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On Contact CR-Submanifolds of a Cosymplectic Manifold

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Abstract. In this paper, we study the differential geometry of contact CR-submanifolds of a cosymplectic manifold. Necessary and sufficient conditions are given for a submanifold to be a contact CR-submanifold in cosymplectic manifolds and cosymplectic space forms. Finally, the induced structures on submanifolds are investigated, these structures are categorized and we discuss these results.

1. Introduction

The study of the differential geometry of contact CR-submanifolds, as a generalization of invariant(holomorphic) and anti-invariant(totally real) submanifolds of an almost contact metric manifold was initiated by A. Bejancu [1] and was followed by several researchers. Some authors studied contact CR-submanifolds of different classes of almost contact metric manifolds given in the references of this paper. Recently, in different studies M. Atçeken et al. [11], [12], [13], [14] and S. Uddin et al. [18], [19], [20] studied contact CR-submanifold and warped product CR-submanifolds in various type manifolds.

The contact CR-submanifolds are rich and interesting subject. Therefore it was continued to work in this subject matter. This study the present paper is organized as follows.

In this paper, contact CR-submanifolds of a cosymplectic manifold were studied. In Section 2, basic formulas and definitions for a cosymplectic manifold and their submanifolds were reviewed. In Section 3, the definition and some basic results of a contact CR-submanifold of a cosymplectic manifold was recalled. In Section 4, some new results for contact CR-submanifolds in a cosymplectic manifold and a cosymplectic space form $\widetilde{M}(c)$ was given.

2. Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold together with an almost contact structure (ϕ, ξ, η), i.e., ξ is a global vector field ϕ is a $(1,1)-$type tensor field and η is a 1-form on $\widetilde{M}$ such that

$$\phi^2 X = -X + \eta(X)\xi, \ \ \phi\xi = 0, \ \eta(\phi X) = 0, \ \eta(\xi) = 1 \tag{1}$$

for any $X, Y \in \Gamma(\widetilde{M})$, where $\Gamma(\widetilde{M})$ denotes the set differentiable vector fields on $\widetilde{M}$.

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The almost contact manifold is called an almost contact metric manifold if there exists a Riemannian metric g satisfying;

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y) \tag{2}$$

for any $X, Y \in \Gamma(\widetilde{M})$. Clearly, in this case, η is dual of ξ,i.e., $\eta(X) = g(X, \xi)$,for any $X, Y \in \Gamma(\widetilde{M})$.

The fundamental 2−form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$, for any $X, Y \in \Gamma(\widetilde{M})$. The $\widetilde{M}$ is called an almost cosymplectic manifold η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where d is exterior differentiable operator [4]. Also, an almost contact metric manifold is called normal if $\left[\phi, \phi\right] + 2d\eta \otimes \xi = 0$, where $\left[\phi, \phi\right]$ is Nijenhuis tensor field which is defined by $\left[\phi, \phi\right](X, Y) = \phi^2 \left[X, Y\right] + \left[\phi X, \phi Y\right] - \phi\left[\phi X, Y\right] - \phi\left[X, \phi Y\right]$.

If $\widetilde{M}$ is almost contact metric manifold is normal, $\widetilde{M}$ is said to be cosymplectic manifold. It is well know that an almost contact metric manifold is cosymplectic if and only if

$$(\widetilde{\nabla}_X \phi)Y = 0 \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$. Then manifolds are locally a product of a Kaehler manifold and real line a circle.

If a cosymplectic manifold $\widetilde{M}$ has constant ϕ- sectional curvature, then it is called a cosymplectic space form $\widetilde{M}(c)$. Then Riemannian curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\begin{aligned}\widetilde{R}(X, Y)Z \;=\; & \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi \\ & -\eta(X)g(Y, Z)\xi + g(\phi Y, Z)\phi X + g(X, \phi Z)\phi Y + 2g(X, \phi Y)\phi Z\}\end{aligned} \tag{4}$$

for any vector fields X, Y, Z tangent to $\widetilde{M}$[15].

Now, let M be an isometrically immersed submanifold in a cosymplectic manifold $\widetilde{M}$. Then the formulas Gauss and Weingarten for M in $\widetilde{M}$ given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{5}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{6}$$

for any vector fields X, Y tangent to M and V normal to M, where, ∇ denotes the induced Levi-Civita connection on M, $\nabla^{\perp}$ is the normal connection , A_V is the shape operator of M with respect to V and σ is second fundamental form of M in $\widetilde{M}$. The second fundamental form σ and shape operator A_V are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V) \tag{7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m}\sigma(e_i, e_i)$, where m is the dimension of M and $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of M. A submanifold M of an contact metric manifold $\widetilde{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y)H, \tag{8}$$

for any $X, Y \in \Gamma(TM)$. A submanifold M is said to be totally geodesic if $\sigma = 0$ and M is said to be minimal if $H = 0$. For any submanifold M of a Riemannian manifold $\widetilde{M}$, the equation of Gauss is given by

$$\widetilde{R}(X, Y)Z = R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X\sigma)(Y, Z) - (\widetilde{\nabla}_Y\sigma)(X, Z), \tag{9}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\widetilde{R}$ and R denote the Riemannian curvature tensor of $\widetilde{M}$ and M, respectively. The covariant derivative $\widetilde{\nabla}\sigma$ of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(\nabla_X Z, Y) \tag{10}$$

for any $X, Y, Z \in \Gamma(TM)$.

Then the Gauss and the Codazzi equations are, respectively, given by

$$(\widetilde{R}(X,Y)Z)^{T} = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \tag{11}$$

and

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z), \tag{12}$$

where $(\widetilde{R}(X,Y)Z)^{\perp}$ denotes the normal part of $\widetilde{R}(X,Y)Z$. If $(\widetilde{R}(X,Y)Z)^{\perp} = 0$, then M is said to be curvature-invariant submanifold of $\widetilde{M}$. The Ricci equation is given by

$$g(\widetilde{R}(X,Y)V,U) = g(\widetilde{R}^{\perp}(X,Y)V,U) + g([A_U, A_V]X, Y), \tag{13}$$

for any $X, Y, \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$, where $\widetilde{R}^{\perp}$ denotes the Riemannian curvature tensor of the normal $T^{\perp}M$ and if $\widetilde{R}^{\perp} = 0$, then the normal connection of M is called flat.

