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On Fully Coupled Nonlocal Multi-point boundary Value Problems of Nonlinear Mixed-order Fractional Differential Equations on an Arbitrary Domain

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Abstract. We investigate the existence and uniqueness of solutions for a mixed-type coupled fractional differential system equipped with nonlocal multi-point coupled boundary conditions on an arbitrary domain by applying standard tools of the fixed point theory. Our results, well illustrated with the aid of examples, are new and enhance the scope of the literature on the topic.

1. Introduction

We introduce and study a new class of coupled systems of mixed-order fractional differential equations equipped with nonlocal multi-point coupled boundary conditions. In precise terms, we consider the following fully coupled system:

$$\begin{cases} D^{\xi}x(t) = \varphi(t, x(t), y(t)), \ \ t \in [a, b], \ \ 0 < \xi < 1, \\ D^{\zeta}y(t) = \psi(t, x(t), y(t)), \ \ t \in [a, b], \ \ 1 < \zeta < 2, \\ px(a) + qy(b) = x_0, \ \ y(a) = 0, \ \ y'(b) = \displaystyle\sum_{i=1}^{m} \delta_i x(\sigma_i), \ a < \sigma_i < b, \end{cases} \tag{1}$$

where D^{χ} is Caputo fractional derivative of order $\chi \in \{\xi, \zeta\}$, $\varphi, \psi : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions $p, q, \delta_i \in \mathbb{R}, i = 1, 2, \ldots, m$.

Here we emphasize that the novelty of the present work lies in the fact that we consider a coupled system of fractional differential equations of different order on an arbitrary domain equipped with coupled nonlocal multi-point boundary conditions. Moreover, several new results appear as special cases of the obtained work. It is imperative to notice that much of the work related to the coupled systems of fractional differential equations deals with the fixed domain. Thus our results are more general and contribute significantly to the existing literature on the topic.

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Fractional differential equations appear in the mathematical modeling of many real world phenomena occurring in engineering and scientific disciplines, for instance, see the works [1]-[6]. Mathematical models based on fractional order integral and differential operators yield more insight into the characteristics of the associated phenomena as such operators are nonlocal in nature in contrast to the classical ones. In particular, coupled systems of fractional order differential equations have received a great attention in view of their great utility in handling and comprehending the practical issues such as synchronization of chaotic systems [7, 8], anomalous diffusion [9], ecological effects [10], etc. For some recent theoretical results on the topic, we refer the reader to a series of papers [11]-[15] and the references cited therein.

An auxiliary result related to a linear variant of the problem (1) is established in Section 2, while the main results are obtained in Section 3. Examples illustrating the main results are discussed in Section 4.

2. Preliminaries

Let us recall some preliminary concepts of fractional calculus [3].

Definition 2.1. *Let h be a locally integrable real-valued function on $-\infty \leq a < t < b \leq +\infty$. The Riemann–Liouville fractional integral I_a^α of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) is defined as*

$$I_a^\alpha h(t) = (h * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ denotes the Euler gamma function.

Definition 2.2. *Let $h \in L^1[a,b]$, $-\infty \leq a < t < b \leq +\infty$ and $h * K_{m-\alpha} \in W^{m,1}[a,b], m = [\alpha]+1, \alpha > 0$, where $W^{m,1}[a,b]$ is the Sobolev space defined as*

$$W^{m,1}[a,b] = \left\{ h \in L^1[a,b] : \frac{d^m}{dt^m} h \in L^1[a,b] \right\}.$$

The Riemann–Liouville fractional derivative D_a^α of order $\alpha > 0$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as

$$D_a^\alpha h(t) = \frac{d^m}{dt^m} I_a^{1-\alpha} h(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} h(s)ds.$$

Definition 2.3. *Let $h \in L^1[a,b]$, $-\infty \leq a < t < b \leq +\infty$ and $h * K_{m-\alpha} \in W^{m,1}[a,b], m = [\alpha], \alpha > 0$. The Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as*

$${}^cD_a^\alpha h(t) = D_a^\alpha \left[h(t) - h(a) - h'(a)\frac{(t-a)}{1!} - \ldots - h^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.4. *If $h \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as*

$${}^cD_a^\alpha [h](t) = I_a^{1-\alpha} h^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} h^{(m)}(s)ds.$$

In the sequel, the Riemann–Liouville fractional integral I_a^α and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a = 0$ are respectively denoted by I^α and ${}^cD^\alpha$.

