Filomat 32:13 (2018), 4711–4729
https://doi.org/10.2298/FIL1813711F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fixed Point Results Via Simulation Functions in the Context of Quasi-metric Space

Andreea Fulga[a], **Ayşegül Taş**[b]

[a]Department of Mathematics and Computer Sciences, Universitatea Transilvania Brasov, Brasov, Romania.
[b]Cankaya University, Department Of Management, Ankara, Turkey

Abstract. In this paper, we investigate the existing non-unique fixed points of certain mappings, via simulation functions in the context of quasi-metric space. Our main results generalize and unify several existing results on the topic in the literature.

1. Introduction and Preliminaries

Quasi metric spaces are one of the interesting topics for fixed-point theory researchers because they generalize the concept of metric space by giving up the symmetry condition. For some results on fixed point theorems related to quasi-dimensional spaces, see e.g. [2], [3], [4], [7]. First, we recall some basic concepts and fundamental results.

Definition 1.1. *A quasi-metric on a set X is a function $q : X \times X \to [0,\infty)$ such that:*

($q1$) $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$;

($q2$) $q(x,z) \leq q(x,y) + q(y,z)$, *for all* $x, y, z \in X$.

The pair (X,q) is called a quasi-metric space.

Any metric space is a quasi-metric space, but the converse is not true in general. Now, we give convergence, completeness and continuity on quasi-metric spaces.

Definition 1.2. *Let (X,q) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0. \tag{1}$$

Remark 1.3. *In a quasi-metric space (X,q), the limit for a convergent sequence is unique. If $x_n \to x$, we have for all $y \in X$*

$$\lim_{n\to\infty} q(x_n, y) = q(x,y) \text{ and } \lim_{n\to\infty} q(y, x_n) = q(y,x).$$

2010 *Mathematics Subject Classification*. Primary 46T99; Secondary 47H10,54H25
Keywords. (quasi-metric spaces, fixed point, self-mappings)
Received: 22 January 2018; Accepted: 13 March 2018
Communicated by Erdal Karapınar
Email addresses: afulga@unitbv.ro (Andreea Fulga), aysegul@cankaya.edu.tr (Ayşegül Taş)

Definition 1.4. *Let (X, q) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $q(x_n, x_m) < \epsilon$ for all $n \geq m > N$.*

Definition 1.5. *Let (X, q) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $q(x_n, x_m) < \epsilon$ for all $m \geq n > N$.*

Definition 1.6. *Let (X, q) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $q(x_n, x_m) < \epsilon$ for all $m, n > N$.*

Remark 1.7. *A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.*

Definition 1.8. *Let (X, q) be a quasi-metric space. We say that:*
1. (X, q) is left-complete if and only if each left-Cauchy sequence in X is convergent.
2. (X, q) is right-complete if and only if each right-Cauchy sequence in X is convergent.
3. (X, q) is complete if and only if each Cauchy sequence in X is convergent.

Definition 1.9. *Let (X, q) be a quasi-metric space. The map $T : X \to X$ is continuous if for each sequence $\{x_n\}$ in X converging to $x \in X$, the sequence $\{Tx_n\}$ converges to Tx, that is,*

$$\lim_{n\to\infty} q(Tx_n, Tx) = \lim_{n\to\infty} q(Tx, Tx_n) = 0 \tag{2}$$

Definition 1.10. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if:*

$(c1)$ *φ is increasing;*

$(c2)$ $\lim_{n\to\infty} \varphi^n(t) = 0$, *for $t \in [0, \infty)$.*

Proposition 1.11. *If φ is a comparison function then:*

(i) *each φ^k is also a comparison function for all $k \in \mathbb{N}$;*

(ii) *φ is continuous at 0;*

(iii) *$\varphi(t) < t$ for all $t > 0$.*

Definition 1.12. *A function $\psi : [0, \infty) \to [0, \infty)$ is called a $c-$comparison function if:*

$(cc1)$ *ψ is monotone increasing;*

$(cc2)$ $\sum_{n=0}^{\infty} \psi^n(t) < \infty$, *for all $t \in (0, \infty)$.*

We denote by Ψ the family of $c-$comparison functions.

Remark 1.13. *If ψ is a $c-$comparison function, then $\psi(t) < t$ for all $t > 0$.*

Remark 1.14. *A $c-$comparison function is a comparison function.*

In order to unify the several existing fixed point results in the literature, [14], Khojasteh *et al.* introduced the notion of *simulation function* and investigated the existence and uniqueness of a fixed point of certain mappings via simulation functions.

Definition 1.15. *A* simulation function *is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) *$\zeta(t, s) < s - t$ for all $t, s > 0$;*

(ζ_2) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{3}$$

Notice that in [14] there was a superfluous condition $\zeta(0,0) = 0$. *Let* $\mathcal{Z}$ *denote the family of all simulation functions* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$. *Due to the axiom* ($\zeta_1$), *we have*

$$\zeta(t,t) < 0 \text{ for all } t > 0. \tag{4}$$

The following example is derived from [2, 14, 15].

Example 1.16. *Let* $\phi_i : [0,\infty) \to [0,\infty)$, $i = 1,2,3$, *be continuous functions with* $\phi_i(t) = 0$ *if, and only if,* $t = 0$. *For* $i = 1,2,3,4,5,6$, *we define the mappings* $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, *as follows*

(*i*) $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ *for all* $t,s \in [0,\infty)$, *where* $\phi_1(t) < t \leq \phi_2(t)$ *for all* $t > 0$.

(*ii*) $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ *for all* $t,s \in [0,\infty)$, *where* $f, g : [0,\infty)^2 \to (0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$.

(*iii*) $\zeta_3(t,s) = s - \phi_3(s) - t$ *for all* $t,s \in [0,\infty)$.

(*iv*) *If* $\varphi : [0,\infty) \to [0,1)$ *is a function such that* $\limsup_{t\to r^+} \varphi(t) < 1$ *for all* $r > 0$, *and we define*

$$\zeta_4(t,s) = s\varphi(s) - t \qquad \text{for all } s,t \in [0,\infty).$$

(*v*) *If* $\eta : [0,\infty) \to [0,\infty)$ *is an upper semi-continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$, *and we define*

$$\zeta_5(t,s) = \eta(s) - t \qquad \text{for all } s,t \in [0,\infty).$$

(*vi*) *If* $\phi : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^\varepsilon \phi(u)du$ *exists and* $\int_0^\varepsilon \phi(u)du > \varepsilon$, *for each* $\varepsilon > 0$, *and we define*

$$\zeta_6(t,s) = s - \int_0^t \phi(u)du \qquad \text{for all } s,t \in [0,\infty).$$

It is clear that each function ζ_i ($i = 1,2,3,4,5,6$) *forms a simulation function.*

In 2012 Samet et al.[16] introduced the notion of $\alpha-$ admissible mappings, concept which is used frequently in several papers to establish various fixed point results.

Definition 1.17. *[16] Let* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0,\infty)$ *be a function. We say that* T *is an* $\alpha-$*admissible if for all* $x, y \in X$ *we have*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

2. Main results

Definition 2.1. *A set* X *is regular with respect to mapping* $\alpha : X \times X \to [0,\infty)$ *if, whenever* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $\alpha(x_{n+1}, x_n) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\left\{x_{n(k)}\right\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *and* $\alpha(x, x_{n(k)}) \geq 1$ *for all* n.

