Filomat 32:13 (2018), 4773–4785
https://doi.org/10.2298/FIL1813773H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on the Stationary Distribution of Stochastic SIS Epidemic Model with Vaccination Under Regime Switching

Junna Hu[a], Zhiming Li[a], Ting Zeng[a], Zhidong Teng[a]

[a]*College of Mathematics and Systems Science, Xinjiang University*

Abstract. In this paper, the stochastic SIS epidemic model with vaccination under regime switching is further investigated. A new threshold R_0^s which is different from that given in [22] is established. A new technique to deal with the nonlinear incidence and vaccination for stochastic epidemic model under regime switching is proposed. When $R_0^s > 0$, the existence of a unique stationary distribution and the ergodic property are obtained by constructing a new stochastic Lyapunov function with Markov switching. The corresponding result which is acquired in [22] is improved and extended.

1. Introduction

It is well known that mathematical models which describe the dynamical behaviors of infectious diseases have played an important role in understanding the mechanism of disease transmission and control in the epidemiological aspect. Owing to our real life is full of randomness and stochasticity, transmissions of many infectious disease are inevitably affected by environmental random perturbations, such as white noise, colored noise and jumps noise, etc. (See [1-13]). In recent years many authors have proposed and investigated various types of stochastic epidemic dynamical models with such disturbances (see, for example, [13-26]). Particularly, we can see that the SIS (susceptible-infected-susceptible) type stochastic epidemic models are proposed and investigated in many articles, see for example [6,13,15,17-20,] and the references cited therein.

In view of the importance of vaccination for the control of some infectious diseases, Zhao and Jiang in [17] proposed and investigated the following stochastic SIS epidemic model with bilinear incidence and vaccination:

$$\begin{cases} dS(t) =[(1-q)A - \beta S(t)I(t) - (\mu+p)S(t) + \gamma I(t) \\ \qquad\qquad + \varepsilon V(t)]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) =[\beta S(t)I(t) - (\mu+\gamma+\alpha)I(t)]dt + \sigma_2 I(t)dB_2(t), \\ dV(t) =[qA + pS(t) - (\mu+\varepsilon)V(t)]dt + \sigma_3 V(t)dB_3(t). \end{cases} \tag{1}$$

2010 *Mathematics Subject Classification*. 60-XX
Keywords. Stochastic SIS epidemic model; stationary distribution; Markov switching; threshold; nonlinear incidence; vaccination.
Received: 30 June 2017; Revised: 03 June 2018; Accepted: 14 August 2018
Communicated by Miljana Jovanović
Research supported by the Natural Science Foundation of Xinjiang [Grant Nos. 2016D03022].
Email addresses: 1375718971@qq.com (Junna Hu), 865972735@qq.com (Zhiming Li), 945151318@qq.com (Ting Zeng), zhidong_tenf@sina.com (Zhidong Teng)

They established the threshold $\bar{R}_0 = R_0 - \frac{\sigma_2^2}{2(\mu+\gamma+\alpha)}$, where $R_0 = \frac{\beta S_0}{\mu+\gamma+\alpha}$ is the threshold of the corresponding deterministic model of model (1) and S_0 is the number of susceptibles in the disease-free equilibrium which is given in [17]. They further obtained that when $\bar{R}_0 < 1$ then the disease I dies out with probability one, and when $\bar{R}_0 > 1$ then the solution $(S(t), I(t), V(t))$ is permanent in the mean with probability one. We easily see that $\bar{R}_0$ is an extension of R_0 and $\bar{R}_0 < 1$ (or > 1) is equivalent to $R_0^* = \beta S_0 - (\mu + \gamma + \alpha + \frac{1}{2}\sigma_2^2) < 0$ (or > 0). Therefore, $\bar{R}_0$ has a very evident epidemiological meaning.

In [22], Zhang et al. introduced environmental colored noise into the above model, and proposed and investigated the following stochastic SIS epidemic model with bilinear incidence and vaccination under regime switching:

$$\begin{cases} dS(t) =[(1 - q_{r(t)})A_{r(t)} - \beta_{r(t)}S(t)I(t) - (\mu_{r(t)} + p_{r(t)})S(t) \\ \qquad\quad + \gamma_{r(t)}I(t) + \varepsilon_{r(t)}V(t)]dt + \sigma_{1r(t)}S(t)dB_1(t), \\ dI(t) =[\beta_{r(t)}S(t)I(t) - (\mu_{r(t)} + \gamma_{r(t)} + \alpha_{r(t)})I(t)]dt + \sigma_{2r(t)}I(t)dB_2(t), \\ dV(t) =[q_{r(t)}A_{r(t)} + p_{r(t)}S(t) - (\mu_{r(t)} + \varepsilon_{r(t)})V(t)]dt + \sigma_{3r(t)}V(t)dB_3(t), \end{cases} \tag{2}$$

where the regime switching is modeled by a continuous time Markov chain $r(t)$ with the values in a finite state space $\mathcal{M} = \{1, 2, \cdots, N\}$. They established the threshold $R_0^s = \sum_{k\in\mathcal{M}} \pi_k R_{0k}$, where $R_{0k} = c_1(k)(1-q_k)A_k + c_2(k)q_kA_k - (\mu_k + \gamma_k + \alpha_k + \frac{1}{2}\sigma_{2k}^2)$, $c_1(k)$ and $c_2(k)$ are the solutions of linear system (3) in [22]. They further proved that if $R_0^s > 0$ then the solution $(S(t), I(t), V(t))$ of model (2) admits a unique ergodic stationary distribution.

Comparing the above two thresholds R_0^* and R_0^s, we see that R_0^s is completely different from R_0^*. However, since there are $c_1(k)$ and $c_2(k)$ in R_0^s, the epidemiological meaning of R_0^s is not very evident. Therefore, an important and interesting problem is to establish a new threshold R_0^s for model (2) which is similar to $\bar{R}_0$ or R_0^*.

On the other hand, the nonlinear incidence rates are very important substances in modelling the dynamics of epidemic systems. In recent years, many authors have investigated various types of stochastic epidemic models with nonlinear incidence (see, for example, [9,24-26]). A stochastic SIS epidemic model with nonlinear incidence $\beta Sg(I)$ is proposed in [24], where the authors established the threshold criteria on the extinction and permanence in the mean in probability meaning. In [26], the authors extended the model in [24] into a general nonlinear incidence rate $f(S, I)$, and the sufficient conditions for the global stability of the disease-free equilibrium, permanence in the mean with probability one and existence of unique stationary distribution are established. However, from the expression of the threshold R_0^s for model (2) we easily see that this threshold only can been used to model (2) with the bilinear incidence. Therefore, another important problem is to extend the results obtained for model (2) to the model with nonlinear incidence by introducing a new threshold.