It is well known that

1. If $\alpha + \beta > 1$, then the equation $(I^\alpha I^\beta u)(t) = (I^{\alpha+\beta}u)(t), t \in J$ is satisfied for $u \in L^1(J, \mathbb{R})$.

2. Let $\beta > \alpha$. Then the equation $(D^{\alpha} I^{\beta} u)(t) = (I^{\beta-\alpha} u)(t), t \in J$ holds for $u \in C(J, \mathbb{R})$.
3. Let $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$. Then $D^{\alpha} t^k = 0$ for $k \in \{0, 1, 2, \ldots, n-1\}$; $D^{\alpha} t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$, $\beta > n$; $I^{\alpha} t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$.

Now we present an important result to analyze the problem (1).

Lemma 2.5. *Let* $\lambda := p + q(b-a) \sum_{i=1}^{m} \delta_i \neq 0$ *and* $\bar{\varphi}, \bar{\psi} \in C[a,b]$. *Then the unique solution of the system*

$$\begin{cases} D^{\xi} x(t) = \bar{\varphi}(t), \quad t \in [a,b], \quad 0 < \xi < 1, \\ D^{\zeta} y(t) = \bar{\psi}(t), \quad t \in [a,b], \quad 1 < \zeta < 2, \\ px(a) + qy(b) = x_0, \quad y(a) = 0, \quad y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i), \end{cases} \tag{2}$$

is given by the following pair of integral equations

$$x(t) = I^{\xi} \bar{\varphi}(t) + \frac{1}{\lambda} \Big[x_0 - q I^{\zeta} \bar{\psi}(b) + q(b-a)(I^{\zeta-1} \bar{\psi}(b) - \sum_{i=1}^{m} \delta_i I^{\xi} \bar{\varphi}(\sigma_i)) \Big], \tag{3}$$

$$y(t) = I^{\zeta} \bar{\psi}(t) - \frac{t-a}{\lambda} \Big[p(I^{\zeta-1} \bar{\psi}(b) - \sum_{i=1}^{m} \delta_i I^{\xi} \bar{\varphi}(\sigma_i)) - \sum_{i=1}^{m} \delta_i (x_0 - q I^{\zeta} \bar{\psi}(b) \Big]. \tag{4}$$

Proof. As argued in [3], the solutions of fractional differential equations in (2) can be written as

$$x(t) = I^{\xi} \bar{\varphi}(t) + c_1 \quad \text{and} \quad y(t) = I^{\zeta} \bar{\psi}(t) + c_2 + c_3(t-a), \tag{5}$$

where $c_i \in \mathbb{R}\,(i = 1, 2, 3)$ are arbitrary constants. Using the condition $y(a) = 0$ in (5), we get $c_2 = 0$, while making use of the conditions $px(a) + qy(b) = x_0$ and $y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i)$ in (5) yields the following system

$$pc_1 + q(b-a)c_3 = x_0 - q I^{\zeta} \bar{\psi}(b),$$
$$\sum_{i=1}^{m} \delta_i \, c_1 - c_3 = I^{\zeta-1} \bar{\psi}(b) - \sum_{i=1}^{m} \delta_i I^{\xi} \bar{\varphi}(\sigma_i).$$

Solving the above system for c_1 and c_3, we find that

$$c_1 = \frac{1}{\lambda} \Big[x_0 - q I^{\zeta} \bar{\psi}(b) + q(b-a)(I^{\zeta-1} \bar{\psi}(b) - \sum_{i=1}^{m} \delta_i I^{\xi} \bar{\varphi}(\sigma_i)) \Big]$$

and

$$c_3 = -\frac{1}{\lambda} \Big[p(I^{\zeta-1} \bar{\psi}(b) - \sum_{i=1}^{m} \delta_i I^{\xi} \bar{\varphi}(\sigma_i)) - \sum_{i=1}^{m} \delta_i (x_0 - q I^{\zeta} \bar{\psi}(b) \Big],$$

which, on substituting in (5), completes the solutions (3) and (4). The converse follows by direct computation. □