Lemma 2.2. *Let $T : X \to X$ be an $\alpha-$admissible function and $x_n = Tx_{n-1}, n \in \mathbb{N}$. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$, then we have*

$$\alpha(x_{n-1}, x_n) \geq 1 \text{ and } \alpha(x_n, x_{n-1}) \geq 1, \text{ for all } n \in \mathbb{N}_0.$$

Proof. By assumption, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. On account of the definition of $\{x_n\} \subset X$ and owing to the fact that T is $\alpha-$ admissible, we derive

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we have

$$\alpha(x_{n-1}, x_n) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{5}$$

We consider now the case where $\alpha(Tx_0, x_0) \geq 1$. By using the same technique as above, we get that

$$\alpha(x_n, x_{n-1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{6}$$

□

Theorem 2.3. *Let (X, q) be a complete quasi-metric space and a map $\alpha : X \times X \to [0, \infty)$. Suppose that there exist $\zeta \in \mathcal{Z}$, $\psi \in \Psi$ and a self-mapping T such that*

$$\zeta(\alpha(x, y)q(Tx, Ty), \psi(q(x, y))) \geq 0, \tag{7}$$

for each $x, y \in X$. Suppose also that

(*i*) *T is $\alpha-$admissible;*
(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
(*iii*) *either, T is continuous, or*
(*iv*) *X is regular with respect to mapping α.*

Then, T has a fixed point.

Proof. By (*ii*), there is $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. By using this initial point, we define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then, x_{n_0} is a fixed point of T, that is, $Tx_{n_0} = x_{n_0}$. From now, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, in other words

$$q(x_{n+1}, x_n) > 0 \text{ and } q(x_n, x_{n+1}) > 0.$$

By replacing $x = x_n$ and $y = x_{n-1}$ in (7) and taking into account (ζ1) we find, for all $n \geq 1$, that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})q(Tx_n, Tx_{n-1}), \psi(q(x_n, x_{n-1}))) \\ &< \psi(q(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})q(Tx_n, Tx_{n-1}) \\ &= \psi(q(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})q(x_{n+1}, x_n). \end{aligned} \tag{8}$$

Consequently, we have

$$\begin{aligned} q(x_{n+1}, x_n) &\leq \alpha(x_n, x_{n-1})q(x_{n+1}, x_n) \leq \psi(q(x_n, x_{n-1})) \\ &< q(x_n, x_{n-1}). \end{aligned} \tag{9}$$

Recursively, we obtain that

$$q(x_{n+1}, x_n) \leq \psi^n(q(x_1, x_0)), \forall n \geq 1. \tag{10}$$

By using the triangle inequality and (10), for all $k \geq 1$, we get

$$\begin{aligned} q(x_{n+k}, x_n) &\leq q(x_{n+k}, x_{n+k-1}) + ... + q(x_{n+1}, x_n) \\ &\leq \sum_{p=n}^{n+k-1} \psi^p(q(x_1, x_0)) \\ &\leq \sum_{p=n}^{\infty} \psi^p(q(x_1, x_0)). \end{aligned} \tag{11}$$

Letting $n \to \infty$ in the above inequality, we derive that $\sum_{p=n}^{\infty} \psi^p(q(x_1, x_0)) \to 0$. Hence, $q(x_{n+k}, x_n) \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ is a left-Cauchy sequence in (X, d).
Analogously, we deduce that $\{x_n\}$ is a right-Cauchy sequence in (X, d).

On account of Remark 1.7, we deduce that the constructed sequence $\{x_n\}$ is Cauchy in the complete quasi-metric space (X, q). It implies that there exists $u \in X$ such that

$$\lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = 0. \tag{12}$$

If T is continuous, then, by using the property $(q1)$, we derive that

$$\lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tx_{n-1}, Tu) = 0, \tag{13}$$

and

$$\lim_{n\to\infty} q(Tu, x_n) = \lim_{n\to\infty} q(Tu, Tx_{n-1}) = 0. \tag{14}$$

Thus, we have

$$\lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tu, x_n) = 0. \tag{15}$$

Keeping (12) and (15) in the mind together with the uniqueness of a limit, we conclude that $u = Tu$, that is, u is a fixed point of T.

If X is regular with respect to α, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(u, x_{n(k)}) \geq 1$ for all k. Applying (7), for all k, and taking into account Remark 1.3 we get that

$$q(Tu, x_{n(k)+1}) = q(Tu, Tx_{n(k)}) \leq \alpha(u, x_{n(k)})q(Tu, Tx_{n(k)}) \leq \psi(q(u, x_{n(k)})). \tag{16}$$

Letting $k \to \infty$ in the above equality, we obtain that

$$q(Tu, u) \leq 0. \tag{17}$$

Thus, we have $q(Tu, u) = 0$, that is $Tu = u$. □

Theorem 2.4. *Let (X, q) be a complete quasi-metric space and a map $\alpha : X \times X \to [0, \infty)$. Suppose that there exist $\zeta \in \mathcal{Z}$, $\psi \in \Psi$ and a self-mapping T such that*

$$\zeta(\alpha(x, y)q(Tx, Ty), \psi(M(x, y))) \geq 0, \tag{18}$$

for each $x, y \in X$, where

$$M(x, y) = \max\{q(x, y), q(Tx, x), q(Ty, y), \frac{1}{2}[q(Tx, y) + q(Ty, x)]\}. \tag{19}$$

Suppose also that

(*i*) *T is α−admissible;*

(*ii*) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(Tx_0, x_0) \geq 1$;
(*iii*) *either, T is continuous, or*
(*iv*) *X is regular with respect to mapping* α.

Then, T has a fixed point.

Proof. Following the lines in the proof of Theorem 2.3, we find a sequence $\{x_n\} \subset X$ which is built by $x_n = Tx_{n-1}$. Further, with the same reasoning in the proof of Theorem 2.3, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,

$$q(x_{n+1}, x_n) > 0 \text{ and } q(x_n, x_{n+1}) > 0.$$

Taking the inequality (18) and Lemma 2.2 into account, we find

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})q(Tx_n, Tx_{n-1}), \psi(M(x_n, x_{n-1}))) \\ &< \psi(M(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})q(Tx_n, Tx_{n-1}) \\ &= \psi(M(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})q(x_{n+1}, x_n). \end{aligned} \tag{20}$$

which yields that

$$q(x_{n+1}, x_n) = q(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1})q(Tx_n, Tx_{n-1}) \leq \psi(M(x_n, x_{n-1})), \tag{21}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{q(x_n, x_{n-1}), q(Tx_n, x_n), q(Tx_{n-1}, x_{n-1}), \tfrac{1}{2}[q(Tx_n, x_{n-1}) + q(Tx_{n-1}, x_n)]\} \\ &= \max\{q(x_n, x_{n-1}), q(x_{n+1}, x_n), q(x_n, x_{n-1}), \tfrac{1}{2}[q(x_{n+1}, x_{n-1}) + q(x_n, x_n)]\} \\ &\leq \max\{q(x_n, x_{n-1}), q(x_{n+1}, x_n)\}. \end{aligned} \tag{22}$$

Since ψ is a nondecreasing function, (21) implies that

$$q(x_{n+1}, x_n) \leq \psi(\max\{q(x_n, x_{n-1}), q(x_{n+1}, x_n)\}), \tag{23}$$

for all $n \geq 1$. We shall examine two cases. Suppose that $q(x_{n+1}, x_n) > q(x_n, x_{n-1})$. Since $q(x_{n+1}, x_n) > 0$, we obtain that

$$q(x_{n+1}, x_n) \leq \psi(q(x_{n+1}, x_n)) < q(x_{n+1}, x_n), \tag{24}$$

is a contradiction. Therefore, we find that $\max\{q(x_n, x_{n-1}), q(x_{n+1}, x_n)\} = q(x_n, x_{n-1})$. Since $\psi \in \Psi$, (23) yields that

$$q(x_{n+1}, x_n) \leq \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}) \tag{25}$$

for all $n \geq 1$. Recursively, we derive that

$$q(x_{n+1}, x_n) \leq \psi^n(q(x_1, x_0)), \quad \forall n \geq 1. \tag{26}$$

Together with (26) and the triangle inequality, for all $k \geq 1$, we get that

$$\begin{aligned} q(x_{n+k}, x_n) &\leq q(x_{n+k}, x_{n+k-1}) + \ldots + q(x_{n+1}, x_n) \\ &\leq \sum_{p=n}^{n+k-1} \psi^p(q(x_1, x_0)) \\ &\leq \sum_{p=n}^{\infty} \psi^p(q(x_1, x_0)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{27}$$

