T_2-space; (d) The open-hereditary property of a semi-T_2-space; (e) Non-hereditary property of a semi-T_2-space.

Proof. We need to prove that if $(X, T) := X$ is a semi-T_2 space and Y is an open subset of (X, T) then the subspace $(Y, T_Y) := Y$ is a semi-T_2-space. To be specific, consider distinct two points $p, q \in Y \subset X$. Then there are two semi-open sets $SO(p)$ and $SO(q)$ in (X, T) such that $SO(p) \cap SO(q) = \emptyset$. Since Y is an open set in (X, T) the sets $SO(p) \cap Y$ and $SO(q) \cap Y$ are semi-open sets in (Y, T_Y) such that $(SO(p) \cap Y) \cap (SO(q) \cap Y) = \emptyset$. Hence the subspace Y of the space X is semi-T_2. $\square$

Example 5.14. (1) Consider the spaces (X, κ^2_X) (see Figure 3(a)) and (Y, γ_Y) (see Figure 3(b)) which are respectively open subspaces in $(\mathbf{Z}^2, \kappa^2)$ and $(\mathbf{Z}^2, \gamma)$. By Proposition 5.12, those are semi-T_2-spaces. To be specific, take the two distinct points p, q in (X, κ^2_X) (see Figure 3(a)). Then we have $SO(p) = \{(1, 1), (1, 2), (2, 2)\}$ and $SO(q) = \{(3, 2), (3, 3)\}$ which are disjoint. Besides, for two points p, q in (Y, γ_Y) then we have $SO(p) = \{(0, 3), (1, 3)\}$ and $SO(q) = \{(2, 3), (2, 4)\}$ which are disjoint.

(2) Consider the subspace (W, κ_W), where $W := [1, 3]_{\mathbf{Z}}$ (see Figure 3(d)). Then the open subspace (W, κ_W) is a semi-T_2-space because for any distinct two points p and q in W there are smallest semi-open sets $SO(p)$ and $SO(q)$ so that $SO(p) \cap SO(q) = \emptyset$, where $SO(1) = \{1\}, SO(2) \in \{\{1, 2\}, \{2, 3\}\}, SO(3) = \{3\}$, which implies that (W, κ_W) is a semi-T_2-space.

6. Concluding Remark and Further Work

We have studied various properties of semi-T_i-separation axioms, $i \in \{\frac{1}{2}, 1, 2\}$. In particular, although semi-T_i-spaces, $i \in \{1, 2\}$ do not have the hereditary property, it turns out that they have the open-hereditary property.

As a further work, motivated by Theorem 4.1, we have the following problems.

Question 6.1. *Under what condition does the $T_{\frac{1}{2}}$-separation axiom imply the semi-T_1-separation axiom?*

Question 6.2. *Under what condition does the T_1-separation axiom imply the semi-T_2-separation axiom?*

Besides, after developing some new digital topological spaces, we need to study some topological properties related to low-level separation axioms and their corresponding semi-separation axioms.

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