Motivated by the above works, in this paper we propose the following stochastic SIS epidemic model with vaccination and nonlinear incidence under regime switching:

$$\begin{cases} dS(t) =[(1 - q_{r(t)})A_{r(t)} - \beta_{r(t)}f(S(t))g(I(t)) - (\mu_{r(t)} + p_{r(t)})S(t) \\ \qquad\quad + \gamma_{r(t)}I(t) + \varepsilon_{r(t)}V(t)]dt + \sigma_{1r(t)}S(t)dB_1(t), \\ dI(t) =[\beta_{r(t)}f(S(t))g(I(t)) - (\mu_{r(t)} + \gamma_{r(t)} + \alpha_{r(t)})I(t)]dt + \sigma_{2r(t)}I(t)dB_2(t), \\ dV(t) =[q_{r(t)}A_{r(t)} + p_{r(t)}S(t) - (\mu_{r(t)} + \varepsilon_{r(t)})V(t)]dt + \sigma_{3r(t)}V(t)dB_3(t), \end{cases} \tag{3}$$

where the regime switching is modeled by a continuous time Markov chain $r(t)$ with values in a finite state space. Our purpose is to establish a new threshold which is similar to threshold R_0^*, and further to obtain a threshold criterion for the existence of a unique stationary distribution and the ergodic property by constructing a new stochastic Lyapunov function with Markov switching. Particularly, we will propose a new technique to deal with the nonlinear incidence functions and vaccination for stochastic epidemic model under regime switching. We will give a considerable improvement and extension for the corresponding results given in [22].

This paper is organized as follows. In Section 2, as preliminaries some notations and useful lemmas are introduced. In Section 3, the main theorem in this paper is stated and proved. In Section 4, numerical examples are given to illustrate the main results. Lastly, a brief conclusion is presented in Section 5.

2. Preliminaries

Denote $R_+ = [0, \infty)$ and $R_+^n = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$. For any bounded function $f(t)$ defined on $[0, \infty)$, we denote $f^u = \sup_{t\geq 0} f(t)$ and $f^l = \inf_{t\geq 0} f(t)$. Let $\mathcal{M} = \{1, 2, \cdots, N\}$. For a vector $g = (g_1, g_2, \cdots, g_N)$, let $\hat{g} = \min_{k\in\mathcal{M}} g_k$ and $\breve{g} = \max_{k\in\mathcal{M}} g_k$.

In model (3), $S(t)$, $I(t)$ and $V(t)$ denote the numbers of susceptible, infectious and immune, respectively. $r(t)$ for $t \geq 0$ be a right-continuous Markov chain with values in a finite space $\mathcal{M}$; $B_i(t)$ $(i = 1, 2, 3)$ are independent standard Browian motion and σ_i^2 represent the intensities of $B_i(t)$. The parameter $A_{r(t)}$ is the input of new members into the susceptible; $q_{r(t)}$ is a fraction of vaccinated for new members; $\beta_{r(t)}$ is the disease transmission coefficient; $\mu_{r(t)}$ is the natural death rate of the total population; $\gamma_{r(t)}$ is the recovery rate of infectious; $p_{r(t)}$ denotes the proportional coefficient of vaccinated for the susceptible and $0 \leq p_{r(t)} < 1$; $\varepsilon_{r(t)}$ is the rate of losing their immunity for vaccinated individuals; $\alpha_{r(t)}$ represents the disease-caused death rate of infectious.

Throughout this paper, we assume that model (1) is defined in a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

The generator $\Gamma = (\gamma_{ij})_{N\times N}$ of Markov chain $r(t)$ is given by

$$\mathcal{P}\{r(t+\vartriangle) = j|r(t) = i\} = \begin{cases} \gamma_{ij} \vartriangle +o(\vartriangle), & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii} \vartriangle +o(\vartriangle), & \text{if} \quad i = j, \end{cases}$$

where $\vartriangle > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\sum_{j=1}^N \gamma_{ij} = 0$. We assume that $B_i(t)$ and $r(t)$ are independent for $i = 1, 2, 3$, and the Markov chain $r(t)$ is irreducible, which means that the model can switch from one environmental regime to another. That is to say, the Markov chain $r(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$. It can be determined the equation $\pi\Gamma = 0$ subject to $\sum_{h=1}^N \pi_h = 1$ and $\pi_h > 0$ for all $h \in \mathcal{M}$. In this paper, we assume $\gamma_{ij} > 0$ for $i \neq j$, and for each $k \in \mathcal{M}$ the parameters A_k, q_k, β_k, μ_k, p_k, ε_k, γ_k and σ_{ik} $(i = 1, 2, 3)$ are nonnegative constants, and $q_k < 1$, $A_k > 0$, $\mu_k > 0$ and $\max\{p_k, q_k\} > 0$.

For functions $f(S)$ and $g(I)$ we introduce the following assumptions.

(H_1) The functions $f(S)$ and $g(I)$ are nonnegative and continuously differentiable for $S \geq 0$ and $I \geq 0$, respectively. $f(0) = g(0) = 0$, and $g'(0) > 0$.

(H_2) $\frac{g(I)}{I}$ is nonincreasing for $I > 0$, and $\max_{I>0}\{\frac{g'(0)}{g(I)} - \frac{1}{I}\} < \infty$.

(H_3) $f'(S) \geq 0$ and $f''(S) \geq 0$ for all $S \geq 0$, and $\sup_{S>0}\{\frac{f'(S)S}{f(S)}\} < \infty$.

Remark 2.1. *When $f(S) = \frac{S^m}{1+\omega_1 S}$ or $f(S) = S$ and $g(I) = \frac{I}{1+\omega_2 I}$ with constants $m \geq 2$, $\omega_1 > 0$ and $\omega_2 \geq 0$, then (H_1), (H_2) and (H_3) are clearly satisfied. Furthermore, from (H_3) we easily obtain $\sup_{S\geq 1} \frac{1}{f(S)} < \infty$.*

We firstly have the following result on the existence of globally positive solution for model (3).

Lemma 2.2. *Assume that $(H_1) - (H_3)$ hold. For any initial value $(S(0), I(0), V(0)) \in R_+^3$, the model (3) has a unique solution $(S(t), I(t), V(t))$ defined on $t \in R_{+0}$ satisfying $(S(t), I(t), V(t)) \in R_+^3$ for all $t \geq 0$ with probability one.*

Lemma 2.2 can be easily proved by using the same method which is given in Theorem 2.1 by Gray et al. in [18]. We hence omit it here.

Lemma 2.3. *The following equation*

$$\begin{cases} (\mu_k + p_k)S_0(k) - \varepsilon_k V_0(k) - (1 - q_k)A_k + \sum_{l\in\mathcal{M}} \gamma_{kl} S_0(l) = 0, \\ (\mu_k + \varepsilon_k)V_0(k) - p_k S_0(k) - q_k A_k + \sum_{l\in\mathcal{M}} \gamma_{kl} V_0(l) = 0 \end{cases} \tag{4}$$

has a unique positive stationary distribution solution $(S_0(k), V_0(k), k \in \mathcal{M})$.