Now, let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$\phi X = TX + NX, \tag{14}$$

where TX is the tangential component and NX is the normal component of ϕX. Similarly for $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi V = tV + nV, \tag{15}$$

where tV is the tangential component and nV is also the normal component of ϕV.
Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(TX, Y) = -g(X, TY)$ and $V, U \in \Gamma(T^{\perp}M)$, we get $g(U, nV) = -g(nU, V)$. These show that T and n are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$g(NX, V) = -g(X, tV), \tag{16}$$

which gives the relation between N and t.

Now, applying ϕ to (14) and (15), we respectively, obtain

$$T^2X = -X + \eta(X)\xi - tNX, \quad NTX + nNX = 0 \tag{17}$$

and

$$TtV + tnV = 0, \quad NtV + n^2V = -V. \tag{18}$$

for any vector fields X tangent to M and V normal to M.

We define the covariant derivatives of the tensor field T, N, t and n by $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y$, $(\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y$, $(\nabla_X t)V = \nabla_X tV - t\nabla^{\perp}_X V$ and $(\nabla_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V$ respectively.
Since M is tangent to ξ, making use of (5), (7) and (14), we obtain

$$\nabla_X\xi = 0, \quad \sigma(X,\xi) = 0, \quad A_V\xi = 0 \tag{19}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

Let X and Y be vector fields tangent to M. Then we obtain

$$(\nabla_X T)Y = A_{NY}X + t\sigma(X,Y) \tag{20}$$

and

$$(\nabla_X N)Y = n\sigma(X,Y) - \sigma(X,TY). \tag{21}$$

Similarly, for any vector field X tangent to M and any vector field V normal to M. Then we have

$$(\nabla_X t)V = A_{nV}X - TA_V X \tag{22}$$

and

$$(\nabla_X n)V = -\sigma(tV,X) - NA_V X. \tag{23}$$

Taking into account (4) and (13), we have

$$\begin{aligned} g(\widetilde{R}^{\perp}(X,Y)V,U) + g([A_U,A_V]X,Y) &= \frac{c}{4}\{g(X,tV)g(U,NY) - g(Y,tV)g(NX,U) \\ &\quad +2g(X,TY)g(nV,U)\} \end{aligned} \tag{24}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$. By using (4) and (9), the Riemanian curvature tensor R of an immersed submanifold M of a cosymplectic space form $\widetilde{M}(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\ &\quad -\eta(X)g(Y,Z)\xi + g(X,\phi Z)\phi Y + g(\phi Y,Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &\quad +A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y + (\widetilde{\nabla}_Y\sigma)(X,Z) - (\widetilde{\nabla}_X\sigma)(Y,Z). \end{aligned} \tag{25}$$

Comparing the tangential and normal parts of the both sides of this equation, we have, following equations of Gauss and Codazzi equation respectively:

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad +\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(X,TZ)TY \\ &\quad +g(TY,Z)TX + 2g(X,TY)TZ\} + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y. \end{aligned} \tag{26}$$

and

$$(\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z) = \frac{c}{4}\{g(X,TZ)NY + g(TY,Z)NX + 2g(X,TY)NZ\}. \tag{27}$$

3. Contact CR-Submanifold of a Cosymplectic Manifold

In this section, we shall define contact CR-submanifolds in a cosymplectic manifold and research fundamental properties of their from theory of submanifold.

Let M be submanifold of an almost contact metric manifold $\widetilde{M}$, then M is called invariant submanifold if $\phi(T_xM) \subseteq T_xM$, $\forall x \in M$. Further, M is said to be anti-invariant submanifold if $\phi(T_xM) \subseteq T_x^{\perp}M$, $\forall x \in M$. Similarly, it can be easily seen that a submanifold M of an almost contact metric manifolds $\widetilde{M}$ is said to be invariant(anti-invariant), if N (or T) are identically zero in (14). Now we give definition of contact CR-submanifold which is a generalization of invariant and anti-invariant submanifolds.

Definition 3.1. *[10]. A submanifold M of a cosymplectic manifold. $\widetilde{M}$ is called contact CR-submanifold if there exists on M a differentiable invariant distribution D whose orthogonal complementary $\phi D^{\perp}$ is anti-invariant, i.e.,*

i) $TM = D \oplus D^{\perp}$, $\xi \in \Gamma(D)$

ii) $\phi D_x = D_x$

iii) $\phi D_x^{\perp} \subseteq T_x^{\perp} M$, *for each* $x \in M$.

A contact CR-submanifold is called anti-invariant(or, totally real) if $D_x = 0$ *and invariant(or, holomorphic) if* $D_x^{\perp} = 0$, *respectively, for any* $x \in M$. *It is called proper contact CR-submanifold if neither* $D_x = 0$ *nor* $D_x^{\perp} = 0$.

Anti-invariant and invariant submanifolds are the special case of contact CR-submanifolds.

If we denote dimensions of the distributions D and $D^{\perp}$ by d_1 and d_2, respectively. Then M is called anti-invariant (resp. invariant) if $d_1 = 0 (resp. d_2 = 0)$.