Therefore, $\{x_n\}$ is a left-Cauchy sequence in (X, q).
Analogously, we shall prove that $\{x_n\}$ is a right-Cauchy sequence in (X, q). From (18) and Lemma 2.2, we derive that

$$\begin{aligned} 0 &\le \zeta(\alpha(x_{n-1}, x_n)q(Tx_{n-1}, Tx_n), \psi(M(x_{n-1}, x_n))) \\ &< \psi(M(x_{n-1}, x_n)) - \alpha(x_{n-1}, x_n)q(Tx_{n-1}, Tx_n) \\ &= \psi(M(x_{n-1}, x_n)) - \alpha(x_{n-1}, x_n)q(Tx_{n-1}, Tx_n). \end{aligned} \tag{28}$$

which implies

$$q(x_n, x_{n+1}) = q(Tx_{n-1}, Tx_n) \le \alpha(x_{n-1}, x_n)q(Tx_{n-1}, Tx_n) \le \psi(M(x_{n-1}, x_n)), \tag{29}$$

for all $n \ge 1$, where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{q(x_{n-1}, x_n), q(Tx_{n-1}, x_{n-1}), q(Tx_n, x_n), \tfrac{1}{2}[q(Tx_{n-1}, x_n) + q(Tx_n, x_{n-1})]\} \\ &= \max\{q(x_{n-1}, x_n), q(x_n, x_{n-1}), q(x_{n+1}, x_n), \tfrac{1}{2}[q(Tx_{n-1}, x_n) + q(Tx_n, x_{n-1})]\} \\ &\le \max\{q(x_{n-1}, x_n), q(x_n, x_{n-1}), q(x_{n+1}, x_n)\}. \end{aligned} \tag{30}$$

Since ψ is a nondecreasing function, the inequality (29) turns into

$$q(x_n, x_{n+1}) \le \psi(\max\{q(x_{n-1}, x_n), q(x_n, x_{n-1}), q(x_{n+1}, x_n)\}), \tag{31}$$

for all $n \ge 1$. We shall examine three cases.

Case 1. Suppose that $\max\{q(x_{n-1}, x_n), q(x_n, x_{n-1}), q(x_{n+1}, x_n)\} = q(x_{n-1}, x_n)$. Since $\psi \in \Psi$, from (30) we find that

$$q(x_n, x_{n+1}) \le \psi(q(x_{n-1}, x_n)) < q(x_{n-1}, x_n) \tag{32}$$

for all $n \ge 1$. Inductively, we get that

$$q(x_n, x_{n+1}) \le \psi^n(q(x_0, x_1)), \forall n \ge 1. \tag{33}$$

By using the triangle inequality and taking (33) into consideration, for all $k \ge 1$, we get

$$\begin{aligned} q(x_n, x_{n+k}) &\le q(x_n, x_{n+1}) + ... + q(x_{n+k-1}, x_{n+k}) \\ &\le \sum_{p=n}^{n+k-1} \psi^p(q(x_0, x_1)) \\ &\le \sum_{p=n}^{\infty} \psi^p(q(x_0, x_1)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{34}$$

Case 2. Assume that $\max\{q(x_{n-1}, x_n), q(x_n, x_{n-1}), q(x_{n+1}, x_n)\} = q(x_n, x_{n-1})$. Regarding $\psi \in \Psi$ and (31), we obtain that

$$q(x_n, x_{n+1}) \le \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}) \tag{35}$$

for all $n \ge 1$. From (18) and Lemma 2.2, we derive that

$$\begin{aligned} q(x_n, x_{n-1}) &= q(Tx_{n-1}, Tx_{n-2}) \\ &\le \alpha(x_{n-1}, x_{n-2})q(Tx_{n-1}, Tx_{n-2}) \\ &\le \psi(M(x_{n-1}, x_{n-2})), \end{aligned} \tag{36}$$

for all $n \ge 1$, where

$$\begin{aligned} M(x_{n-1}, x_{n-2}) &= \max\{q(x_{n-1}, x_{n-2}), q(Tx_{n-1}, x_{n-1}), q(Tx_{n-2}, x_{n-2}), \tfrac{1}{2}[q(Tx_{n-1}, x_{n-2}) + q(Tx_{n-2}, x_{n-1})]\} \\ &= \max\{q(x_{n-1}, x_{n-2}), q(x_n, x_{n-1}), q(x_{n-1}, x_{n-2}), \tfrac{1}{2}[q(x_n, x_{n-2}) + q(x_{n-1}, x_{n-1})]\} \\ &\le \max\{q(x_{n-1}, x_{n-2}), q(x_n, x_{n-1})\}. \end{aligned} \tag{37}$$

Since ψ is a nondecreasing function, (21) implies that

$$q(x_n, x_{n-1}) \leq \psi(\max\{q(x_{n-1}, x_{n-2}), q(x_n, x_{n-1})\}), \tag{38}$$

for all $n \geq 1$.
We shall examine two cases. Suppose that $q(x_n, x_{n-1}) > q(x_{n-1}, x_{n-2})$. Since $q(x_n, x_{n-1}) > 0$, we obtain that

$$q(x_n, x_{n-1}) \leq \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}), \tag{39}$$

is a contradiction. Therefore, we find that $\max\{q(x_{n-1}, x_{n-2}), q(x_n, x_{n-1}) = q(x_{n-1}, x_{n-2})$. Since $\psi \in \Psi$, (38) yields that

$$q(x_n, x_{n-1}) \leq \psi(q(x_{n-1}, x_{n-2})) < q(x_{n-1}, x_{n-2}) \tag{40}$$

for all $n \geq 1$. Recursively, we derive that

$$q(x_n, x_{n-1}) \leq \psi^{n-1}(q(x_1, x_0)), \quad \forall n \geq 1. \tag{41}$$

If we combine the inequalities (35) and (41), we derive that

$$q(x_n, x_{n+1}) \leq \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}) \leq \psi^{n-1}(q(x_1, x_0)), \quad \forall n \geq 1. \tag{42}$$

Together with (42) and the triangle inequality, for all $k \geq 1$, we get that

$$\begin{aligned} q(x_n, x_{n+k}) &\leq q(x_n, x_{n+1}) + ... + q(x_{n+k-1}, x_{n+k}) \\ &< q(x_n, x_{n-1}) + ... + q(x_{n+k-1}, x_{n+k-2}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^{p-1}(q(x_1, x_0)) \\ &\leq \sum_{p=n}^{\infty} \psi^{p-1}(q(x_1, x_0)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{43}$$

Case 3. Assume that $\max\{q(x_{n-1}, x_n), q(x_n, x_{n-1}), q(x_{n+1}, x_n)\} = q(x_{n+1}, x_n)$. Since $q(x_{n+1}, x_n) > 0$ we have

$$q(x_n, x_{n+1}) \leq \psi(q(x_{n+1}, x_n)) < q(x_{n+1}, x_n), \tag{44}$$

and, as in the previous case, we get that $q(x_n, x_{n+k}) \to 0$ as $n \to \infty$. Therefore, by (34) and (43), we conclude that $\{x_n\}$ is a right-Cauchy sequence in (X, q).

From Remark 1.7, $\{x_n\}$ is a Cauchy sequence in complete quasi-metric space (X, q). This implies that there exists $u \in X$ such that

$$\lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = 0. \tag{45}$$

Then, using the continuity of T we obtain

$$\lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tx_{n-1}, Tu) = 0 \tag{46}$$

and

$$\lim_{n\to\infty} q(Tu, x_n) = \lim_{n\to\infty} q(Tu, Tx_{n-1}) = 0. \tag{47}$$

Thus, we have

$$\lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tu, x_n) = 0. \tag{48}$$

It follows from (45) and (48) that $u = Tu$, that is, u is a fixed point of T.