Proof Equation (4) can be rewritten as

$$PY = Q, \tag{5}$$

where $Y = (S_0(1), \cdots, S_0(N), V_0(1), \cdots, V_0(N))$, $Q = ((1-q_1)A_1, \cdots, (1-q_N)A_N, q_1A_1, \cdots, q_NA_N)^T$, and

$$P = \begin{bmatrix} \mu_1+p_1+\gamma_{11} & \cdots & \gamma_{1N} & -\varepsilon_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N1} & \cdots & \mu_N+p_N+\gamma_{NN} & 0 & \cdots & -\varepsilon_N \\ -p_1 & \cdots & 0 & \mu_1+\varepsilon_1+\gamma_{11} & \cdots & \gamma_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -p_N & \gamma_{N1} & \cdots & \mu_N+\varepsilon_N+\gamma_{NN} \end{bmatrix}$$

For $k = 1, 2, \cdots, N$, the leading principal submatrixs of P are

$$P_k = \begin{bmatrix} \mu_1+p_1+\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1k} \\ \gamma_{21} & \mu_2+p_2+\gamma_{22} & \cdots & \gamma_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{k1} & \gamma_{k2} & \cdots & \mu_k+p_k+\gamma_{kk} \end{bmatrix},$$

$$P_{N+k} = \begin{bmatrix} \mu_1+p_1+\gamma_{11} & \cdots & \gamma_{1N} & -\varepsilon_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N1} & \cdots & \mu_N+p_N+\gamma_{NN} & 0 & \cdots & -\varepsilon_k \\ -p_1 & \cdots & 0 & \mu_1+\varepsilon_1+\gamma_{11} & \cdots & \gamma_{1k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -p_k & \gamma_{k1} & \cdots & \mu_k+\varepsilon_k+\gamma_{kk} \end{bmatrix}.$$

We see that each column of sub-matrix P_k has the sum $\mu_i + p_i + \sum_{j=1}^{k} \gamma_{ji} \geq \mu_i > 0$ for $i = 1, 2, \cdots, k$, and for sub-matrix P_{N+k}, the sum of i-th column is $\mu_i + p_i + \sum_{j=1}^{N} \gamma_{ji} - p_i = \mu_i > 0$ $(1 \leq i \leq N)$ and $\mu_i + \varepsilon_i + \sum_{j=1}^{k} \gamma_{ji} - \varepsilon_i \geq \mu_i > 0$ $(N < i \leq N+k)$. Lemma 5.3 in [27] implies $\det P_k > 0$ for $k = 1, 2, \cdots, 2N$. By Theorem 2.10 in [27], P is a nonsingular M-matrix, Hence, for the vector Q, equation (5) has a unique positive solution $Y = (S_0(k), V_0(k), k \in \mathcal{M})$. This completes the proof.

Lemma 2.4. *The following equation*

$$\begin{cases} \sum_{l \in \mathcal{M}} \gamma_{kl} c_1(l) - (\mu_k + p_k) c_1(k) + p_k c_2(k) + \beta_k f'(S_0(k)) g'(0) = 0, \\ \sum_{l \in \mathcal{M}} \gamma_{kl} c_2(l) - (\mu_k + \varepsilon_k) c_2(k) + \varepsilon_k c_1(k) = 0. \end{cases} \tag{6}$$

has a unique positive solution $(c_1(k), c_2(k), k \in \mathcal{M})$.

The proof of the lemma is similar to Lemma 2.3, so we here omit it.

3. Existence of stationary distribution

Define $R_0^s = \sum_{k \in \mathcal{M}} \pi_k R_{0k}$, with

$$R_{0k} = \beta_k f(S_0(k)) g'(0) - (\mu_k + \gamma_k + \alpha_k + \frac{1}{2}\sigma_{2k}^2). \tag{7}$$

Theorem 3.1. *Assume that* $(H_1)-(H_3)$ *hold and* $R_0^s > 0$. *Then for any initial value* $(S(0), I(0), V(0)) \in R_+^3$, *solution* $(S(t), I(t), V(t))$ *of model (3) admits a unique ergodic stationary distribution. That is to say, there exists a unique invariant probability measure* $\mu(\cdot, \cdot)$ *such that for any Borel measurable function* $h(\cdot, \cdot): R_+^3 \times \mathcal{M} \to R$ *satisfying*

$$\sum_{k=1}^{N} \int_{R_+^3} |h(x,k)| \mu(dx, k) < \infty,$$

one has

$$P\Big(\lim_{t\to\infty}\frac{1}{t}\int_0^t h((S(s),I(s),V(s)),r(s))ds=\sum_{k=1}^{n}\int_{R_+^3}h(x,k)\mu(dx,k)\Big)=1.$$

Proof We have $\gamma_{ij}>0$ for any $i\neq j$. Obviously, diffusion matrix $D(x,k)=\mathrm{diag}\{\sigma_{1k}S,\sigma_{2k}I,\sigma_{3k}V\}$ is positive define. Let $\sigma_k^2=\max\{\sigma_{1k}^2,\sigma_{2k}^2,\sigma_{3k}^2\}$. Define a C^2-function $W(S,I,V,k)=W_1+W_2+W_3+MW_4$, where

$$W_1=\frac{1}{\theta+1}(S+I+V)^{\theta+1},\ W_2=-\int_1^S\frac{1}{f(S)}dS,\ W_3=-\ln V,$$
$$W_4=-c_1(k)(S+I-S_0(k))-c_2(k)(V-V_0(k))-\int_1^I\frac{g'(0)}{g(I)}dI-\omega_k,$$

and $c_1(k)$, $c_2(k)$ are the positive solutions of equation (6) in Lemma 2,the Markov chain ω_k ($k\in\mathcal{M}$) will be determined later, and $\theta\in(0,1)$ and $M>0$ satisfy

$$\hat{\mu}-\frac{\theta}{2}\check{\sigma}^2>0,\quad f_1^u+f_3^u-MR_0^s\leq-2,\tag{8}$$

where $f_1(x)$, $f_3(x)$ will be determined later. From (H_2) and (H_3), we have $\frac{1}{f(S)}\leq M_1$ for all $S\geq1$ and $\frac{g'(0)}{g(I)}\leq\frac{1}{I}+M_2$ for all $I>0$, where $M_1=\sup_{S\geq1}\frac{1}{f(S)}$ and $M_2=\sup_{I>0}\{\frac{g'(0)}{g(I)}-\frac{1}{I}\}$. Hence, for any $S\geq1$, $I>0$, $V>0$ and $k\in\mathcal{M}$ we have

$$W(S,I,V,k)\geq P(S)+Q(I)+R(V),$$

where

$$\begin{aligned}P(S)=&\ \frac{1}{\theta+1}S^{\theta+1}-M_1S-M\check{c}_1S+M\hat{c}_1\hat{S}_0,\\ Q(I)=&\ \frac{1}{\theta+1}I^{\theta+1}-M\ln I-MM_2I-M\check{c}_1I-M\check{\omega},\\ R(V)=&\ \frac{1}{\theta+1}V^{\theta+1}-\ln V-M\check{c}_2V+M\hat{c}_2\hat{V}_0.\end{aligned}$$

Thus, we can easily obtain that $W(S,I,V,k)$ satisfies

$$\inf\{W(S,I,V,k):\max\{S,I,V\}\geq H,k\in\mathcal{M}\}\to\infty\quad\text{as}\quad H\to\infty.$$

On the other hand, from assumption (H_1), since

$$\lim_{S\to0^+}\int_1^S\frac{1}{f(s)}ds=-\infty,\ \lim_{I\to0^+}\int_1^I\frac{1}{g(i)}di=-\infty,\ \lim_{V\to0^+}\ln V=-\infty,$$

we further have

$$\inf\{W(S,I,V,k):\min\{S,I,V\}\leq h,k\in\mathcal{M}\}\to\infty\quad\text{as}\quad h\to0^+.$$