Let us denote the orthogonal projections on D and $D^{\perp}$ by $\omega_1 : \Gamma(TM) \to D$ and $\omega_2 : \Gamma(TM) \to D^{\perp}$ respectively. Then we have

$$X = \omega_1 X + \omega_2 X + \eta(X)\xi$$

for any $X \in \Gamma(TM)$, where $\omega_1 X \in \Gamma(D)$ and $\omega_2 X \in \Gamma(D^{\perp})$. From (14) and (15), we have
and

$$\phi X = TX + NX = \phi\omega_1 X + \phi\omega_2 X = T\omega_1 X + N\omega_1 X + T\omega_2 X + N\omega_2 X$$

it is clear that

$$N\omega_1 = 0 \text{ and } T\omega_2 = 0,$$

$$N = N\omega_2 \text{ and } T = T\omega_1.$$

Proposition 3.2. *Let M be an isometrically immersed submanifold of a cosymplectic manifold $\widetilde{M}$. Then the invariant distribution D has an almost contact metric structure (T, ξ, η, g) and so $dım(D_p)$ =odd for each $p \in M$.*

We denote the orthogonal subbundle $\phi D^{\perp}$ in $T^{\perp}M$ by υ, then we have direct sum

$$T^{\perp}M = \phi D^{\perp} \oplus \nu \text{ and } \phi D^{\perp} \perp \nu.$$

Here we note that ν is an invariant subbundle with respect to ϕ and so dim(ν)=even.
Also,

$$t(T^{\perp}M) = D^{\perp} \textit{ and } n(T^{\perp}M) \subset \nu.$$

Example 3.3. *Thus $(\mathbb{R}^9, \varphi, \xi, \eta, g)$ is an almost contact metric structure on $\mathbb{R}^9$. We call the usual contact metric structure of $\mathbb{R}^9$. Then we have*

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{4} y_i dx_i), \quad \xi = 2\frac{\partial}{\partial z}$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{4}(dx_i \otimes dx_i + dy_i \otimes dy_i)$$

$$\phi(\sum_{i=1}^{4}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{4}(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}),$$

where $(x_i, y_i, z), i = 1, 2, 3, 4$ *are the cartesian coordinates.*

Now, let M *be a submanifold of* $\mathbb{R}^9$ *defined by the following equation*

$$\chi(u, w, v, s, z) = (2(u, 0, w, 0, v, 0, 0, s, z).$$

We can easily to see that the tangent bundle of M *is spanned by the tangent vectors*

$$E_1 = 2(\frac{\partial}{\partial x_1} + y_i\frac{\partial}{\partial z}),\ E_2 = 2\frac{\partial}{\partial y_1},\ E_3 = 2(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}),\ E_4 = 2\frac{\partial}{\partial y_4}, E_5 = 2\frac{\partial}{\partial z} = \xi.$$

For the almost contact structure ϕ *of* $\mathbb{R}^9$*. We obtain,*

$$\phi E_1 = -E_2,\ \phi E_2 = E_1,\ \phi E_3 = -2\frac{\partial}{\partial y_3},\ \phi E_4 = 2\frac{\partial}{\partial x_4},\ \phi E_5 = 2\frac{\partial}{\partial z} = 0.$$

By direct calculations, we can infer $D = span\{E_1, E_2, E_5\}$ *is invariant distribution. Since* $g(\phi E_3, E_i) = 0,\ i = 1, 2, 4, 5$ *and* $g(\phi E_4, E_j) = 0,\ j = 1, 2, 3, 5,\ \phi E_3,\phi E_4$ *are orthogonal to* M, $D^\perp = span\{E_3, E_4\}$ *is an anti-invariant distribution. Thus* M *is a* 5*-dimensional proper contact CR-submanifold of* $\mathbb{R}^9$ *with it's usual almost contact metric structure.*

Proposition 3.4. *Let* M *be a Contact CR-submanifold of a cosymplectic manifold* $\widetilde{M}$*. For any vector fields* X *tangent to* D *(resp.*$D^\perp$ *is necessary and sufficient that* $NX = 0$ *(resp.*$TX = 0$*).*

Furthermore, taking account of (1) and proposition 3.2, we have

$$T^2X = -X + \eta(X)\xi \tag{28}$$

for any vector field X in D. Moreover

$$g(TX, TY) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y in D.

Proposition 3.5. *Let* M *be a contact CR-submanifold of a cosymplectic manifold* $\widetilde{M}$*. Then, we have*

$$\nabla^\perp_W\phi Z - \nabla^\perp_Z\phi W \in \phi(D^\perp)$$

for any $Z, W \in \Gamma(D^\perp)$.

Proof. For any $Z, W \in \Gamma(D^\perp), V \in \Gamma(\nu)$. Then (3), Gauss and Weingarten formulas, we have

$$\begin{aligned}
g(\nabla^\perp_W\phi Z - \nabla^\perp_Z\phi W, V) &= g(A_{\phi Z}W + \widetilde{\nabla}_W\phi Z - A_{\phi W}Z - \widetilde{\nabla}_Z\phi W, V)\\
&= g(\widetilde{\nabla}_W\phi Z - \widetilde{\nabla}_Z\phi W, V)\\
&= g((\widetilde{\nabla}_W\phi)Z + \phi\widetilde{\nabla}_W Z - (\widetilde{\nabla}_Z\phi)W - \phi\widetilde{\nabla}_Z W, V)\\
&= g(\phi\widetilde{\nabla}_W Z - \phi\widetilde{\nabla}_Z W, V) = g(\widetilde{\nabla}_Z W - \widetilde{\nabla}_W Z, \phi V)\\
&= g(\sigma(Z, W) - \sigma(Z, W), \phi V) = 0.
\end{aligned}$$

Thus the proof is complete. □

Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then for any $Z, W \in \Gamma(D^\perp)$ and $U \in \Gamma(TM)$, also by using (3), (5) and (7), we have

$$\begin{aligned}
g(A_{NZ}W - A_{NW}Z, U) &= g(\sigma(W, U), NZ) - g(\sigma(Z, U), NW)\\
&= g(\widetilde{\nabla}_U W, \phi Z) - g(\widetilde{\nabla}_U Z, \phi W)\\
&= g(\phi\bar{\nabla}_U Z, W) - g(\phi\bar{\nabla}_U W, Z)\\
&= -g(A_{NZ}U, W) + g(A_{NW}U, Z)\\
&= g(A_{NW}Z - A_{NZ}W, U).
\end{aligned}$$

It follows that

$$A_{NZ}W = A_{NW}Z, \tag{29}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Theorem 3.6. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the tensor n is parallel if and only if the shape operator A_V of M satisfies the condition*

$$A_V tW = A_W tV, \tag{30}$$

for all $W, V \in \Gamma(T^{\perp}M)$.