In the following, for brevity, we use the notations:

$$L_1 = \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{q(b-a)}{|\lambda| \Gamma(\xi+1)} \sum_{i=1}^{m} |\delta_i| (\sigma_i - a)^{\xi}, \; M_1 = \frac{q}{|\lambda|} \frac{(b-a)^{\zeta}(\zeta+1)}{\Gamma(\zeta+1)}, \tag{6}$$

$$L_2 = \frac{p(b-a)}{|\lambda| \Gamma(\xi+1)} \sum_{i=1}^{m} |\delta_i| (\sigma_i - a)^{\xi}, \; M_2 = \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)} \left(1 + \frac{p\zeta}{|\lambda|} + q(b-a) \sum_{i=1}^{m} |\delta_i| \right). \tag{7}$$

3. Main Results

In view of Lemma 2.5, we define an operator $T : X \times X \to X \times X$ by

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix},$$

where $(X \times X, \|(x,y)\|)$ is a Banach space equipped with norm $\|(x,y)\| = \|x\| + \|y\|, x, y \in X$ ($X = \{x(t)|x(t) \in C([a,b],\mathbb{R})\}$ is a Banach space associated with the norm $\|x\| = \sup\{|x(t)|, t \in [a,b]\}$),

$$T_1(x,y)(t) = I^{\xi}\bar{\varphi}(t) + \frac{1}{\lambda}\Big[x_0 - qI^{\zeta}\bar{\psi}(b) + q(b-a)(I^{\zeta-1}\bar{\psi}(b) - \sum_{i=1}^{m}\delta_i I^{\xi}\bar{\varphi}(\sigma_i))\Big],$$

$$T_2(x,y)(t) = I^{\zeta}\bar{\psi}(t) - \frac{t-a}{\lambda}\Big[p(I^{\zeta-1}\bar{\psi}(b) - \sum_{i=1}^{m}\delta_i I^{\xi}\bar{\varphi}(\sigma_i)) - \sum_{i=1}^{m}\delta_i(x_0 - qI^{\zeta}\bar{\psi}(b)\Big],$$

and $\bar{\varphi}(t) = \varphi(t, x(t), y(t))$ and $\bar{\psi}(t) = \psi(t, x(t), y(t))$.

In our first result, we establish the uniqueness of solutions for the system (1) by applying contraction mapping principle due to Banach.

Theorem 3.1. *Assume that*

(H_1) $\varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist positive constants* ℓ_1 *and* ℓ_2 *such that for all* $t \in [a,b]$ *and* $x_i, y_i \in \mathbb{R},\ i = 1,2,$ *we have*

$$|\varphi(t,x_1,x_2) - \varphi(t,y_1,y_2)| \le \ell_1(|x_1 - y_1| + |x_2 - y_2|),$$

$$|\psi(t,x_1,x_2) - \psi(t,y_1,y_2)| \le \ell_2(|x_1 - y_1| + |x_2 - y_2|).$$

If

$$(L_1 + L_2)\ell_1 + (M_1 + M_2)\ell_2 < 1, \tag{8}$$

where L_1, M_1 *and* L_2, M_2 *are respectively given by (6) and (7), then the system (1) has a unique solution on* $[a,b]$.

Proof. Define $\sup_{t\in[a,b]} \varphi(t,0,0) = N_1 < \infty$ and $\sup_{t\in[a,b]} \psi(t,0,0) = N_2 < \infty$ and $r > 0$ such that

$$r > \frac{\frac{|x_0|}{|\lambda|}\Big(1 + (b-a)\sum_{i=1}^{m}|\delta_i|\Big) + (L_1 + L_2)N_1 + (M_1 + M_2)N_2}{1 - (L_1 + L_2)\ell_1 - (M_1 + M_2)\ell_2}.$$