Now, suppose that X is regular with respect to α. Then, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(u, x_{n(k)}) \geq 1$ for all k. Applying (18), for all k, we get that

$$q(Tu, x_{n(k)+1}) = q(Tu, Tx_{n(k)}) \leq \alpha(u, x_{n(k)})q(Tu, Tx_{n(k)}) \leq \psi(M(u, x_{n(k)})) < M(u, x_{n(k)}), \tag{49}$$

where

$$M(u, x_{n(k)}) = \max\{q(u, x_{n(k)}), q(Tu, u), q(Tx_{n(k)}, x_{n(k)}), \frac{1}{2}[q(Tu, x_{n(k)}) + q(Tx_{n(k)}, u)]\}.$$

Thus,

$$q(Tu, x_{n(k)+1}) < \max\{q(u, x_{n(k)}), q(Tu, u), q(x_{n(k)+1}, x_{n(k)}), \frac{1}{2}[q(Tu, x_{n(k)}) + q(x_{n(k)+1}, u)]\}. \tag{50}$$

Letting $k \to \infty$ in the above inequality, we obtain that

$$q(Tu, u) < q(Tu, u) \tag{51}$$

which is a contradiction. Thus, we have $q(Tu, u) = 0$, that is $Tu = u$. □

Example 2.5. *Let $X = [0, \infty)$ be equipped with a quasi-metric $q : X \times X \to \mathbb{R}_0^+$ such that $q(x, y) = \max\{x - y, 0\}$. Consider the self mapping $T : X \to X$ such that*

$$Tx = \begin{cases} \frac{x}{8} & \text{if } x \in \left[0, \frac{1}{2}\right), \\ 1 - x & \text{if } x \in \left[\frac{1}{2}, 1\right], \\ \frac{x^2+8}{3} & \text{if } x \in (1, \infty), \end{cases}$$

and functions $\zeta \in \mathcal{Z}$, defined by $\zeta(s, t) = \frac{1}{2}s - t$, respectively $\psi \in \Psi$, $\psi(t) = \frac{t}{3}$. We define $\alpha : X \times X \to \mathbb{R}_0^+$ such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right) \\ 2 & \text{if } (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the self-mapping T is not continuous at $x = \frac{1}{2}$ and $x = 1$. We have to consider the following cases:

(a) If $0 \leq x < y < \frac{1}{2}$ because $q(Tx, Ty) = q(\frac{x}{8}, \frac{y}{8}) = 0$, inequality (18) becomes

$$0 = \alpha(x, y)q(Tx, Ty) \leq \frac{\psi(M(x, y))}{2}$$

which is obviously true for any function $\psi \in \Psi$.

(b) For $0 \leq \frac{y}{8} \leq y \leq \frac{x}{8} < x < \frac{1}{2}$ we have

$$M(x, y) = \max\left\{q(x, y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), \frac{1}{2}\left[q(\frac{x}{8}, y) + q(\frac{y}{8}, x)\right]\right\} = \max\left\{x - y, 0, 0, \frac{1}{2}\left(\frac{x}{8} - y\right)\right\} = x - y$$

and $q(Tx, Ty) = q\left(\frac{x}{8}, \frac{y}{8}\right) = \frac{x}{8} - \frac{y}{8}$. Taking into account the properties of the function ζ, we get

$$\alpha(x, y)q(Tx, Ty) = \frac{x - y}{8} < \frac{1}{2} \cdot \frac{x - y}{3} = \frac{1}{2}\psi(M(x, y)).$$

(c) For $0 \leq \frac{y}{8} \leq \frac{x}{8} \leq y \leq x < \frac{1}{2}$

$$M(x, y) = \max\left\{q(x, y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), \frac{1}{2}\left[q(\frac{x}{8}, y) + q(\frac{y}{8}, x)\right]\right\} = \max\{x - y, 0, 0, 0\} = x - y$$

and $q(Tx,Ty) = q\left(\frac{x}{8},\frac{y}{8}\right) = \frac{x}{8} - \frac{y}{8}$. *So,*

$$\alpha(x,y)q(Tx,Ty) = \frac{x-y}{8} < \frac{1}{2}\cdot\frac{x-y}{3} = \frac{1}{2}\psi(M(x,y)).$$

(*d*) *If* $x \in \left[0,\frac{1}{2}\right)$ *and* $y = \frac{1}{2}$ *we have* $q(Tx,T\frac{1}{2}) = q(\frac{x}{8},\frac{1}{2}) = 0$ *and* $M(x,\frac{1}{2}) = \frac{1}{2}\left(\frac{1}{2}-x\right)$, *and (18) is true.*

(*e*) *For* $x = y = \frac{1}{2}$, *we get* $q(T\frac{1}{2},T\frac{1}{2}) = q(\frac{1}{2},\frac{1}{2}) = 0$ *and* $M\left(\frac{1}{2},\frac{1}{2}\right) = 0$, *so, also, (18) is fulfilled. Notice that for any other possibilities, the result is provided easily from the fact that* $\alpha(x,y) = 0$. *Let us check that T is* $\alpha-$ *admissible. From the definition of function* α, *for any* $(x,y) \in \left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right)$ *we have*

$$\alpha(x,y) = 1 \Rightarrow \alpha(Tx,Ty) = 1.$$

and for $x = y = \frac{1}{2}$,

$$\alpha(\frac{1}{2},\frac{1}{2}) = 2 \geq 1 \Rightarrow \alpha(T\frac{1}{2},T\frac{1}{2}) = \alpha(\frac{1}{2},\frac{1}{2}) = 2 \geq 1.$$

Thus, the first condition (*i*) *of Theorem (2.4) is satisfied. The second condition* (*ii*) *of Theorem is also fulfilled. Indeed, for* $x_0 = 0$, *we have* $\alpha(0,T0) = \alpha(0,T0) = \alpha(0,0) = 1 \geq 1$. *It is also easy to see that* (X,q) *is regular. Indeed let* $\{x_n\}$ *be a sequence in X such that for all n and* $x_n \to x$ *as* $n \to \infty$. *Since* $\alpha(x_n,x_{n+1}) \geq 1$ *for all n, by the definition of* α, *we have* $x_n \in \left[0,\frac{1}{2}\right)$ *for all n and* $x \in \left[0,\frac{1}{2}\right)$. *Then,*

$$\alpha(x_n,x) = 1 \geq 1.$$

If $x = \frac{1}{2}$, *then* $x_n = \frac{1}{2}$ *and* $\alpha(x_n,x) = 2 \geq 1$. *It is clear that T satisfies all the conditions of Theorem (2.4) for any choice of* $\zeta \in S$ *and T has two distinct fixed points, namely,* $x = 0$ *and* $x = \frac{1}{2}$.

Theorem 2.6. *Let* (X,q) *be a complete quasi-metric space and a map* $\alpha : X\times X \to [0,\infty)$. *Suppose that there exist* $\zeta \in \mathcal{Z}$, $\psi \in \Psi$ *and a self-mapping T such that*

$$\zeta(\Gamma(x,y),\psi(q(x,y))) \geq 0, \tag{52}$$

for each $x,y \in X$, *where*

$$\Gamma(x,y) = \alpha(x,y)\left[\min\{q(Tx,Ty),q(x,Tx),q(y,Ty)\} - \min\{q(Ty,x),q(Tx,y)\}\right].$$

Suppose also that

(*i*) *T is* $\alpha-$*admissible;*
(*ii*) *there is a constant* $C > 1$ *such that* $\frac{1}{C}q(x,y) \leq q(y,x) \leq Cq(x,y)$ *for all* $x,y \in X$,
(*iii*) *there exists* $x_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$ *and* $\alpha(Tx_0,x_0) \geq 1$;
(*iv*) *either, T is continuous, or*
(*iv*′) *X is regular with respect to mapping* α.