Proof. For all $W, V \in \Gamma(T^{\perp}M)$, from (7), (16) and (23), we have

$$\begin{aligned} g((\nabla_X n)V, W) &= -g(\sigma(tV, X), W) - g(NA_V X, W) \\ &= -g(A_W tV, X) + g(A_V X, tW) \\ &= g(A_V tW - A_W tV, X), \end{aligned}$$

for all $X \in \Gamma(TM)$. The proof is complete. □

Theorem 3.7. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is completely integrable and its maximal integral submanifold is an anti-invariant submanifold of $\widetilde{M}$.*

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, By using (2) and (3), we have

$$\begin{aligned} g([Z, W], X) &= g(\widetilde{\nabla}_Z W, X) - g(\widetilde{\nabla}_W Z, X) \\ &= g(\widetilde{\nabla}_W X, Z) - g(\widetilde{\nabla}_Z X, W) \\ &= g(\phi\widetilde{\nabla}_W X, \phi Z) - g(\phi\widetilde{\nabla}_Z X, \phi W) \\ &= g(\widetilde{\nabla}_W \phi X - (\widetilde{\nabla}_W \phi)X, \phi Z) - g(\widetilde{\nabla}_Z \phi X - (\widetilde{\nabla}_Z \phi)X, \phi W) \end{aligned}$$

Here, By using (5), (7) and (29), we obtain

$$\begin{aligned} g([Z, W], X) &= g(\widetilde{\nabla}_W \phi X, \phi Z) - g(\widetilde{\nabla}_Z \phi X, \phi W) \\ &= g(\sigma(\phi X, W), \phi Z) - g(\sigma(\phi X, Z), \phi W) \\ &= g(A_{\phi Z} W - A_{\phi Z} W, \phi X) = 0. \end{aligned}$$

Thus $[Z, W] \in \Gamma(D^{\perp})$ for any $Z, W \in \Gamma(D^{\perp})$, that is, $D^{\perp}$ is integrable. Thus the proof is complete. □

Theorem 3.8. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the invariant distribution D is integrable if and only if the second fundamental form of M satisfies*

$$\sigma(X, \phi Y) = \sigma(\phi X, Y) \tag{31}$$

for any $X, Y \in \Gamma(D)$.

Proof. For any vector field X, Y in D, making use of (3), we have

$$\begin{aligned} \phi[X, Y] &= \phi(\nabla_X Y - \nabla_Y X) = \phi(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X) \\ &= \widetilde{\nabla}_X \phi Y - \widetilde{\nabla}_Y \phi X + (\widetilde{\nabla}_Y \phi)X - (\widetilde{\nabla}_X \phi)Y \end{aligned}$$

Here, by using (5), we have

$$\begin{aligned} \phi[X, Y] &= \widetilde{\nabla}_X \phi Y - \widetilde{\nabla}_Y \phi X \\ &= \nabla_X \phi Y - \nabla_Y \phi X + \sigma(X, \phi Y) - \sigma(\phi X, Y) \end{aligned} \tag{32}$$

From the normal components of (32), we conclude

$$N[X,Y] = \sigma(X,\phi Y) - \sigma(\phi X, Y).$$

Thus D is integrable if and only if (31) is satisfied. □

Theorem 3.9. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the invariant distribution D is completely integrable and its maximal integral submanifold is an invariant submanifold of$\widetilde{M}$ if and only if the shape operator A_V of M satisfies*

$$A_V T + T A_V \in \Gamma(D^\perp), \tag{33}$$

for any $V \in \Gamma(T^\perp M)$.

Proof. For any vector field X, Y in D and $V \in \Gamma(T^\perp M)$ by using (31), we have

$$\begin{aligned} g(\sigma(X,\phi Y),V) - g(\sigma(\phi X,Y),V) &= g(A_V X,\phi Y) - g(A_V\phi X, Y) \\ &= -g(TA_V X, Y) - g(A_V T X, Y) = 0. \end{aligned}$$

Hence

$$TA_V X + A_V T X \in \Gamma(D^\perp)$$

Thus D is integrable if and only if (33) is satisfied. □

Definition 3.10. *A contact CR-submanifold M of cosymplectic manifold $\widetilde{M}$ is said to be D-geodesic (resp. $D^\perp$-geodesic) if $\sigma(X,Y) = 0$ for $X, Y \in \Gamma(D)$ (resp. $\sigma(Z,W) = 0$ for $Z, W \in \Gamma(D^\perp)$). If $\sigma(X,Z) = 0$, the M is called mixed geodesic submanifold, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.*

Theorem 3.11. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the anti-invariant distribution $D^\perp$ is totally geodesic in M if and only if $\sigma(Z,X) \in \Gamma(\nu)$ for any $Z \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$.*

Proof. For any $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$, we have

$$\begin{aligned} g(\nabla_Z W,\phi X) &= -g(\widetilde{\nabla}_Z \phi X, W) \\ &= -g((\widetilde{\nabla}_Z\phi)X + \phi\widetilde{\nabla}_Z X, W) \\ &= g(\widetilde{\nabla}_Z X,\phi W) = g(\sigma(Z,X),\phi W) \end{aligned}$$

Thus $\nabla_Z W \in \Gamma(D^\perp)$ if and only if $\sigma(Z,X) \in \Gamma(\nu)$. □