In the first step, we show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in X \times X : \|(x,y)\| \le r\}$. By the assumption (H_1), for $(x,y) \in B_r,\ t \in [a,b]$, we have

$$\begin{aligned} |\varphi(t,x(t),y(t))| &\le |\varphi(t,x(t),y(t)) - \varphi(t,0,0)| + |\varphi(t,0,0)| \\ &\le \ell_1(|x(t)| + |y(t)|) + N_1 \\ &\le \ell_1(\|x\| + \|y\|) + N_1 \le \ell_1 r + N_1. \end{aligned} \tag{9}$$

Similarly, we can get

$$|\psi(t,x(t),y(t))| \le \ell_2(\|x\| + \|y\|) + N_2 \le \ell_2 r + N_2. \tag{10}$$

Using (9) and (10), we obtain

$$\begin{aligned}
|T_1(x,y)(t)| &\leq \frac{(b-a)^{\xi}}{\Gamma(\xi+1)}\|\bar{\varphi}\| + \frac{1}{|\lambda|}\Bigg[|x_0| + q\frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)}\|\bar{\psi}\| \\
&\quad + q(b-a)\Big(\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)}\|\bar{\psi}\| + \sum_{i=1}^{m}|\delta_i|\frac{(\sigma_1-a)^{\xi}}{\Gamma(\xi+1)}\|\bar{\varphi}\|\Big)\Bigg] \\
&\leq \frac{|x_0|}{|\lambda|} + \left[\frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{q(b-a)}{|\lambda|\Gamma(\xi+1)}\sum_{i=1}^{m}|\delta_i|(\sigma_i-a)^{\xi}\right](\ell_1 r + N_1) \\
&\quad + \frac{q}{|\lambda|}\frac{(b-a)^{\zeta}(\zeta+1)}{\Gamma(\zeta+1)}(\ell_2 r + N_2) \\
&= \frac{|x_0|}{|\lambda|} + (L_1\ell_1 + M_1\ell_2)r + L_1N_1 + M_1N_2,
\end{aligned}$$

which implies that

$$\|T_1(x,y)\| \leq \frac{|x_0|}{|\lambda|} + (L_1\ell_1 + M_1\ell_2)r + L_1N_1 + M_1N_2,$$

where we have taken the norm for $t \in [a,b]$. Likewise, we can find that

$$\|T_2(x,y)\| \leq \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^{m}|\delta_i| + (L_2\ell_1 + M_2\ell_2)r + L_2N_1 + M_2N_2.$$

Consequently,

$$\begin{aligned}
\|T(x,y)\| &\leq \frac{|x_0|}{|\lambda|} + \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^{m}|\delta_i| \\
&\quad + [(L_1+L_2)\ell_1 + (M_1+M_2)\ell_2]r + (L_1+L_2)N_1 + (M_1+M_2)N_2 \leq r.
\end{aligned}$$

Now, for $(x_1,y_1),(x_2,y_2) \in X \times X$ and for any $t \in [a,b]$, we get

$$\begin{aligned}
&|T_1(x_2,y_2)(t) - T_1(x_1,y_1)(t)| \\
\leq\ & \left[\frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{q(b-a)}{|\lambda|\Gamma(\xi+1)}\sum_{i=1}^{m}|\delta_i|(\sigma_i-a)^{\xi}\right]\ell_1(\|x_2-x_1\| + \|y_2-y_1\|) \\
& + \frac{q}{|\lambda|}\frac{(b-a)^{\zeta}(\zeta+1)}{\Gamma(\zeta+1)}\ell_2(\|x_2-x_1\| + \|y_2-y_1\|) \\
=\ & (L_1\ell_1 + M_1\ell_2)(\|x_2-x_1\| + \|y_2-y_1\|),
\end{aligned}$$

which implies that

$$\|T_1(x_2,y_2) - T_1(x_1,y_1)\| \leq (L_1\ell_1 + M_1\ell_2)(\|x_2-x_1\| + \|y_2-y_1\|). \tag{11}$$

Similarly, we find that

$$\|T_2(x_2,y_2)(t) - T_2(x_1,y_1)\| \leq (L_2\ell_1 + M_2\ell_2)(\|x_2-x_1\| + \|y_2-y_1\|). \tag{12}$$