With the generalized $I\hat{t}o's$ formula (See [27]), we have

$$\begin{aligned}\mathcal{L}W_1\leq&(S+I+V)^{\theta}(A_k-\mu_k(S+I+V))+\frac{\theta}{2}(S+I+V)^{\theta-1}\sigma_k^2(S^2+I^2+V^2)\\ \leq&-(\hat{\mu}-\frac{\theta}{2}\check{\sigma}^2)(S^{\theta+1}+I^{\theta+1}+V^{\theta+1})+3^{\theta}\check{A}(S^{\theta}+I^{\theta}+V^{\theta}),\end{aligned}\tag{9}$$

$$\begin{aligned}\mathcal{L}W_2=&-\frac{(1-q_k)A_k}{f(S)}+\beta_kg(I)+(\mu_k+p_k)\frac{S}{f(S)}-\gamma_k\frac{I}{f(S)}-\varepsilon_k\frac{V}{f(S)}+\frac{1}{2}\sigma_{1k}^2f'(S)(\frac{S}{f(S)})^2\\ \leq&\frac{1}{f(S)}(-(1-\check{q})\hat{A}+(\check{\mu}+\check{p}+\frac{1}{2}\check{\sigma}_1^2M_3)S)+\check{\beta}g'(0)I,\end{aligned}\tag{10}$$

where $M_3 = \sup_{S>0}\{\frac{f'(S)S}{f(S)}\}$ from (H_3), and

$$\begin{aligned}\mathcal{L}W_3 =& -\frac{q_k A_k}{V} - p_k\frac{S}{V} + \mu_k + \varepsilon_k + \frac{1}{2}\sigma_{3k}^2 \\ \leq& \begin{cases} -q_k\dfrac{A_k}{V} + \mu_k + \varepsilon_k + \dfrac{1}{2}\sigma_{3k}^2, & q_k > 0, \\ -p_k\dfrac{S}{V} + \mu_k + \varepsilon_k + \dfrac{1}{2}\sigma_{3k}^2, & q_k = 0, p_k > 0, \end{cases} \\ \leq& -\hat{\lambda}\frac{1}{V}\min\{\hat{A}, S\} + \check{\mu} + \check{\varepsilon} + \frac{1}{2}\check{\sigma}_3^2,\end{aligned} \tag{11}$$

where $\hat{\lambda} = \min\{p_i, q_j : i \in N_1, j \in N_2\}$ with $N_1 = \{i : p_i > 0\}$ and $N_2 = \{j : q_j > 0\}$. Furthermore, by Lemmas 2 and 3 we also have

$$\begin{aligned}\mathcal{L}W_4 =& -c_1(k)[-(\mu_k + p_k)S + \varepsilon_k V - (\mu_k + \alpha_k)I + (\mu_k + p_k)S_0(k) - \varepsilon_k V_0(k)] \\ &- \sum_{l\in\mathcal{M}}\gamma_{kl}c_1(l)(S + I - S_0(k)) - c_2(k)[p_k S - (\mu_k + \varepsilon_k)V - p_k S_0(t) \\ &+ (\mu_k + \varepsilon_k)V_0(k)] - \sum_{l\in\mathcal{M}}\gamma_{kl}c_2(l)(V - V_0(k)) - \beta_k g'(0)(f(S_0(k)) \\ &+ f(S) - f(S_0(k))) + (\mu_k + \gamma_k + \alpha_k)g'(0)\frac{I}{g(I)} \\ &+ \frac{1}{2}\sigma_{2k}^2 g'(0)g'(I)(\frac{I}{g(I)})^2 - \sum_{l\in\mathcal{M}}\gamma_{kl}\omega_l.\end{aligned} \tag{12}$$

Using the mean value theorem and then by (H_2), we have that there exist $\zeta_1 \in (S, S_0(t))$ such that

$$f(S) - f(S_0(k)) = f'(\zeta_1)(S - S_0(k)) \geq f'(S_0(k))(S - S_0(k)),$$

and we also have $g'(I) \leq \frac{g(I)}{I} \leq g'(0)$ for $I > 0$. Hence, from (12) and Lemma 3 we have

$$\begin{aligned}\mathcal{L}W_4 \leq& -\beta_k g'(0)f(S_0(k)) + (\mu_k + \gamma_k + \alpha_k + \frac{1}{2}\sigma_{2k}^2) - \sum_{l\in\mathcal{M}}\gamma_{kl}w_l + [c_1(k)\alpha_k + p_k c_2(k) + \beta_k f'(S_0(k))g'(0)]I \\ &+ (\mu_k + \gamma_k + \alpha_k + \frac{1}{2}\sigma_{2k}^2)(g'(0)\frac{I}{g(I)} - 1) \\ =& -R_{0k} - \sum_{l\in\mathcal{M}}\gamma_{kl}\omega_l + [c_1(k)\alpha_k + p_k c_2(k) + \beta_k f'(S_0(k))g'(0)]I + (\mu_k + \gamma_k + \alpha_k + \frac{1}{2}\sigma_{2k}^2)(g'(0)\frac{I}{g(I)} - 1).\end{aligned} \tag{13}$$

Since the generator matrix Γ is irreducible, for $\overline{R} = (R_{01}, R_{02}, \cdots, R_{0N})$ we can determine a set $\omega = (\omega_1, \omega_2, \cdots, \omega_N)$ satisfying the following Poisson system

$$\Gamma\omega = (\sum_{h=1}^{N}\pi_h R_{0h})\vec{1} - \overline{R}.$$

Consequently, we further have

$$-R_{0k} - \sum_{l\in\mathcal{M}}\gamma_{kl}\omega_l = -\sum_{l\in\mathcal{M}}\pi_k R_{0k} = -R_0^s.$$

Substituting it into (13), we have

$$\mathcal{L}W_4 \leq -R_0^s + [\check{c}_1^*\check{\alpha} + \check{p}\check{c}_2^* + \check{\beta}f'(\check{S}_0)g'(0)]I + (\check{\mu} + \check{\gamma} + \check{\alpha} + \frac{1}{2}\check{\sigma}_2^2)M_2 I. \tag{14}$$

Combining (9), (10), (11) and (14), we have

$$\mathcal{L}W = \mathcal{L}W_1 + \mathcal{L}W_2 + \mathcal{L}W_3 + M\mathcal{L}W_4 \le f_1(S) + f_2(I) + f_3(V),$$

where

$$\begin{aligned} f_1(S) =& -(\hat{\mu} - \frac{\theta}{2}\check{\sigma}^2)S^{\theta+1} + 3^\theta \check{A}S^\theta + \check{\mu} + \check{\varepsilon} + \frac{1}{2}\check{\sigma}_3^2 \\ &+ \frac{1}{f(S)}(-(1-\check{q})\hat{A} + (\check{\mu} + \check{p} + \frac{1}{2}\check{\sigma}_1^2 M_3)S), \\ f_2(I) =& -(\hat{\mu} - \frac{\theta}{2}\check{\sigma}^2)I^{\theta+1} + \check{\beta}g'(0)I + 3^\theta \check{A}I^\theta + M(-R_0^s + [\check{c}_1^*\check{\alpha} + \check{p}\check{c}_2^* \\ &+ \check{\beta}f'(\check{S}_0)g'(0)]I + (\check{\mu} + \check{\gamma} + \check{\alpha} + \frac{1}{2}\check{\sigma}_2^2)M_2 I), \\ f_3(V) =& -(\hat{\mu} - \frac{\theta}{2}\check{\sigma}^2)V^{\theta+1} + 3^\theta \check{A}V^\theta - \hat{\lambda}\frac{1}{V}\min\{\hat{A}, S\}. \end{aligned}$$