Theorem 3.12. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the invariant distribution D is totally geodesic in M if and only if $\sigma(Z,W) \in \Gamma(\nu)$ for any $Z, W \in \Gamma(D)$.*

Proof. For any $Z, W \in \Gamma(D)$ and $X \in \Gamma(D^\perp)$, we have

$$\begin{aligned} g(\nabla_Z \phi W, X) &= g((\widetilde{\nabla}_Z\phi)W + \phi\widetilde{\nabla}_Z W, X) \\ &= -g(\widetilde{\nabla}_Z W,\phi X) = -g(\sigma(Z,W),\phi X), \end{aligned}$$

thus $\nabla_Z W \in \Gamma(D)$ if and only if $\sigma(Z,W) \in \Gamma(\nu)$. This completes of the prof. □

Let $e_1, e_2, ..., e_p, e_{p+1} = \phi e_1, e_{p+2} = \phi e_2, ..., e_{2p} = \phi e_p, e_{2p+1} = \xi, e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_1, e_2, ..., e_p, e_{p+1}, .., e_{2p}, e_{2p+1} = \xi$ are tangent to $\Gamma(D)$ and $e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}$ are tangent to $\Gamma(D^{\perp})$.

The mean curvature vector field H of M in $\widetilde{M}$ is defined by

$$H = \frac{1}{2p+q+1} \sum_{i,j=1}^{2p+q+1} \sigma(e_i, e_j).$$

If $H = 0$, then M is said to be minimal. Now we shall define

$$H_D = \frac{1}{2p+1} \sum_{i=1}^{2p+1} \sigma(e_i, e_i), \quad H_{D^{\perp}} = \frac{1}{q} \sum_{j=2p+2}^{2p+q+1} \sigma(e_j, e_j).$$

If $H_D = 0$, then the contact $CR-$submanifold M is said to be $D-$ minimal and If $H_{D^{\perp}} = 0$, then the contact $CR-$submanifold M is said to be $D^{\perp}-$ minimal.

Theorem 3.13. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. If D is integrable, then M is D_H-minimal submanifold in $\widetilde{M}$.*

Proof. Let $\{e_1, e_2, ..., e_p, e_{p+1} = \phi e_1, e_{p+2} = \phi e_2, ..., e_{2p} = \phi e_p, e_{2p+1} = \xi\}$ be an orthonormal frame of $\Gamma(D)$ and we denote the second fundamental form of M in $\widetilde{M}$ by σ. Then the mean curvature tensor H of M can be written as

$$H_D = \frac{1}{2p+1} \sum_{i=1}^{2p+1} \left\{\sigma(e_i, e_i) + \sigma(\phi e_i, \phi e_i) + \sigma(\xi, \xi)\right\},$$

By using (28)and (31) we mean that $\sigma(\xi, \xi) = 0$, we have

$$\begin{aligned} H_D &= \frac{1}{2p+1} \sum_{i=1}^{2p+1} \left\{\sigma(e_i, e_i) + \sigma(\phi^2 e_i, e_i)\right\} = \frac{1}{2p+1} \sum_{i=1}^{2p+1} \left\{\sigma(e_i, e_i) + \sigma(T^2 e_i, e_i)\right\} \\ &= \frac{1}{2p+1} \sum_{i=1}^{2p+1} \{\sigma(e_i, e_i) + \sigma(-e_i + \eta(e_i)\xi, e_i)\} = \frac{1}{2p+1} \sum_{i=1}^{2p+1} \{\sigma(e_i, e_i) - \sigma(e_i, e_i)\} = 0. \end{aligned}$$

This proves our assertion. □

Theorem 3.14. *Let M be a proper contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. If N is parallel on D, then either M is a $D-$geodesic submanifold or $\sigma(X, Y)$ is an eigenvector of n^2 with eigenvalue -1, for any $X,Y\in \Gamma(D)$.*

Proof. Since $(\nabla_X N)Y = 0$, for any $X, Y \in \Gamma(D)$, from (21) we have

$$n\sigma(X, Y) = \sigma(X, TY). \tag{34}$$

On the other hand, since D is a invariant distribution and $T\xi = 0$, we obtain

$$n\sigma(X, -Y + \eta(Y)\xi) = \sigma(X, T(-Y + \eta(Y)\xi)) \tag{35}$$

that is,

$$n\sigma(X, Y - \eta(Y)\xi) = \sigma(X, TY). \tag{36}$$

Now, applying n to (36), we have

$$n^2\sigma(X, Y - \eta(Y)\xi) = n\sigma(X, TY).$$

By interchanging of Y and TY in (34) , we have

$$n\sigma(X, TY) = \sigma(X, T^2Y). \tag{37}$$

Hence, by using (28), (36) and (37), we obtain

$$n^2\sigma(X, Y - \eta(Y)\xi) = n\sigma(X, TY) = \sigma(X, T^2Y) = -\sigma(X, Y - \eta(Y)\xi).$$

This implies that either σ vanishes on D or σ is an eigenvector of n^2 with eigenvalue -1. □

Theorem 3.15. *Let M be a totally umbilical non-trivial contact CR-submanifold of cosymplectic manifold $\widetilde{M}$. If $dim(D^{\perp}) > 1$, then M is totally geodesic submanifold in $\widetilde{M}$.*