It follows from (11) and (12) that

$$\|T(x_2,y_2) - T(x_1,y_1)\| \leq [(L_1+L_2)\ell_1 + (M_1+M_2)\ell_2](\|x_2-x_1\| + \|y_2-y_1\|).$$

From the above inequality, we deduce that T is a contraction in view of the condition (8). Hence it follows by Banach's fixed point theorem that there exists a unique fixed point for the operator T, which corresponds to a unique solution of problem (1) on $[a,b]$. □

In the following result, we apply Leray-Schauder alternative ([16] p. 4) to prove the existence of solutions for the problem (1).

Theorem 3.2. *Assume that*

(H_2) $\varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist real constants* $k_i, \gamma_i \geq 0$, $(i = 1,2)$ *and* $k_0 > 0, \gamma_0 > 0$ *such that* $\forall x, y \in \mathbb{R}$,

$$|\varphi(t,x,y)| \leq k_0 + k_1|x| + k_2|y|,$$

$$|\psi(t,x,y)| \leq \gamma_0 + \gamma_1|x| + \gamma_2|y|.$$

Then, the system (1) has at least one solution on $[a,b]$ *provided that*

$$(L_1 + L_2)k_1 + (M_1 + M_2)\gamma_1 < 1 \;\; \text{and} \;\; (L_1 + L_2)k_2 + (M_1 + M_2)\gamma_2 < 1, \tag{13}$$

where L_1, M_1 *and* L_2, M_2 *are respectively given by (6) and (7).*

Proof. Observe that continuity of the operator $T : X \times X \to X \times X$ follows from continuity of functions φ and ψ. Next, let $\Omega \subset X \times X$ be bounded such that

$$|\varphi(t, x(t), y(t))| \leq K_1, \quad |\psi(t, x(t), y(t))| \leq K_2, \;\; \forall (x,y) \in \Omega,$$

for positive constants K_1 and K_2. Then for any $(x,y) \in \Omega$, we have

$$\begin{aligned} |T_1(x,y)(t)| &\leq \frac{|x_0|}{|\lambda|} + \left[\frac{(b-a)^\xi}{\Gamma(\xi+1)} + \frac{q(b-a)}{|\lambda|\Gamma(\xi+1)}\sum_{i=1}^m |\delta_i|(\sigma_i - a)^\xi\right]K_1 \\ &\quad + \frac{q}{|\lambda|}\frac{(b-a)^\zeta(\zeta+1)}{\Gamma(\zeta+1)}K_2 \\ &= \frac{|x_0|}{|\lambda|} + L_1K_1 + M_1K_2, \end{aligned}$$

which implies that

$$\|T_1(x,y)\| \leq \frac{|x_0|}{|\lambda|} + L_1K_1 + M_1K_2.$$

Similarly, we can be shown that

$$\|T_2(x,y)\| \leq \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^m |\delta_i| + L_2K_1 + M_2K_2.$$

From the foregoing arguments, we deduce that the operator T is uniformly bounded, that is,

$$\|T(x,y)\| \leq \frac{|x_0|}{|\lambda|} + \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^m |\delta_i| + (L_1 + L_2)K_1 + (M_1 + M_2)K_2.$$

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then we have

$$\begin{aligned} & |T_1(x(t_2), y(t_2)) - T_1(x(t_1), y(t_1))| \\ \leq\; & K_1\left|\frac{1}{\Gamma(\xi)}\int_a^{t_2}(t_2 - s)^{\xi-1}ds - \frac{1}{\Gamma(\xi)}\int_a^{t_1}(t_1 - s)^{\xi-1}ds\right| \\ \leq\; & K_1\left\{\frac{1}{\Gamma(\xi)}\int_a^{t_1}[(t_2 - s)^{\xi-1} - (t_1 - s)^{\xi-1}]ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2 - s)^{\xi-1}ds\right\} \\ \leq\; & \frac{K_1}{\Gamma(\xi+1)}[2(t_2 - t_1)^\xi + |t_2^\xi - t_1^\xi|]. \end{aligned}$$