By (H_3), it is clear that $f_1^u < \infty$, $f_2^u < \infty$ and $f_3^u < \infty$. Since $f_1(S) \to -\infty$ as $S \to +\infty$ or $S \to 0^+$, there is a constant $\eta_1 > 0$ such that when $0 < S < \eta_1^{-1}$ or $S > \eta_1$ one has

$$f_1(S) + +f_2^u + f_3^u < -1. \tag{15}$$

Since $f_2(I) \to -\infty$ as $I \to +\infty$ and $f_2(I) \to -MR_0^s$ as $I \to 0^+$, from (8) we have that there is a constant $\eta_2 > 0$ such that when $0 < I < \eta_2^{-1}$ or $I > \eta_2$ one has

$$f_1^u + f_2(I) + f_3^u < -1. \tag{16}$$

Since when $\eta_1^{-1} \le S \le \eta_1$, $f_2(V) \to -\infty$ as $V \to +\infty$ or $V \to 0^+$, there is a constant $\eta_3 > 0$ such that when $\eta_1^{-1} \le S \le \eta_1$, $0 < V < \eta_3^{-1}$ or $V > \eta_3$ one has

$$f_1^u + f_2^u + f_3(V) < -1. \tag{17}$$

Let $Q = [\eta_1^{-1}, \eta_1] \times [\eta_2^{-1}, \eta_2] \times [\eta_3^{-1}, \eta_3]$, then from (15)-(17) we further have

$$\mathcal{L}W(S, I, V, k) \le -1, \quad (S, I, V, k) \in Q^c \times \mathcal{M},$$

where $Q^c = R_+^3 \backslash Q$. Thus, from Lemma 2.1 in [28] we finally have that solution $(S(t), I(t),$ $V(t))$ of model (3) has a unique ergodic stationary distribution. This completes the proof.

When $f(S) = \frac{S^m}{1+\omega_1 S}$ and $g(I) = \frac{I}{1+\omega_2 I}$, then $R_0^s = \sum_{k\in\mathcal{M}} \pi_k R_{0k}$ with

$$R_{0k} = \beta_k \frac{S_0^m(k)}{1 + \omega_1 S_0(k)} - \mu_k - \gamma_k - \alpha_k - \frac{1}{2}\sigma_{2k}^2.$$

From Remark 1, we have the following result as a corollary of Theorem 1.

Corollary 3.2. *Assume $f(S) = \frac{S^m}{1+\omega_1 S}$ and $g(I) = \frac{I}{1+\omega_2 I}$ with constants $m \ge 2$, $\omega_1 > 0$ and $\omega_2 \ge 0$. If $R_0^s > 0$, then for any initial value $(S(0), I(0), V(0)) \in R_+^3$, solution $(S(t), I(t), V(t))$ of model (3) admits a unique ergodic stationary distribution.*

When $f(S) = S$ and $g(I) = I$, then $R_0^s = \sum_{k\in\mathcal{M}} \pi_k R_{0k}$ with

$$R_{0k} = \beta_k S_0(k) - \mu_k - \gamma_k - \alpha_k - \frac{1}{2}\sigma_{2k}^2. \tag{18}$$

From Theorem 1, we have the following corollary.

Corollary 3.3. *Assume $f(S) = S$ and $g(I) = I$. If $R_0^s > 0$, then for any initial value $(S(0), I(0), V(0)) \in R_+^3$, solution $(S(t), I(t), V(t))$ of model (2) admits a unique ergodic stationary distribution.*

Remark 3.4. *From (18) we easily see that a new threshold R_0^s is established in this paper for model (2), which is different from the threshold R_0^s given in [22]. Furthermore, we also see that the threshold R_0^s given in [22] only is propitious to model (2) in the case of bilinear incidence $\beta_{r(t)}SI$. But, from Theorem 1 and Corollary 1 we see that the threshold R_0^s established in this paper can be used to model (2) with the general nonlinear incidence $\beta_{r(t)}f(S)g(I)$. This shows that the result obtained in this paper is a considerable improvement and generalization of the corresponding result given in [22].*

4. Numerical examples

In this section, we introduce some numerical examples to illustrate the main results established in this paper, and will further find some new dynamical properties.

Example 4.1. *Take in model (3) the incidence functions $f(S) = \frac{S^3}{1+S}$ and $g(I) = \frac{I}{1+1.5I}$, the Markov chain $r(t)$ with the finite values in the state space $\mathcal{M} = \{1, 2, 3\}$ and the generator*

$$\Gamma = \begin{pmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 3 & 1 & -4 \end{pmatrix}.$$

Furthermore, take the parameters $A_{r(t)} = (A_1, A_2, A_3) = (1.5, 1.6, 1.55)$, $q_{r(t)} = (q_1, q_2, q_3) = (0.2, 0.15, 0.3)$, $\beta_{r(t)} = (\beta_1, \beta_2, \beta_3) = (0.2, 0.15, 0.3)$, $\mu_{r(t)} = (\mu_1, \mu_2, \mu_3) = (0.06, 0.05, 0.08)$, $\alpha_{r(t)} = (\alpha_1, \alpha_2, \alpha_3) = (0.8, 0.9, 0.7)$, $p_{r(t)} = (p_1, p_2, p_3) = (0.4, 0.3, 0.2)$, $\gamma_{r(t)} = (\gamma_1, \gamma_2, \gamma_3) = (0.02, 0.35, 0.4)$ and $\varepsilon_{r(t)} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0.25, 0.35, 0.3)$.

By calculating, the Markov chain $r(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{3}{7}, \frac{2}{7}, \frac{2}{7})$. Solving equation (4) in Lemma 3, we have $S_0(k) = (S_0(1), S_0(2), S_0(3)) = (10.3, 22.8, 11.9)$ and $V_0(k) = (V_0(1), V_0(2), V_0(3)) = (14.7, 9.2, 7.5)$. Thus, from (7) we further have $R_{01} = 1.6600$, $R_{02} = -1.2360$ and $R_{03} = 2.6150$. Therefore, $R_0^s = 1.1054 > 0$.

By Corollary 1, the solution $(S(t), I(t), V(t))$ of model (3) with initial value $(S(0), I(0), V(0)) \in R_+^3$ has a unique ergodic stationary distribution. The numerical simulations in Figure 1 indicate that $(S(t), I(t), V(t))$ not only has a unique stationary distribution, but is also permanent with probability one.

Here, the solution $(S(t), I(t), V(t))$ is said to be permanent in the mean with probability one if there exist two constants $M > m > 0$ which are independent of solution $(S(t), I(t), V(t))$ such that

$$\begin{aligned} m \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t S(s)ds \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t S(s)ds \le M \ a.s., \\ m \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds \le M \ a.s., \\ m \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t V(s)ds \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t V(s)ds \le M \ a.s.. \end{aligned}$$

The solution $(S(t), I(t), V(t))$ is said to be permanent with probability one, if there exist two constants $M > m > 0$ which are independent of solution $(S(t), I(t), V(t))$ such that

$$\begin{aligned} m \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le M \ a.s., \\ m \le \liminf_{t\to\infty} I(t) \le \limsup_{t\to\infty} I(t) \le M \ a.s., \\ m \le \liminf_{t\to\infty} V(t) \le \limsup_{t\to\infty} V(t) \le M \ a.s.. \end{aligned}$$

From the above definitions, it is clear that the permanence implies the permanence in the mean.