Proof. We first prove that $tH = 0$, where H is the mean curvature vector of M. Since (29) holds for any $X \in \Gamma(D^{\perp})$, we have

$$A_{NX}tH = A_{NtH}X. \tag{38}$$

Taking into account of M being totally umbilical submanifold, we obtain from (8)

$$\begin{aligned} g(A_{NX}tH, X) &= g(A_{NtH}X, X) \\ g(\sigma(tH, X), NX) &= g(\sigma(X, X), NtH) \\ g(tH, X)g(H, NX) &= g(X, X)g(H, NtH) \end{aligned}$$

by equation (16), we have

$$g(H, NX)g(H, NX) = g(X, X)\, \|tH\|^2 . \tag{39}$$

Since $dim(D^{\perp}) > 1$, we can choose X in such that, furthermore, because of $X \in \Gamma(D)$, $NX = 0$ is already zero.
On the other hand, from (22) we have

$$\begin{aligned} g((\nabla_X t)H, Y) &= -g(t\nabla_X H, Y) = g(A_{nH}X, Y) - g(TA_H X, Y) \\ &= g(\sigma(X, Y), nH) + g(A_H X, TY) \\ &\quad\; g(X, Y)g(nH, H) + g(\sigma(X, TY), H) \\ &= g(X, TY)\, \|H\|^2 \end{aligned}$$

for any vector fields X and Y tangent to M. Putting $Y = TX$ in this equation, we have

$$g(X, T^2X)\, \|H\|^2 = 0,$$

from which

$$\begin{aligned} g(X, -X + \eta(X)\xi + tNX)\, \|H\|^2 &= 0 \\ \{-g(X, X) + \eta(X)\eta(X) - g(NX, NX)\}\, \|H\|^2 &= 0. \end{aligned}$$

Since M is non-trivial, we can choose an X in D such that $NX = 0$. Hence,
$g(TX, TX)\, \|H\|^2 = 0$, then we have $H = 0$, we hence M is totally geodesic submanifold. □

For a contact CR-submanifold M, if the invariant distribution D and$D^{\perp}$ are totally geodesic in M, then M is called contact CR-product. The following theorems characterize contact CR-products in cosymplectic manifolds.

Theorem 3.16. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then M is contact CR-product if and only if the shape operator A of M satisfies the condition*

$$A_{\phi W}\phi X = 0, \tag{40}$$

for all $X \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Proof. Let us assume that M is a contact CR-submanifold of $\widetilde{M}$. Then by using(2), (3) and(5), we obtain

$$\begin{aligned} g(A_{\phi W}\phi X, Y) &= g(\sigma(\phi X, Y), \phi W) = g(\widetilde{\nabla}_Y \phi X, \phi W) = g((\widetilde{\nabla}_Y \phi)X + \phi\widetilde{\nabla}_Y X, \phi W) \\ &= g(\phi\widetilde{\nabla}_Y X, \phi W) = g(\widetilde{\nabla}_Y X, W) = g(\nabla_Y X, W) \end{aligned}$$

and

$$\begin{aligned} g(A_{\phi W}\phi X, Z) &= g(\sigma(\phi X, Z), \phi W) = g(\widetilde{\nabla}_Z \phi X, \phi W) = g((\widetilde{\nabla}_Y \phi)X + \phi\widetilde{\nabla}_Z X, \phi W) \\ &= g(\phi\widetilde{\nabla}_Z X, \phi W) = g(\widetilde{\nabla}_Z X, W) = -g(\nabla_Z W, X) \end{aligned}$$

for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. So $\nabla_Y X \in \Gamma(D)$ and $\nabla_Z W \in \Gamma(D^\perp)$ if and only if (40) is satisfied. This proves our assertion. □

Theorem 3.16 can be expressed in a different way as shown as Theorem 3.17 below.

Theorem 3.17. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then M is contact CR-product if and if only if*

$$t\sigma(X, U) = 0, \tag{41}$$

for any $U \in \Gamma(M)$ and $X \in \Gamma(D)$.

Proof. For contact CR-product M in[12], it was proved that $A_{\phi W} X = 0$, for any $X \in \Gamma(D)$ and $W \in \Gamma(D^\perp)$. This condition implies (41).

Conversely, we suppose that (41) is satisfied. Then we have

$$\begin{aligned} g(\nabla_X Y, W) &= g(\phi\widetilde{\nabla}_X Y, \phi W) = g(\widetilde{\nabla}_X \phi Y, \phi W) - g((\widetilde{\nabla}_X \phi)Y, \phi W) \\ &= g(\sigma(X, TY), \phi W) = -g(t\sigma(X, TY), W) \end{aligned}$$

and

$$\begin{aligned} g(\nabla_Z W, \phi X) &= -g(\widetilde{\nabla}_Z \phi X, W) = -g((\widetilde{\nabla}_Z \phi)X, W) - g(\phi\widetilde{\nabla}_Z X, W) \\ &= g(\widetilde{\nabla}_Z X, \phi W) = -g(t\sigma(X, Z), W), \end{aligned}$$

for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. This proves our assertion. □

4. Contact CR-Submanifolds in Cosymplectic Space Forms

In section, some new results for contact CR-submanifolds in a cosymplectic manifold and a cosymplectic space form $\widetilde{M}(c)$ was given.

Theorem 4.1. *Let M be a contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$ such that $c \neq 0$. If M is a curvature-invariant contact CR-submanifold, then either M is invariant or anti-invariant submanifold.*

Proof. We suppose that M is a curvature-invariant contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$ such that $c \neq 0$. Then from (27), we have

$$\frac{c}{4}\{g(X, TZ)NY + g(TY, Z)NX + 2g(X, TY)NZ\} = 0, \tag{42}$$

for any $X, Y, Z \in \Gamma(TM)$. Taking $Z = X$ in equation (42), we have

$$g(TY, X)NX = 0.$$

This implies that $T = 0$ or $N = 0$, that is, either M is a invariant or an anti-invariant submanifold. Thus the proof is complete. □

Thus we have the following corollary.