Analogously, we can obtain

$$\begin{aligned} & |T_2(x(t_2), y(t_2)) - T_2(x(t_1), y(t_1))| \\ \leq \; & \frac{K_2}{\Gamma(\zeta+1)}[2(t_2-t_1)^{\zeta} + |t_2^{\zeta} - t_1^{\zeta}|] \\ & + \frac{|t_2-t_1|}{|\lambda|}\Big[p\Big(\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)}K_2 + \sum_{i=1}^{m}|\delta_i|\frac{(\sigma_1-a)^{\xi}}{\Gamma(\xi+1)}K_1\Big) + \sum_{i=1}^{m}|\delta_i| q \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)}K_2\Big]. \end{aligned}$$

This shows that the operator $T(x,y)$ is equicontinuous. In consequence, we deduce that the operator $T(x,y)$ is completely continuous.

Finally, we consider the set $\mathcal{P} = \{(x,y) \in X \times X | (x,y) = \lambda T(x,y),\ 0 \leq \lambda \leq 1\}$ and show that it is bounded. Let $(x,y) \in \mathcal{P}$ with $(x,y) = \lambda T(x,y)$. For any $t \in [a,b]$, we have $x(t) = \lambda T_1(x,y)(t)$, $y(t) = \lambda T_2(x,y)(t)$. Then

$$\begin{aligned} |x(t)| & \leq \frac{|x_0|}{|\lambda|} + L_1(k_0 + k_1|x| + k_2|y|) + M_1(\gamma_0 + \gamma_1|x| + \gamma_2|y|) \\ & = \frac{|x_0|}{|\lambda|} + L_1k_0 + M_1\gamma_0 + (L_1k_1 + M_1\gamma_1)|x| + (L_1k_2 + M_1\gamma_2)|y|, \end{aligned}$$

and

$$\begin{aligned} |y(t)| & \leq \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^{m}|\delta_i| + L_2(k_0 + k_1|x| + k_2|y|) + M_2(\gamma_0 + \gamma_1|x| + \gamma_2|y|) \\ & = \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^{m}|\delta_i| + L_2k_0 + M_2\gamma_0 + (L_2k_1 + M_2\gamma_1)|x| + (L_2k_2 + M_2\gamma_2)|y|. \end{aligned}$$

In consequence of the foregoing arguments, we deduce that

$$\|x\| \leq \frac{|x_0|}{|\lambda|} + L_1k_0 + M_1\gamma_0 + (L_1k_1 + M_1\gamma_1)\|x\| + (L_1k_2 + M_1\gamma_2)\|y\|$$

and

$$\|y\| \leq \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^{m}|\delta_i| + L_2k_0 + M_2\gamma_0 + (L_2k_1 + M_2\gamma_1)\|x\| + (L_2k_2 + M_2\gamma_2)\|y\|,$$

which imply that

$$\begin{aligned} \|x\| + \|y\| \leq \; & \frac{|x_0|}{|\lambda|} + \frac{(b-a)|x_0|}{|\lambda|}\sum_{i=1}^{m}|\delta_i| + (L_1+L_2)k_0 + (M_1+M_2)\gamma_0 \\ & + [(L_1+L_2)k_1 + (M_1+M_2)\gamma_1]\|x\| + [(L_1+L_2)k_2 + (M_1+M_2)\gamma_2)]\|y\|. \end{aligned}$$

Thus

$$\|(x,y)\| \leq \frac{1}{M_0}\Big[\frac{|x_0|}{|\lambda|}\Big(1 + (b-a)\sum_{i=1}^{m}|\delta_i|\Big) + (L_1+L_2)k_0 + (M_1+M_2)\gamma_0\Big],$$

where $M_0 = \min\{1 - [(L_1+L_2)k_1 + (M_1+M_2)\gamma_1], 1 - [(L_1+L_2)k_2 + (M_1+M_2)\gamma_2)]\}$. Hence the set $\mathcal{P}$ is bounded. Thus, by Leray-Schauder alternative ([16] p. 4), we deduce that the operator T has at least one fixed point, which corresponds to the fact that the problem (1) has at least one solution on $[a,b]$. □