Figure 2 reflects the sample means of $(S(t), I(t), V(t))$ and the distribution of the the switching times of $r(t)$.

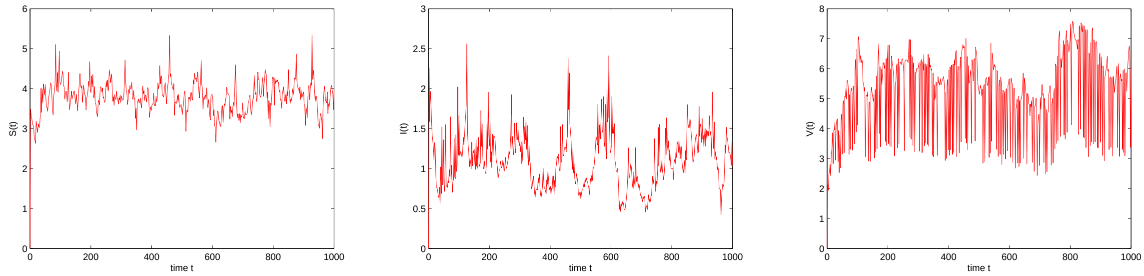

Figure 1: Simulations of the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 4$, $I(0) = 3$ and $V(0) = 2$.

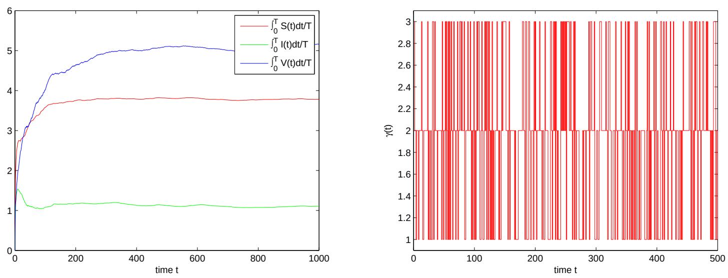

Figure 2: (i) Simulations of the sample means for the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 4$, $I(0) = 3$ and $V(0) = 2$; (ii) Simulations of the switching times of $r(t)$.

Example 4.2. *Take in model (3) the incidence functions* $f(S) = \frac{S^3}{1+S}$ *and* $g(I) = \frac{I}{1+1.5I}$, *the Markov chain* $r(t)$ *with the finite values in the state space* $\mathcal{M} = \{1, 2, 3\}$ *and the generator*

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 2 & -3 \end{pmatrix}.$$

Furthermore, take the parameters $A_{r(t)} = (A_1, A_2, A_3) = (0.8, 0.9, 0.6)$, $q_{r(t)} = (q_1, q_2, q_3) = (0.2, 0.8, 0.6)$, $\beta_{r(t)} = (\beta_1, \beta_2, \beta_3) = (0.1, 0.04, 0.02)$, $\mu_{r(t)} = (\mu_1, \mu_2, \mu_3) = (0.3, 0.2, 0.3)$, $\alpha_{r(t)} = (\alpha_1, \alpha_2, \alpha_3) = (0.06, 0.7, 0.8)$, $p_{r(t)} = (p_1, p_2, p_3) = (0.2, 0.3, 0.2)$, $\gamma_{r(t)} = (\gamma_1, \gamma_2, \gamma_3)$ $= (1.2, 0.2, 0.3)$ *and* $\varepsilon_{r(t)} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0.2, 0.4, 0.3)$.

By calculating, the Markov chain $r(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{7}{12}, \frac{3}{12}, \frac{5}{12})$. Solving equation (4), we have $S_0(k) = (S_0(1), S_0(2), S_0(3)) = (1.7, 2.2, 1.05)$ and $V_0(k) = (V_0(1), V_0(2), V_0(3)) = (1, 2.3, 0.95)$. Thus, from (7) we further have $R_{01} = 0.4433$, $R_{02} = -0.5867$ and $R_{03} = -1.0283$. Therefore, $R_0^s = -0.0594 < 0$.

Let $(S(t), I(t), V(t))$ be the solution of model (3) with initial value $(S(0), I(0), V(0))$ $= (5, 3, 1)$. The numerical simulations in Figure 3 indicate that $S(t)$ and $V(t)$ not only has a unique stationary distribution, but also is permanent with probability one, and $I(t)$ is extinct with probability one. That is, $\lim_{t\to\infty} I(t) = 0$ *a.s.* The computer simulations of the sample means of the solution $(S(t), I(t), V(t))$ and the switching times of $r(t)$ are given in Figure 4.

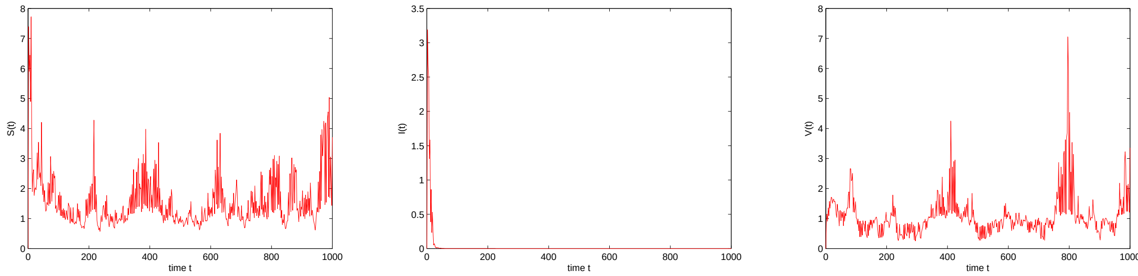

Figure 3: The simulations of the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 5$, $I(0) = 3$ and $V(0) = 1$.

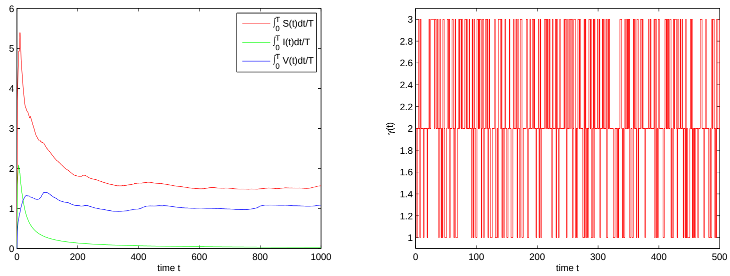

Figure 4: (i) The simulations of the sample means for the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 5$, $I(0) = 3$ and $V(0) = 1$; (ii) The distribution for the switching times of $r(t)$.