Then for each $x_0 \in X$ *the sequence* (T^nx_0) *converges to a fixed point of T.*

Proof. By verbatim of the first lines in the proof of Theorem 2.3, we get a constructive sequence $\{x_n\} \subset X$. Further, with the same reasoning in the proof of Theorem 2.3, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,

$$q(x_{n+1},x_n) > 0 \text{ and } q(x_n,x_{n+1}) > 0.$$

Taking the inequality (52), the axiom (ζ_1) and Lemma 2.2 into account, we find

$$0 \leq \zeta(\Gamma(x_{n-1},x_n),\psi(q(x_{n-1},x_n))) < \psi(q(x_{n-1},x_n)) - \Gamma(x_{n-1},x_n), \tag{53}$$

for all $n \geq 1$. In conclusion, we have

$$\Gamma(x_{n-1}, x_n) \leq \psi(q(x_{n-1}, x_n)), \tag{54}$$

where

$$\begin{aligned}\Gamma(x_{n-1}, x_n) &= \alpha(x_{n-1}, x_n)\left[\min\{q(Tx_{n-1}, Tx_n), q(x_{n-1}, Tx_{n-1}), q(x_n, Tx_n)\} - \right.\\ &\quad \left. - \min\{q(Tx_n, x_{n-1}), q(Tx_{n-1}, x_n)\}\right]\\ &= \alpha(x_{n-1}, x_n)\left[\min\{q(x_n, x_{n+1}), q(x_{n-1}, x_n), q(x_n, x_{n+1})\} - \right.\\ &\quad \left. - \min\{q(x_{n+1}, x_{n-1}), q(x_n, x_n)\}\right]\\ &= \alpha(x_{n-1}, x_n)\min\{q(x_n, x_{n+1}), q(x_{n-1}, x_n)\}\end{aligned} \tag{55}$$

By Lemma 2.2, together with (55) and (5) we obtain that

$$\min\{q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} \leq \alpha(x_{n-1}, x_n)\min\{q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} \leq \psi(q(x_{n-1}, x_n)). \tag{56}$$

To understand the inequality (56), we consider two cases. For the first case, we suppose that $\min\{q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_{n-1}, x_n)$. Since $\psi(t) < t$ for all $t \geq 0$ we have

$$q(x_{n-1}, x_n) \leq \psi(q(x_{n-1}, x_n)) < q(x_{n-1}, x_n),$$

which is a contradiction. Therefore, $\min\{q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_n, x_{n+1})$ and thus we have

$$q(x_n, x_{n+1}) \leq \psi(q(x_{n-1}, x_n)) < q(x_{n-1}, x_n). \tag{57}$$

Applying recurrently Remark 1.13 we find that

$$q(x_n, x_{n+1}) \leq \psi(q(x_{n-1}, x_n)) < ... < \psi^n(q(x_0, x_1). \tag{58}$$

Now, we show that $\{x_n\}$ is right-Cauchy sequence. Together with (58) and the triangle inequality, for all $k \geq 1$, we get that

$$\begin{aligned}q(x_n, x_{n+k}) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + ... + q(x_{n+k-1}, x_{n+k})\\ &= \sum_{p=n}^{n+k-1} q(x_p, x_{p+1}) \leq \sum_{p=n}^{n+k-1} \psi^p(q(x_0, x_1)\\ &= \sum_{p=n}^{\infty} \psi^p(q(x_0, x_1) \to 0 \text{ as } n \to \infty.\end{aligned} \tag{59}$$

We conclude that the sequence $\{x_n\}$ is right-Cauchy in (X, q). Analogously, we shall prove that $\{x_n\}$ is left-Cauchy in (X, q). If substitute $x = x_n$ and $y = x_{n-1}$ in (52), we get

$$\Gamma(x_n, x_{n-1}) \leq \psi(q(x_n, x_{n-1}))$$

or, using Lemma (2.2)

$$\min\{q(x_{n+1}, x_n), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} \leq \psi(q(x_n, x_{n-1})) \tag{60}$$

We shall examine three cases:

Case 1. Obviously, if $\min\{q(x_{n+1}, x_n), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_{n+1}, x_n)$. Since $\psi \in \Psi$, inequality (60) yields

$$q(x_{n+1}, x_n) \leq \psi(q(x_n, x_{n-1})) \tag{61}$$

for all $n \geq 1$. Recursively, we derive

$$q(x_{n+1}, x_n) \leq \psi(q(x_n, x_{n-1})) \leq ... \leq \psi^n(q(x_1, x_0)), \qquad \forall n \geq 1. \tag{62}$$

Together with (62) and the triangle inequality, we get, for all $k \geq 1$

$$\begin{aligned} q(x_{n+k}, x_n) &\leq q(x_{n+k}, x_{n+k-1}) + q(x_{n+k-1}, x_{n+k-2}) + ... + q(x_{n+1}, x_n) \\ &\leq \sum_{p=n}^{n+k-1} \psi^p(q(x_1, x_0)) \\ &\leq \sum_{p=n}^{\infty} \psi^p(q(x_1, x_0)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{63}$$

Therefore, $\{x_n\}$ is a left-Cauchy sequence in (X, q).

Case 2. If $\min\{q(x_{n+1}, x_n), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_{n-1}, x_n)$ then (60) becomes

$$q(x_{n-1}, x_n) \leq \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}) \tag{64}$$

for all $n \in \mathbb{N}$. On the other hand, by (ii), there is a constant $C > 1$ such that

$$q(x_{n-1}, x_n) \leq \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}) \leq Cq(x_{n-1}, x_n). \tag{65}$$

By using the (58) and (59) we get, we conclude that it is left Cauchy.

Case 3. If $\min\{q(x_{n+1}, x_n), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_n, x_{n+1})$ then we conclude that the sequence $\{x_n\}$ is left Cauchy by the same reasons in Case 2.

By Remark (1.7), we deduce that $\{x_n\}$ is a Cauchy sequence in complete quasi-metric space (X, q). It implies that there exists $u \in X$ such that

$$\lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = 0. \tag{66}$$

We shall prove that $Tu = u$. Since from (iv) T is continuous, we obtain

$$\lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tx_{n-1}, Tu) = 0 \tag{67}$$

and respectively,

$$\lim_{n\to\infty} q(Tu, x_n) = \lim_{n\to\infty} q(Tu, Tx_{n-1}) = 0 \tag{68}$$

Thus we have

$$\lim_{n\to\infty} q(Tx_n, u) = \lim_{n\to\infty} q(u, Tx_n) = 0. \tag{69}$$

From (66), (69) and together with the uniqueness of the limit, we conclude that $u = Tu$, that is, u is a fixed point of T. Next, we will show that u is the fixed point of T using the alternative hypothesis (iv'). Then, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Substituting $x = x_{n(k)}$ and $y = u$ in (52) we obtain

$$\zeta(\Gamma(x_{n(k)}, u), \psi(q(x_{n(k)}, u))) \geq 0, \tag{70}$$

or, equivalent $\Gamma(x_{n(k)}, u) \leq \psi(q(x_{n(k)}, u)))$. We have,

$$\begin{aligned} \min &\left\{q(Tx_{n(k)}, Tu), q(x_n, Tx_{n(k)}), q(u, Tu)\right\} - \min\left\{q(x_{n(k)}, Tu), q(Tx_{n(k)}, u)\right\} \\ &\leq \alpha(x_{n(k)}, u)\left[\min\left\{q(Tx_{n(k)}, Tu), q(x_{n(k)}, Tx_{n(k)}), q(u, Tu)\right\} - \right. \\ &\left. - \min\left\{q(x_{n(k)}, Tu), q(Tx_{n(k)}, u)\right\}\right] \\ &\leq \psi(q(x_{n(k)}, u)) \end{aligned} \tag{71}$$

Then it follows that

$$\begin{aligned} \min \ & \left\{q(x_{n(k)+1}, Tu), q(x_{n(k)}, x_{n(k)+1}), q(u, Tu)\right\} - \min\left\{q(x_{n(k)}, Tu), q(x_{n(k)+1}, u)\right\} \\ & \leq \psi(q(x_{n(k)}, u)) < q(x_{n(k)}, u). \end{aligned} \tag{72}$$

Taking limit as $n \to \infty$, and using Remark (1.13), respectively (66)we obtain

$$q(u, Tu) < 0$$

It is a contradiction. Hence, we conclude that $u = Tu$, that is, u is a fixed point of T. □

On account of the condition ($\zeta 2$) and taking $\alpha(x, y) = 1$ in Theorem (2.6), we get the following result:

Theorem 2.7. *Let (X, q) be a complete quasi-metric space, such that the condition (ii) from Theorem (2.6) is satisfied. Let a function $\psi \in \Psi$ and a map $T : X \to X$, such that*

$$\min\{q(Tx, Ty), q(x, Tx), q(y, Ty)\} - \min\{q(x, Ty), q(Tx, y)\} \leq \psi(q(x, y)). \tag{73}$$

Then for each $x \in X$ the sequence $(T^n x)$ converges to a fixed point of T.