Corollary 4.2. *There is not any curvature-invariant proper contact CR-submanifold of cosymplectic space form $\widetilde{M}(c)$ such that $c \neq 0$.*

Theorem 4.3. *Let M be a contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$ with flat normal connection such that $c \neq 0$. If $TA_V = A_VT$ for any vector V normal to M, then M is either an anti-invariant or it is a generic submanifold of $\widetilde{M}(c)$.*

Proof. If the normal connection of M is flat, then from (24), we have

$$\begin{aligned} g([A_U, A_V]X, Y) &= \frac{c}{4}\{g(X, \phi V)g(U, \phi Y) - g(Y, \phi V)g(\phi X, U) \\ &\quad +2g(X, \phi Y)g(\phi V, U)\} \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Here, choosing $U = nV$ and $Y = TX$, by direct calculations, we can state

$$g([A_V, A_{nV}]X, TX) = -\frac{c}{2}\{g(TX, TX)g(nV, nV)\},$$

that is,

$$g(A_{nV}A_VTX - A_VA_{nV}TX, X) = -\frac{c}{2}\{g(TX, TX)g(nV, nV)\}, \tag{43}$$

from which

$$tr(A_{nV}A_VT) - tr(A_VA_{nV}T) = \frac{c}{2}tr(T^2)g(nV, nV).$$

If $TA_V = A_VT$, then we conclude that $tr(A_{nV}A_VT) = tr(A_VA_{nV}T)$ and thus

$$tr(T^2)g(nV, nV) = 0,$$

This tells us that which proves our assertion. $T = 0$ or $n = 0$, that is, either M an anti-invariant or generic submanifold of $\widetilde{M}(c)$. □

Now, let M be a contact $CR-$ product of cosymplectic space forms $\widetilde{M}(c)$, we shall calculate bisectional curvature of cosymplectic manifold $\widetilde{M}$. By using (11) and (12) and considering Theorem3.11 and Theorem 3.12, we have

$$\begin{aligned} -H_t(X, Z) &= g(R(X, \phi X)Z, \phi Z) = g((\widetilde{\nabla}_X\sigma)(\phi X, Z) - (\widetilde{\nabla}_{\phi X}\sigma)(X, Z), \phi Z) \\ &= g((\nabla^{\perp}_X\sigma)(\phi X, Z) - \sigma(\nabla_X\phi X, Z) - \sigma(\nabla_X Z, \phi X), \phi Z) \\ &\quad -g((\nabla^{\perp}_{\phi X}\sigma)(X, Z) - \sigma(\nabla_{\phi X}X, Z) - \sigma(\nabla_{\phi X}Z, X), \phi Z) \\ &= Xg(\sigma(\phi X, Z), \phi Z) - g(\widetilde{\nabla}_X\phi Z, \sigma(\phi X, Z)) \\ &\quad -\phi Xg(\sigma(X, Z), \phi Z) + g(\widetilde{\nabla}_{\phi X}Z, \sigma(X, Z)) \\ &= -g((\widetilde{\nabla}_X\phi)Z + \phi\widetilde{\nabla}_X Z, \sigma(\phi X, Z)) + g((\widetilde{\nabla}_{\phi X}\phi)Z + \phi\widetilde{\nabla}_{\phi X}Z, \sigma(X, Z)) \\ &= -g((\phi\widetilde{\nabla}_X Z, \sigma(\phi X, Z)) + g(\phi\widetilde{\nabla}_{\phi X}Z, \sigma(X, Z)) \\ &= -g((\phi\sigma(X, Z), \sigma(\phi X, Z)) + g(\phi\sigma(\phi X, Z), \sigma(X, Z)) \\ &= 2g(\phi\sigma(\phi X, Z), \sigma(X, Z)) = -2g((\widetilde{\nabla}_Z\phi)X + \phi\widetilde{\nabla}_Z X, \phi\sigma(X, Z)) \\ &= -2g(\phi\widetilde{\nabla}_Z X, \phi\sigma(X, Z)) = -2g(\phi\sigma(X, Z), \phi\sigma(X, Z)) \\ &= 2g(\sigma(X, Z), \phi^2\sigma(X, Z)) = -2g(\sigma(X, Z), \sigma(X, Z)) \\ &= -2\,\|\sigma(X, Z)\|^2, \end{aligned}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. So we get

$$H_t(X, Z) = 2\,\|\sigma(X, Z)\|^2 . \tag{44}$$

Thus we have following the Theorem.

Theorem 4.4. *Let M be a contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$ with constant $\phi-$holomorphic sectional curvature c. Then there do not exist contact CR-product in a cosymplectic space form $\widetilde{M}(c)$ such that $c < 0$.*

Proof. We suppose that M is a contact CR-product in a cosymplectic space form $\widetilde{M}(c)$. Then from (5) and (6), we know $\sigma(Z, \xi) = 0$. By using (4) and (44), we have

$$\begin{aligned} g(R(X, \phi X)\phi Z, Z) &= \frac{c}{2}\left\{g(X, X) - \eta^2(X)\right\} g(Z, Z) \\ 2\,\|\sigma(X, Z)\|^2 &= \frac{c}{2}\left\{g(\phi X, \phi X)\right\} g(Z, Z), \end{aligned}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. So we have

$$\|\sigma(X, Z)\|^2 = \frac{c}{4}\left\{g(\phi X, \phi X)\right\} g(Z, Z) \tag{45}$$

This equality is impossible for $c < 0$. This proves our assertion. □

Theorem 4.5. *Let M be a contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$ such that $c < 0$. Then we have*

$$\|\sigma\|^2 \geq \frac{c}{2} pq,$$

where $\dim D = 2p + 1$ *and* $\dim(D^{\perp}) = q$.