Example 4.3. *Take in model (3) the incidence functions* $f(S) = \frac{S^3}{1+S}$ *and* $g(I) = I^2$, *the Markov chain* $r(t)$ *with the finite values in the state space* $\mathcal{M} = \{1, 2, 3\}$ *and the generator*

$$\Gamma = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}.$$

Furthermore, take the parameters $A_{r(t)} = (A_1, A_2, A_3) = (2.4, 2.5, 2.6)$, $q_{r(t)} = (q_1, q_2, q_3) = (0.5, 0.2, 0.15)$, $\beta_{r(t)} = (\beta_1, \beta_2, \beta_3) = (0.08, 0.05, 0.04)$, $\mu_{r(t)} = (\mu_1, \mu_2, \mu_3) = (0.12, 0.05, 0.4)$, $\alpha_{r(t)} = (\alpha_1, \alpha_2, \alpha_3) = (0.5, 1.6, 0.5)$, $p_{r(t)} = (p_1, p_2, p_3) = (0.6, 0.3, 0.7)$, $\gamma_{r(t)} = (\gamma_1, \gamma_2, \gamma_3) = (0.02, 0.3, 0.02)$ *and* $\varepsilon_{r(t)} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0.5, 0.4, 0.3)$.

Obviously, $g(I)$ does not satisfy assumption (H_2). Hence, Corollary 1 is not applicable. By calculating, the Markov chain $r(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{3}{12}, \frac{6}{12}, \frac{3}{12})$. Solving equation (4), we have $S_0(k) = (S_0(1), S_0(2), S_0(3)) = (8.5, 29.3, 3)$ and $V_0(k) = (V_0(1), V_0(2), V_0(3)) = (11.5, 20.7, 3.5)$. Thus, from (7) we further have $R_{01} = 2.6931$, $R_{02} = -0.0467$ and $R_{03} = 0.7466$. Therefore, $R_0^s = 0.8366 > 0$.

Let $(S(t), I(t), V(t))$ be the solution of model (3) with initial value $(S(0), I(0), V(0))$ $= (5, 3, 1)$. From the numerical simulations in Figure 5 we see that there is an indicate of existence of a stationary distributions for $S(t)$ and $V(t)$, but $I(t)$ may not have the stationary distribution. The corresponding sample means of $(S(t), I(t), V(t))$ and the distribution of $r(t)$ are illustrated in Figure 6. From Figures 5 and 6, we also see that $S(t)$ and $V(t)$ are permanent with probability one, but $I(t)$ only is permanent in the mean with probability one, but not permanent with probability one.

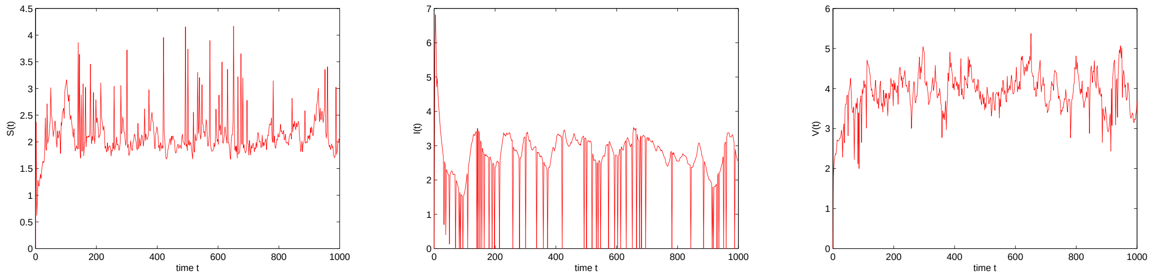

Figure 5: Simulations of the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 5$, $I(0) = 3$ and $V(0) = 1$.

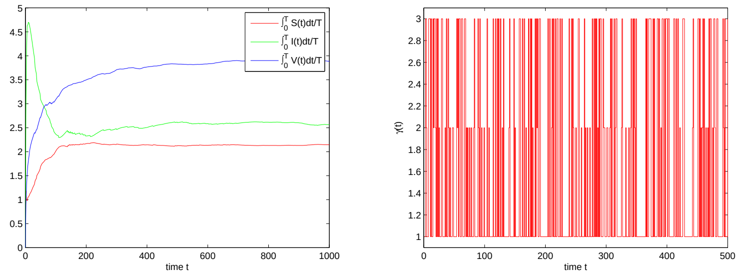

Figure 6: (i) Simulations of the sample means for the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 5$, $I(0) = 3$ and $V(0) = 1$; (ii) Simulations of the switching times of $r(t)$.

Example 4.4. *Take in model (3) the incidence functions* $f(S) = \frac{S}{1+S}$ *and* $g(I) = \frac{I}{1+1.5I}$*, the Markov chain* $r(t)$ *with the finite values in the state space* $\mathcal{M} = \{1, 2, 3\}$ *and the generator*

$$\Gamma = \begin{pmatrix} -4 & 3 & 1 \\ 2 & -4 & 2 \\ 1 & 3 & -4 \end{pmatrix}.$$

Furthermore, take the parameters $A_{r(t)} = (A_1, A_2, A_3) = (0.8, 0.7, 0.6)$, $q_{r(t)} = (q_1, q_2, q_3) = (0.3, 0.2, 0.3)$, $\beta_{r(t)} = (\beta_1, \beta_2, \beta_3) = (1.5, 1.2, 1.6)$, $\mu_{r(t)} = (\mu_1, \mu_2, \mu_3) = (0.1, 0.2, 0.2)$, $\alpha_{r(t)} = (\alpha_1, \alpha_2, \alpha_3) = (0.4, 0.5, 0.6)$, $p_{r(t)} = (p_1, p_2, p_3) = (0.02, 0.3, 0.02)$, $\gamma_{r(t)} = (\gamma_1, \gamma_2, \gamma_3)$ $= (0.65, 0.04, 0.02)$ *and* $\varepsilon_{r(t)} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0.2, 0.4, 0.3)$.

Since $f'(S) = \frac{1}{(1+S)^2}$ is decreasing for $S \geq 0$, assumption (H_3) is not satisfied. Hence, Corollary 1 is not applicable. By calculating, the Markov chain $r(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{3}{7}, \frac{3}{7}, \frac{2}{7})$. Solving equation (4), we have $S_0(k) = (S_0(1), S_0(2), S_0(3)) = (7, 2.2, 2.5)$ and $V_0(k) = (V_0(1), V_0(2), V_0(3)) = (1, 1.3, 0.5)$. Thus, from (7) we further have $R_{01} = -0.1700$, $R_{02} = 0.0587$ and $R_{03} = 0.2017$. Therefore, $R_0^s = 0.0342 > 0$.

Let $(S(t), I(t), V(t))$ be the solution of model (3) with initial value $(S(0), I(0), V(0))$ $= (5, 3, 1)$. From the numerical simulations given in Figure 7 we see that $S(t)$ and $V(t)$ not only seem to confirm a stationary distribution, but also is permanent with probability one, and $I(t)$ may be extinct with probability one. The simulations of the corresponding sample means and the distribution of $r(t)$ are shown in Figure 8.

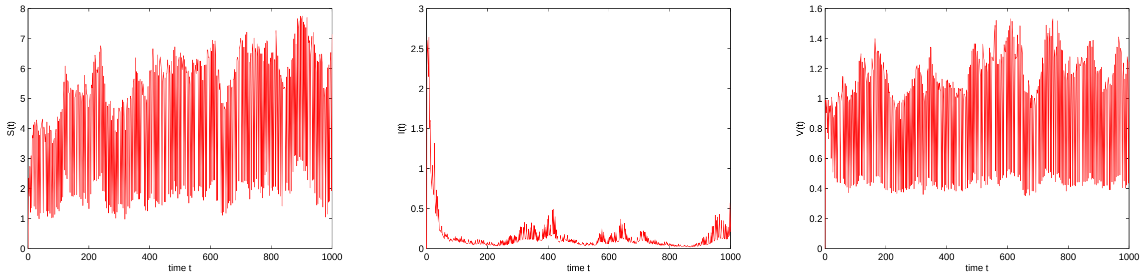

Figure 7: Simulations of the path $(S(t), I(t), V(t))$ with the initial values $S(0) = 2$, $I(0) = 4$ and $V(0) = 1$.