Corollary 2.8. *Let (X, q) be a complete quasi-metric space, $k \in [0, 1)$ and a map $T : X \to X$, such that*

$$\min\{q(Tx, Ty), q(x, Tx), q(y, Ty)\} - \min\{q(x, Ty), q(Tx, y)\} \leq \ k \cdot q(x, y)). \tag{74}$$

Then for each $x \in X$ the sequence $(T^n x)$ converges to a fixed point of T.

Proof. It is sufficient to take $\psi(t) = kt$, where $k \in [0, 1)$, in Theorem (2.7). □

Example 2.9. *Let $X = A \cup B$ where $A = \{a, b, c, d\}$ and $B = [1, 2]$. Consider the self mapping $T : X \to X$ such that*

$$Tx = \begin{cases} a & \text{if } x \in \{a, b\} \cup B, \\ d & \text{if } x \in \{c, d\}. \end{cases}$$

Define a quasi-metric $q : X \times X \to \mathbb{R}_0^+$ as

$$q(x, y) = \begin{cases} \frac{1}{16} & \text{if } (x, y) = (a, c), \\ \frac{1}{6} & \text{if } (x, y) = (c, a), \\ \frac{1}{8} & \text{if } (x, y) \in \{(a, d), (d, a), (b, d), (d, b), (c, d), (d, c)\}, \\ \frac{1}{4} & \text{if } (x, y) \in \{(a, b), (b, a), (b, c), (c, b)\} \cup A \times B \cup B \times A, \\ \frac{|x-y|}{2} & \text{otherwise.} \end{cases}$$

Define $\alpha : X \times X \to \mathbb{R}_0^+$ such that

$$\alpha(x, y) = \begin{cases} 2 & \text{if } (x, y) \in \{(a, c), (c, a), (a, d), (d, a), (a, a), (d, d), (a, b), (b, a), (c, d), (d, c)\} \\ 0 & \text{otherwise.} \end{cases}$$

Let us first notice that, from $Ta = a$, $Td = d$, we get that $q(a, Ta) = q(a, a) = 0$, $q(d, Td) = q(d, d) = 0$ and

$$\Gamma(x, y) = \alpha(x, y)\left[\min\{q(Tx, Ty), q(x, Tx), q(y, Ty)\} - \min\{q(Ty, x), q(Tx, y)\}\right] \leq 0$$

for any $(x, y) \in A_1 = \{(a, c), (c, a), (a, d), (d, a), (a, a), (d, d), (a, b), (b, a), (c, d), (d, c)\}$. Then, the condition

$$0 \leq \zeta(\Gamma(x, y), \psi(q(x, y))) < \psi(q(x, y) - \Gamma(x, y), \tag{75}$$

is fulfilled trivially for $(x, y) \in A_1$ *and for any choice of* $\psi \in \Psi$ *and* $\zeta \in \mathcal{Z}$*. Now, it is easy to get that* T *is* $\alpha-$*admissible, because when* $x \in A$ *we have that* $Tx \in \{a, d\}$*. Hence,*

$$\alpha(x, y) = 2 \geq 1 \Rightarrow \alpha(Tx, Ty) = 2 \geq 1$$

for any $(x, y) \in A_1$*. Thus, the condition* (i) *from Theorem (2.6) is satisfied. From the definition of the quasi-metric* q*, condition* (ii) *holds for any* $C > 1$ *and* (x, y) *except* (a, c) *and* (c, a)*. Let's check for these two cases. For* $C = 4$ *we have*

$$\frac{1}{4} \cdot \frac{1}{16} = \frac{1}{4} \cdot q(a, c) \leq \frac{1}{6} \leq 4 \cdot \frac{1}{16} = 4 \cdot q(a, c)$$

and

$$\frac{1}{4} \cdot \frac{1}{6} = \frac{1}{4} \cdot q(a, c) \leq \frac{1}{16} \leq 4 \cdot \frac{1}{6} = 4 \cdot q(a, c).$$

The condition (iii) *is also satisfied. Indeed, for any* $x_0 \in A$*, we have* $\alpha(x_0, Tx_0) = 2 \geq 1$ *and* $\alpha(Tx_0, x_0) = 2 \geq 1$*. It is also easy to see that* (X, q) *is regular, because, whatever the initial* $x_0 \in \{a, b\}$ *chosen, the sequence* $\{x_n\}$ *tends to* a*, and*

$$\alpha(a, b) \geq 1, \alpha(b, a) \geq 1 \text{ and } \alpha(a, a) \geq 1.$$

Analoguosly, if $x_0 \in \{c, d\}$*, then the sequence* $\{x_n\}$ *tends to* d*, and*

$$\alpha(c, d) \geq 1, \alpha(d, c) \geq 1 \text{ and } \alpha(d, d) \geq 1.$$

Thus, all conditions of Theorem (2.6) are provided. Notice that $Ta = a$ *and* $Td = d$ *are the fixed points of* T*.*

Theorem 2.10. *Let* (X, d) *be a complete quasi-metric space and a map* $\alpha : X \times X \to [0, \infty)$*. Suppose that there exist* $\zeta \in \mathcal{Z}$*,* $\psi \in \Psi$*,* $a \geq 0$ *and a self-mapping* T *such that*

$$\zeta(P(x, y), \psi(S(x, y))) \geq 0, \tag{76}$$

for each $x, y \in X$*, where*

$$P(x, y) = \alpha(x, y)\left(K(x, y) - a \cdot Q(x, y)\right),$$

$$K(x, y) = \min\left\{q(Tx, Ty), q(y, Ty)\right\},$$

$$Q(x, y) = \min\left\{q(x, Ty), q(y, Tx)\right\}$$

and

$$S(x, y) = \max\left\{q(x, y), q(x, Tx), q(y, Ty)\right\}.$$

Suppose also that

(i) *T is* $\alpha-$*admissible;*
(ii) *there is a constant* $C > 1$ *such that* $\frac{1}{C}q(x, y) \leq q(y, x) \leq Cq(x, y)$ *for all* $x, y \in X$*,*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(Tx_0, x_0) \geq 1$*;*
(iv) *either, T is continuous, or*
(iv') *X is regular with respect to mapping* α*.*

Then for each $x_0 \in X$ *the sequence* $(T^n x_0)$ *converges to a fixed point of* T*.*

Proof. For an arbitrary $x \in X$, we shall construct an iterative sequence $\{x_n\}$ as follows:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{77}$$

We suppose that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{78}$$

Indeed, if for some $n \in \mathbb{N}$ we have the inequality $x_n = Tx_{n-1} = x_{n-1}$, then, the proof is completed.