4. Examples

Let us consider the following mixed-type coupled fractional differential systems

$$\begin{cases} D^{3/4}x(t) = \varphi(t, x(t), y(t)),\ \ D^{3/2}y(t) = \psi(t, x(t), y(t)),\ \ t \in [1,2], \\ x(a) + \dfrac{1}{2}y(b) = 1,\ \ y(1) = 0,\ \ y'(2) = \displaystyle\sum_{i=1}^{3}\delta_i x(\sigma_i), \end{cases} \tag{14}$$

where $\xi = 3/4, \zeta = 3/2, p = 1, q = 1/2, x_0 = 1, \delta_1 = 1/4, \delta_2 = 1/2, \delta_3 = 3/4, \sigma_1 = 5/4, \sigma_2 = 3/2, \sigma_3 = 7/4$, With the given data, it is found that $L_1 \simeq 1.395874, L_2 \simeq 0.615619, M_1 \simeq 0.537323, M_2 \simeq 1.961230$.
(1) In order to illustrate Theorem 3.1, we take

$$\varphi(t,x,y) = \frac{e^{-t}}{\sqrt{3+t^2}}\Big(\frac{|x|}{1+|x|} + \tan^{-1} y\Big) + \cos t, \;\; \psi(t,x,y) = \frac{1}{(5+t^4)}\Big(\sin x + |y|\Big) + e^{-t}, \tag{15}$$

which clearly satisfy the condition (H_1) with $\ell_1 = 1/2e$ and $\ell_2 = 1/6$. Moreover $(L_1 + L_2)\ell_1 + (M_1 + M_2)\ell_2 < 0.786419$. Thus the hypothesis of Theorem 3.1 holds true and consequently there exists a unique solution of the problem (14) with $\varphi(t,x,y)$ and $\psi(t,x,y)$ given by (15) on $[1,2]$.
(2) In order to illustrate Theorem 3.2, we take

$$\begin{aligned}
&\varphi(t,x,y) = e^{-2t} + \frac{1}{8}x\cos y + \frac{e^{-t}}{3}\sin y,\\
&\psi(t,x,y) = t\sqrt{t^2+3} + \frac{e^{-t}}{3\pi}x\tan^{-1} y + \frac{1}{\sqrt{48+t^2}}y.
\end{aligned} \tag{16}$$

It is easy to check that the conditions (H_2) is satisfied with $k_0 = 1/2e, k_1 = 1/8, k_2 = 1/3e, \gamma_0 = 2\sqrt{7}, \gamma_1 = 1/6e, \gamma_2 = 1/7$. Furthermore, $(L_1 + L_2)k_1 + (M_1 + M_2)\gamma_1 \simeq 0.404631 < 1$ and $(L_1 + L_2)k_2 + (M_1 + M_2)\gamma_2 \simeq 0.603598 < 1$. Clearly the hypotheses of Theorem 3.2 are satisfied and hence the conclusion of Theorem 3.2 applies to problem (14) with $\varphi(t,x,y)$ and $\psi(t,x,y)$ given by (16).

5. Conclusions

We have developed the criteria ensuring the existence and uniqueness of solutions for a new class of nonlocal multi-point boundary value problems of mixed-type coupled fractional differential equations on an arbitrary domain. The introduction of arbitrary domain extends the scope of the present work as it can be specialized to any fixed domain. Moreover, some special results follow for the mixed coupled system on the arbitrary domain by fixing the parameters involved in the boundary conditions. For instance, by taking $q = 0, p \neq 0$, our results correspond to the nonlocal conditions of the form: $x(a) = x_0/p, y(a) = 0, y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i)$. In case we we take $p = 0, q \neq 0$, we get the results for the boundary conditions of the form: $y(b) = x_0/q,\;\; y(a) = 0,\;\; y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i)$. Letting $\delta_i = 0, i = 1,2,\ldots,m$ in the results of this paper, we obtain the results for the boundary condition: $px(a) + qy(b) = x_0, y(a) = 0, y'(b) = 0$. It is imperative to mention that aforementioned special cases for the mixed coupled fractional differential system on the arbitrary domain are all new results.

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