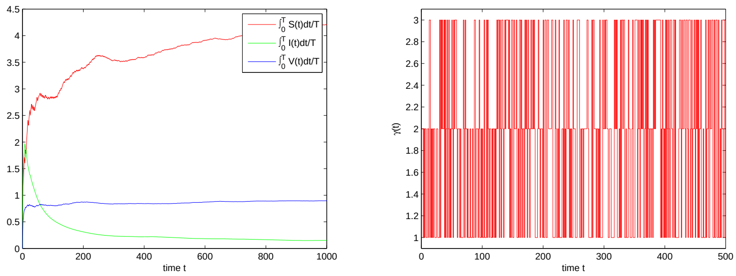

Figure 8: (i) Simulations of the sample means for the solution $(S(t), I(t), V(t))$ with the initial values $S(0) = 2$, $I(0) = 4$ and $V(0) = 1$; (ii) Simulations the switching times of $r(t)$.

5. Conclusion

In this paper, we investigated the stationary distribution for a stochastic SIS epidemic model with vaccination and nonlinear incidence under regime switching. We see that a new threshold is introduced which is different from that given in [22]. A new sufficient condition on the existence of unique ergodic stationary distribution is established. A new technique to deal with the nonlinear incidence and vaccination for the stochastic epidemic models under regime switching is proposed. The corresponding results given in [22] are considerably improved and generalized.

The assumptions (H_2) and (H_3) are introduced for the nonlinear incidence functions $f(S)$ and $g(I)$. We find that they are used only in the proofs of Theorem 3.1. However, from Examples 3 and 4 we see that the assumptions (H_2) and (H_3) may be necessary to ensure the existence of a stationary distribution. Furthermore, from the numerical examples we also find that when the threshold $R_0^s > 0$ the solution of model (3) also is permanent or permanent in the mean with probability one, and when $R_0^s < 0$ the disease I in model (3) will be extinct with probability one.

In the future, some new problems should be further investigated for this model, for instance, stochastic extinction, persistence and asymptotic behaviors of positive solutions. In addition, it is also important and interesting that whether the results and methods established in this paper can be extended to stochastic SIR and SEIR type epidemic models with vaccination and nonlinear incidence under regime switching.

References

[1] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic, Dordrecht, 1992.
[2] N.H. Du, R. Kon, K. Sato, Y. Takeuchi, Dynamical behaviour of Lotka-Volterra competition systems: nonautonomous bistable case and the effect of telegraph noise, J. Comput. Appl. Math. 170 (2004) 399-422.
[3] Q. Lu, Stability of SIRS systems with random perturbations, Phycica A 388 (2009) 3677-3686.
[4] Y. Takeuchi, Global Dynamical Properties of Lotka-Volterra Systems, World Scientific Publishing Company, Singapore, 1996.
[5] X. Mao, Stability of stochastic Differential Equations with Respect to Semimartingales, Longman Scientific and Technical, London, 1991.
[6] M.E. Gilpin, Predator-Prey Communities, Princeton University Press, Princeton, 1975.
[7] M. Slatkin, The dynamics of a population in a Markovian environment, Ecology 59 (1978) 249-256.
[8] Y. Takeuchi, N.H. Du, N.T. Hieu, K. Sato, Evolution of predator-prey systems described by a Lotka-Volterra equation under random environment, J. Math. Anal. Appl. 233 (2006) 938-957.
[9] D.K. Padilla, S.C. Adolph, Plastic inducible morphologies are not always asaptive: the importance of time delay in a stochastic environment, Evol. Ecol. 10 (1996) 105-117.
[10] T. Gard, Stability for multispecies population in random environments, Nonlinear Anal. 10 (1986) 1411-1419.
[11] M. Siakalli, Stability properties of stochastic differential equations driven by Lévy noise (Ph.D. thesis), University of Sheffield, 2009.
[12] D. Applebaum, M. Siakalli, Asymptotic stability of a stochastic differential equations driven by Lévy noise, J. Appl. Probab. 46 (2009) 1116-1129.
[13] Y. Zhou, S. Yuan, D. Zhao, Thereshold of a stochastic SIS model with Lévy Jumps, Appl. Math. Comput. 275 (2016) 255-267.
[14] H. Schurz, K. Tosun, Stochastic asymptotic stability of SIR model with variable diffusion rates, J. Dyn. Diff. Equat. 27 (2015) 69-82.
[15] X. Meng, S. Zhao, T. Feng, T. Zhang, Dynamics of a novel nonlinear stochastic SIS epidemic model with double epidemic hyothesis, J. Math. Anal. Appl. 433 (2016) 227-242.
[16] T. Tang, Z. Teng, Z. Li, Threshold behavion in a chass of stochastic SIRS epidemic models with nonlinear incidence, Stoch. Anal. Appl. 33 (2015) 9940-1019.
[17] Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, Appl. Math. Comput. 243 (2014) 718-727.
[18] A. Gray, D. Greenhalgh, X. Mao, J. Pan, The SIS epidemic model with Markovian switching, J. Math. Anal. Appl. 394 (2012) 496-516.
[19] C. Chen, Y. Kang, Dynamics of a stochastic multi-strain SIS epidemic model driven by Lévy noise, Comm. Nonl. Sci. Numer. Simul. 42 (2017) 379-395.
[20] C. Xu, Global threshold dynamics of a stochastic differential equation SIS model, J. Math. Anal. Appl. 447 (2017) 736-757.
[21] A. Lahrouz, A. Settati, Necessary and sufficient condition for extinction and persistence of SIRS system with random perturbation, Appl. Math. Comput. 233 (2014) 10-19.
[22] X. Zhang, D. Jiang, A. Alsaedi, T. Hayat, Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching, Appl. Math. Letters 59 (2016) 87-93.
[23] Y. Cai, Y. Kang, M. Banerjee, W. Wang, A stochastic SIRS epidemic model with infectious fore under intervention strategies, J. Diff. Equations 259 (2015) 7463-7502.
[24] Z. Teng, L. Wang, Persistence and extinction for a class of stochastic SIS epidemic models with nonlinear incidence rate, Physica A 451 (2016) 507-518.
[25] D. Zhao, T. Zhang, S. Yuan, The threshold of a stochastic SIVS epidemic model with nonlinear saturated incidence, Physica A 443 (2016) 372-379.
[26] R. Rifhat, Q. Ge, Z. Teng, The dynamical behaviors in a stochastic SIS epidemic model with nonlinear incidence, Comput. Math. Meth. Medic. 2016 (2016) 5218163.
[27] X. Mao, C. Yuan, Stochastic Differential Equations with Markov Switching, Imperial College Press, London, 2006.
[28] H. Liu, X. Li, Q. Yang, The ergodic property and positive recurrence of a multi-group Lotka-Volterra mutualistic system with regime switching, Syst. Control Letters 62 (2013) 805-810.