By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (76), we derive that

$$0 \leq \zeta(P(x_{n-1}, x_n), \psi(S(x_{n-1}, x_n))) < \psi(S(x_{n-1}, x_n)) - P(x_{n-1}, x_n). \tag{79}$$

or, equivalent,

$$P(x_{n-1}, x_n) \leq \psi(S(x_{n-1}, x_n)) \tag{80}$$

where

$$K(x_{n-1}, x_n) = \min\{q(Tx_{n-1}, Tx_n), q(x_n, Tx_n)\} = \min\{q(x_n, x_{n+1}), q(x_n, x_{n+1})\} = q(x_n, x_{n+1})$$

$$Q(x_{n-1}, x_n) = \min\{q(x_{n-1}, Tx_n), q(x_n, Tx_{n-1})\} = \min\{q(x_{n-1}, x_{n+1}), q(x_n, x_n)\} = 0$$

$$P(x_{n-1}, x_n) = \alpha(x_{n-1}, x_n)[K(x_{n-1}, x_n) - a \cdot Q(x_{n-1}, x_n)] = \alpha(x_{n-1}, x_n)q(x_n, x_{n+1}).$$

and

$$\begin{aligned} S(x_{n-1}, x_n) &= \max\{q(x_{n-1}, x_n), q(x_{n-1}, Tx_{n-1}), q(x_n, Tx_n)\} \\ &= \max\{q(x_{n-1}, x_n), q(x_{n-1}, x_n), q(x_n, x_{n+1})\} \end{aligned}$$

Taking Lemma (2.2) into account, the inequality (80) becomes

$$q(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)q(x_n, x_{n+1}) \leq \psi\left(\max\{q(x_{n-1}, x_n), q(x_n, x_{n+1})\}\right). \tag{81}$$

Since $\psi(t) < t$ for all $t > 0$, in the case of $\max\{q(x_{n-1}, x_n), q(x_n, x_{n+1})\} = q(x_n, x_{n+1})$, inequality (81) turns into

$$q(x_n, x_{n+1}) \leq \psi\left(q(x_n, x_{n+1})\right) < q(x_n, x_{n+1}),$$

which is a contradiction. Hence, inequality (81) yields that

$$q(x_n, x_{n+1}) \leq \psi\left(q(x_{n-1}, x_n)\right) < q(x_{n-1}, x_n), \tag{82}$$

and, recursively

$$q(x_n, x_{n+1}) \leq \psi^n\left(q(x_0, x_1)\right) \tag{83}$$

In the following we shall prove that the sequence $\{x_n\}$ is right-Cauchy. By using the triangle inequality, for all $k \geq 1$ we get the following approximation

$$\begin{aligned} q(x_n, x_{n+k}) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+k}) \\ &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + ... + q(x_{n+k-1}, x_{n+k}). \end{aligned} \tag{84}$$

Combining (83) and (84) we derive that

$$\begin{aligned} q(x_n, x_{n+k}) &\leq \psi^n(q(x_0, x_1)) + \psi^{n+1}q(x_0, x_1) + ... + \psi^{n+k-1}(q(x_0, x_1)) \\ &\leq \sum_{p=n}^{n+k-1} \psi^p(q(x_0, x_1)) \\ &\leq \sum_{p=n}^{\infty} \psi^p(q(x_0, x_1)). \end{aligned} \tag{85}$$

Letting $n \to \infty$ in the above inequality, we derive that $\sum_{p=n}^{\infty} \psi^p(q(x_0, x_1)) \to 0$. Hence, $q(x_n, x_{n+k}) \to 0$ as $n \to \infty$. We conclude that the sequence $\{x_n\}$ is right-Cauchy in (X, q). Analogously, we shall prove that $\{x_n\}$ is a left-Cauchy sequence in (X, q). For $x = x_n$ and $y = x_{n-1}$, together with Lemma (2.2) we get:

$$\zeta(P(x_n, x_{n-1}), \psi(S(x_n, x_{n-1}))) \geq 0, \tag{86}$$

or, equivalent, using (ζ1),

$$P(x_n, x_{n-1}) \leq \psi(S(x_n, x_{n-1})), \tag{87}$$

where

$$K(x_n, x_{n-1}) = \min\{q(Tx_n, Tx_{n-1}), q(x_{n-1}, Tx_{n-1})\} = \min\{q(x_{n+1}, x_n), q(x_{n-1}, x_n)\}$$

$$Q(x_n, x_{n-1}) = \min\{q(x_n, Tx_{n-1}), q(x_{n-1}, Tx_n)\} = \min\{q(x_n, x_n), q(x_{n-1}, x_n)\} = 0$$

$$\begin{aligned} P(x_n, x_{n-1}) &= \alpha(x_n, x_{n-1})\,[K(x_n, x_{n-1}) - a \cdot Q(x_n, x_{n-1})] \\ &= \alpha(x_n, x_{n-1}) \min\{q(x_{n+1}, x_n), q(x_{n-1}, x_n)\}. \end{aligned}$$

and

$$\begin{aligned} S((x_n, x_{n-1}) &= \max\{q(x_n, x_{n-1}), q(x_n, Tx_n), q(x_{n-1}, Tx_{n-1})\} \\ &= \max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} \end{aligned}$$

Since ψ is a nondecreasing function, (87) implies that

$$\begin{aligned} \min\{q(x_{n+1}, x_n), q(x_{n-1}, x_n)\} &\leq \alpha(x_n, x_{n-1}) \min\{q(x_{n+1}, x_n), q(x_{n-1}, x_n)\} \\ &\leq \psi\left(\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\}\right) \\ &< \max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\}. \end{aligned} \tag{88}$$

We shall examine two cases:

Case 1. If $\min\{q(x_{n+1}, x_n), q(x_{n-1}, x_n)\} = q(x_{n+1}, x_n)$ we have

$$q(x_{n+1}, x_n) \leq \psi\left(\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\}\right) \tag{89}$$

(1.a.) If $\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_n, x_{n-1})$, then (89) becomes

$$q(x_{n+1}, x_n) < \psi\left(q(x_n, x_{n-1})\right) < \dots < \psi^n\left(q(x_1, x_0)\right) \tag{90}$$

Using the triangle inequality, for all $k \geq 1$

$$\begin{aligned} q(x_{n+k}, x_n) &\leq q(x_{n+k}, x_{n+k-1}) + q(x_{n+k-1}, x_{n+k-2}) + \dots + q(x_{n+1}, x_n) \\ &\leq \psi^n\left(q(x_1, x_0)\right) + \dots + \psi^{n+k-1}\left(q(x_1, x_0)\right) \\ &= \sum_{p=n}^{n+k-1} \psi^p\left(q(x_1, x_0)\right) < \sum_{p=n}^{\infty} \psi^p\left(q(x_1, x_0)\right) \to 0, \end{aligned} \tag{91}$$

as $n \to \infty$, which proves that $\{x_n\}$ is left Cauchy.

(1.b.) If $\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_n, x_{n+1})$ then, from Remark 1.14 inequality (89) becomes

$$q(x_{n+1}, x_n) \leq \psi\left(q(x_n, x_{n+1})\right) < q(x_n, x_{n+1}). \tag{92}$$

Considering triangle inequality, together with (92), for any $k \geq 1$, we get

$$\begin{aligned} q(x_{n+k}, x_n) &\leq q(x_{n+k}, x_{n+k-1}) + q(x_{n+k-1}, x_{n+k-2}) + ... + q(x_{n+1}, x_n) \\ &< q(x_{n+k-1}, x_{n+k}) + q(x_{n+k-2}, x_{n+k-1}) + ... + q(x_n, x_{n+1}). \end{aligned} \tag{93}$$

Using (83) and (85) we conclude that $\{x_n\}$ is left Cauchy.
(1.c.) If $\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_{n-1}, x_n)$ then, from Remark 1.14 inequality (89) becomes

$$q(x_{n+1}, x_n) \leq \psi(q(x_{n-1}, x_n)) < q(x_{n-1}, x_n). \tag{94}$$

Using (83) and like above we can show also, that $\{x_n\}$ is left Cauchy.