Proof. Let $\left\{e_1, e_2, ..., e_p, e_{p+1} = \phi e_1, e_{p+2} = \phi e_2, ..., e_{2p} = \phi e_p, e_{2p+1} = \xi, e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}\right\}$ be an orthonormal basis of $\Gamma(TM)$ such that $\left\{e_1, e_2, e_3, ..., e_{2p}, e_{2p+1} = \xi\right\}$ is tangent to D distribution and $\left\{e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}\right\}$ is tangent to $D^{\perp}$ distribution. Then norm of the second fundamental form $\|\sigma\|^2$ is defined by,

$$\begin{aligned} \|\sigma\|^2 &= \sum_{i,\,j=1}^{2p} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) + \sum_{r,\,s=2p+2}^{2p+q+1} g(\sigma(e_r, e_s), \sigma(e_r, e_s)) \\ &\quad +2\sum_{i=1}^{2p}\sum_{r=2p+2}^{2p+q+1} g(\sigma(e_i, e_r), \sigma(e_i, e_r)) \end{aligned}$$

Taking $X = e_1, e_2, ..., e_p, e_{p+1} = \phi e_1, e_{p+2} = \phi e_2, ..., e_{2p} = \phi e_p, e_{2p+1} = \xi$ and $Z = e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}$ in (45), then we obtain

$$\|\sigma\|^2 \geq \frac{c}{2} pq.$$

□

Theorem 4.6. *Let M be a contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then the Ricci tensor S of M is given by*

$$\begin{aligned} S(X, W) &= \frac{c}{4}\{(2p + q + 4)g(X, W) - (2p + q + 2)\eta(X)\eta(W)\} \\ &\quad +(2p + q)g(\sigma(X, W), H) - \sum_{m=1}^{2p+q} g(\sigma(e_m, W), \sigma(X, e_m)) \end{aligned} \tag{46}$$

for any $X, W \in \Gamma(TM)$.

Proof. For any $X, Y, Z, W \in \Gamma(TM)$, by using (25), we have

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &\quad + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) + g(X,TZ)g(TY,W) - g(Y,TZ)g(TX,W) \\ &\quad + 2g(X,TY)g(TZ,W)\} + g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(Y,W),\sigma(X,Z)). \end{aligned}$$

Now, let $e_1, e_2, ..., e_p, e_{p+1} = \phi e_1, e_{p+2} = \phi e_2, ..., e_{2p} = \phi e_p, e_{2p+1} = \xi, e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_1, e_2, ..., e_p, e_{p+1}, .., e_{2p}, e_{2p+1} = \xi$ are tangent to $\Gamma(D)$ and $e_{2p+2}, e_{2p+3}, e_{2p+4}, ..., e_{2p+q+1}$ are tangent to $\Gamma(D^\perp)$. Hence, taking $Y = Z = e_i, e_j$ and $1 \leq i \leq 2p+1, 2p+2 \leq j \leq 2p+q+1$ then we obtain

$$S(X,W) = \sum_{i=1}^{p} g(R(X,e_i)e_i,W) + \sum_{i=p+1}^{2p} g(R(X,\phi e_i)\phi e_i,W) + g(R(X,\xi)\xi,W) + \sum_{j=2p+2}^{2p+q+1} g(R(X,e_j)e_j,W).$$

It follows that

$$\begin{aligned} S(X,W) &= \frac{c}{4}\{(2p+q+4)g(X,W) - (2p+q+2)\eta(X)\eta(W)\} + (2p+q)g(\sigma(X,W),H) \\ &\quad - \sum_{i=1}^{p} g(\sigma(e_i,W),\sigma(X,e_i)) + \sum_{i=p+1}^{2p} g(\sigma(\phi e_i,W),\sigma(X,\phi e_i)) \\ &\quad - g(\sigma(\xi,W),\sigma(X,\xi)) - \sum_{j=2p+2}^{2p+q+1} g(\sigma(e_j,W),\sigma(X,e_j)). \end{aligned} \tag{47}$$

Hence, the proof follows from the above relation. □

Theorem 4.7. *Let M be a contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then the scalar curvature tensor τ of M is given by*

$$\tau = \frac{c}{4}\left\{(2p+q)^2 + 3(2p+q)\right\} + (2p+q)^2\|H\|^2 - \|\sigma\|^2 \tag{48}$$

Proof. By using (46), we have

$$\tau = \sum_{i=1}^{p} S(e_i,e_i) + \sum_{i=p+1}^{2p} S(\phi e_i,\phi e_i) + S(\xi,\xi) + \sum_{j=2p+2}^{2p+q+1} S(e_j,e_j)$$

which gives (48). Thus the proof is complete. □

Thus we have the following corollary.

Corollary 4.8. *Let M be a minimal contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then the scalar curvature tensor τ of M is given by*

$$\tau = \frac{c}{4}(2p+q)^2 + 3(2p+q) - \|\sigma\|^2 \tag{49}$$

Theorem 4.9. *Let M be a totally umbilical contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then the Ricci tensor S of M is given by*

$$S(X,W) = \frac{c}{4}\{(2p+q+4)g(X,W) - (2p+q+2)\eta(X)\eta(W)\} \tag{50}$$

for any $X, W \in \Gamma(TM)$.

Proof. From (46) by using (8), we obtain

$$\begin{aligned} S(X,W) &= \frac{c}{4}\{(2p+q+4)g(X,W)-(2p+q+2)\eta(X)\eta(W)\} \\ &\quad +(2p+q)g(\sigma(X,W),H)-\sum_{m=1}^{2p+q} g(\sigma(e_m,W),\sigma(X,e_m)) \end{aligned} \tag{51}$$

Thus, the proof follows from the above relations, which proves the theorem completely. □

Thus we have the following corollary.

Corollary 4.10. *Every totally umbilical contact CR-submanifold M of a cosymplectic space form $\widetilde{M}(c)$ is an η-Einstein submanifold.*

Theorem 4.11. *Let M be a totally umbilical contact CR-submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then the scalar curvature tensor τ of M is given by*

$$\tau = \frac{c}{4}\left\{(2p+q)^2+3(2p+q)\right\} \tag{52}$$

Proof. By using (50), we have

$$\tau = \sum_{i=1}^{p} S(e_i,e_i)+\sum_{i=p+1}^{2p} S(\phi e_i,\phi e_i)+S(\xi,\xi)+\sum_{j=2p+2}^{2p+q+1} S(e_j,e_j)$$

which gives (52). Thus the proof is complete. □

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