Case 2. If $\min\{q(x_{n+1}, x_n), q(x_{n-1}, x_n)\} = q(x_{n-1}, x_n)$ we have

$$q(x_{n-1}, x_n) \leq \psi(\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\}) \tag{95}$$

(2.a.) If $\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_n, x_{n-1})$, then (95) becomes

$$q(x_{n-1}, x_n) \leq \psi(q(x_n, x_{n-1})) < q(x_n, x_{n-1}). \tag{96}$$

and, by (ii) we have

$$q(x_{n-1}, x_n) < q(x_n, x_{n-1}) \leq Cq(x_{n-1}, x_n), \tag{97}$$

where $C > 1$. By using the (58) and (59) we get, we conclude that it is left Cauchy.
(2.b.) If $\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_n, x_{n+1})$, then (95) becomes

$$q(x_{n-1}, x_n) \leq \psi(q(x_n, x_{n+1})) < q(x_n, x_{n+1}). \tag{98}$$

From (83) and since $\psi \in \Psi$ we get

$$q(x_{n-1}, x_n) \leq \psi(q(x_n, x_{n+1})) < q(x_n, x_{n+1}) \leq \psi(q(x_{n-1}, x_n)) < q(x_{n-1}, x_n), \tag{99}$$

which is a contradiction.
(2.c.) If $\max\{q(x_n, x_{n-1}), q(x_n, x_{n+1}), q(x_{n-1}, x_n)\} = q(x_{n-1}, x_n)$, since $\psi(t) < t$ for all $t \geq 1$, we get

$$q(x_{n-1}, x_n) \leq \psi(q(x_{n-1}, x_n)) < q(x_{n-1}, x_n.) \tag{100}$$

This is a contradiction. Using Remark 1.7, we deduce that x_n is a Cauchy sequence in complete quasi-metric space (X, q). It implies that there exists $u \in X$ such that

$$\lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = 0 \tag{101}$$

and using the property (iv), (the continuity of T) we obtain

$$\lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tx_{n-1}, Tu) = 0 \tag{102}$$

and

$$\lim_{n\to\to\infty} q(Tu, x_n) = \lim_{n\to\infty} q(Tu, Tx_{n-1}) = 0. \tag{103}$$

Thus, we have

$$\lim_{n\to\infty} q(Tu, x_n) = \lim_{n\to\infty} q(x_n, Tu) = 0. \tag{104}$$

It follows from (101) and (104), $Tu = u$, that is, u is a fixed point of T.

If X is regular with respect to α, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Substituting $x = x_{n(k)}$ and $y = u$ in (76) we obtain

$$\zeta(P(x_{n(k)}, u), \psi(S(x_{n(k)}, u))) \geq 0, \tag{105}$$

where

$$\begin{aligned} P(x_{n(k)}, u) &= \alpha(x_{n(k)}, u)\left[K(x_{n(k)}, u) - a \cdot Q(x_{n(k)}, u)\right] \\ &= \alpha(x_{n(k)}, u)\left[\min\left\{q(Tx_{n(k)}, Tu), q(u, Tu)\right\} - a \cdot \min\left\{q(x_{n(k)}, Tu), q(u, Tx_{n(k)})\right\}\right]. \end{aligned} \tag{106}$$

Since ψ is a nondecreasing function, the inequality (87) turns into

$$\begin{aligned} \min &\left\{q(x_{n(k)+1}, Tu), q(u, Tu)\right\} - a\min\left\{q(x_{n(k)}, Tu), q(u, x_{n(k)+1})\right\} \\ &\leq \alpha(x_{n(k)}, u)\left[\min\left\{q(x_{n(k)+1}, Tu), q(u, Tu)\right\} - a\min\left\{q(x_{n(k)}, Tu), q(u, x_{n(k)+1})\right\}\right] \\ &\leq \psi\left(\max\left\{q(x_{n(k)}, u), q(x_{n(k)}, Tx_{n(k)}), q(u, Tu)\right\}\right) \\ &< \max\left\{q(x_{n(k)}, u), q(x_{n(k)}, x_{n(k)+1}), q(u, Tu)\right\}. \end{aligned} \tag{107}$$

Taking the limit as $k \to \infty$ in the above inequality and using Remark 1.3 we obtain

$$q(u, Tu) < q(u, Tu) \tag{108}$$

which is a contradiction. Therefore, we find $q(u, Tu) = 0$, that is, $Tu = u$. □

Theorem 2.11. *Let (X, q) be a complete quasi-metric space which satisfied (ii) from Theorem (2.10). Suppose that there exist $\psi \in \Psi$ and a self-mapping T, which satisfies*

$$K(x, y) - aQ(x, y) \leq \psi(S(x, y)) \tag{109}$$

for all distinct $x, y \in X$, $a \geq 0$, where $K(x, y)$, $Q(x, y)$ and $S(x, y)$ are defined as in Theorem (2.10). Then for each $x_0 \in X$ the sequence $(T^n x_0)$ converges to a fixed point of T.

Corollary 2.12. *Let (X, q) be a complete quasi-metric space which satisfied (ii) from Theorem (2.10). Suppose that there exist $a \geq 0$, $k \in [0, 1)$ and a self-mapping T which satisfies*

$$K(x, y) - aQ(x, y) \leq k \cdot (S(x, y)) \tag{110}$$

for all distinct $x, y \in X$, where $K(x, y)$, $Q(x, y)$ and $S(x, y)$ are defined as in Theorem (2.10). Then for each $x_0 \in X$ the sequence $(T^n x_0)$ converges to a fixed point of T.

References

[1] J. Achari, On Cirics non-unique fixed points, Mat. Vesnik, 13 (28)no. 3, 255-257 (1976)

[2] H.H. Alsulami, E. Karapınar, F. Khojasteh, A.F. Roldán-López-de-Hierro, A proposal to the study of contractions in quasi-metric spaces Discrete Dynamics in Nature and Society 2014, Article ID 269286, 10 pages.

[3] H. Aydi, M. Jellali, E. Karapinar, On fixed point results for α-implicit contractions in quasi-metric spaces and consequences, Nonlinear Analysis: Modelling and Control, 21 (1) (2016), 40-56.

[4] H. Aydi, A. Felhi, E. Karapinar, F.A. Alojail, Fixed points on quasi-metric spaces via simulation functions and consequences, Journal of Mathematical Analysis

[5] H. Alsulamia, E. Karapınar, V. Rakocevic, Ciric type nonunique fixed point theorems on b-metric spaces Filomat 31:11 (2017), 3147-3156.

[6] H. Aydi, M. Bota, E. Karapnar, and S. Mitrovic, A fixed point theorem for set-valued quasi-contractions in $b-$ metric spaces, Fixed Point Theory and Appl., 2012 (2012), Article Id 88.

[7] N. Bilgili, E. Karapınar, B. Samet, Generalized $\alpha-\psi$ contractive mappings in quasi-metric spaces and related fixed-point theorems, Journal of Inequalities and Applications 2014, 2014:36.

[8] M. Bota, C. Chifu, E. Karapınar, Fixed point theorems for generalized $(\alpha - \psi)$-Ciric-type contractive multivalued operators in $b-$metric spaces J. Nonlinear Sci. Appl. 9 (2016), Issue: 3 pages 1165-1177.
[9] Berinde,V: Sequences of operators and fixed points in quasimetric spaces, Mathematica, 41(4) (1996), 23-27.
[10] L.B. Ciric, On some maps with a nonunique fixed point. Publ. Inst. Math. 17, 52-58 (1974).
[11] E. Karapınar, A New Non-Unique Fixed Point Theorem, J. Appl. Funct. Anal. , 7 (2012),no:1-2, 92-97.
[12] E. Karapınar, Some Nonunique Fixed Point Theorems of Ćiric type on Cone Metric Spaces, Abstr. Appl. Anal., vol. 2010, Article ID 123094, 14 pages (2010).
[13] E. Karapınar, R.P. Agarwal, A note on Ciric type nonunique fixed point theorems, Fixed Point Theory and Applications (2017) 2017:20 DOI 10.1186/s13663-017-0614-z.
[14] F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, Filomat 29:6 (2015), 1189-1194.
[15] A.F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, J. Comput. Appl. Math. 275 (2015) 345-355.
[16] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$ -contractive type mappings, Nonlinear Anal., Theory Methods Appl., Ser. A, 75(4):2154-2165, 2012.