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Numerical Solution of Two Dimensional Nonlinear Fuzzy Fredholm Integral Equations of Second Kind Using Hybrid of Block-Pulse Functions and Bernstein Polynomials

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Abstract. In this paper, first, we introduce a successive approximation method in terms of a combination of Bernstein polynomials and block-pulse functions. The proposed method is given for solving two dimensional nonlinear fuzzy Fredholm integral equations of the second kind. Then, we present the convergence of the proposed method. Also we investigate the numerical stability of the method with respect to the choice of the first iteration. Finally, two numerical examples are presented to show the accuracy of the method.

1. Introduction

The fuzzy integral equations (FIE) are very useful for solving many problems in several applied fields such as mathematical economics; electrical engineering; medicine ; biology and optimal control theory. Since these equations usually can not be solved explicitly, so it is required to obtain approximate solutions. The Banach fixed point theorem is the main tool in studying the existence and uniqueness of the solution for fuzzy integral equations which can be found in [5, 21]. Numerical procedures for solving fuzzy integral equations of the second kind, based on the successive approximation method and other iterative techniques, have been investigated in [6, 24]. Recently, Bica and Popescu [7, 8] applied the successive approximation method to the fuzzy Hammerstein integral equation. Ezzati and Ziari [9] proved the convergence of the successive approximation method for solving nonlinear fuzzy Fredholm integral equations of the second kind, and they proposed an iterative procedure based on the trapezoidal quadrature. Mirzaee [18] obtained an approximate solution for the linear Fredholm fuzzy integral equations of the second kind by hybrid of block-pulse function and Taylor series (HBT). Ezzati and Baghmeshe [4] obtained an approximate solution for the nonlinear Fredholm fuzzy integral equations of the second kind by hybrid of block-pulse function and Taylor series. Zarrini and Torkaman [25] obtained the numerical solution of fuzzy Fredholm integral equations by hybrid orthonormal Bernstein and block-pulse functions.

In this paper, first, we approximate the fuzzy function by hybrid block-pulse functions and Bernstein polynomials (HBB) and estimate the error approximation. Then, we propose an iterative procedure based

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on HBB for solving two dimensional nonlinear fuzzy Fredholm integral equations

$$u(x,y) = f(x,y) \oplus (FR)\int_a^b (FR)\int_c^d K(x,y,s,t) \odot G(u(s,t))dsdt, (x,y) \in [a,b]\times[c,d], \tag{1}$$

where $K(x,y,s,t)$ is an arbitrary positive crisp kernel in $[a,b]\times[c,d]\times[a,b]\times[c,d]$ and $f : [a,b]\times[c,d] \to R_F$, $u(x,y)$ is a fuzzy real valued function and $G : R_F \to R_F$ is continuous. We assume K is continuous and therefore uniformly continuous with respect to x, y. This property implies that there exists $M > 0$ such that

$$M = \max |K(x,y,s,t)|.$$

The rest of the paper is organized as follows: In Section 2, we review some elementary concepts of the fuzzy set theory, fuzzy Reiman integrable function and modulus of continuity. In Section 3, hybrid of Bernstain polynomials and block - pulse functions is introduced.The error estimate, convergence and numerical stability analysis of the proposed method are presented in Section 4. We present two examples in section 5 to show the efficiency of the proposed method. Finally, we present our concluding remarks in Section 6.

2. Preliminaries

Definition 1 (See[2]). A fuzzy number is a function $u : R \to [0,1]$ having the properties:

(1) u is normal , that is $\exists\, x_0 \in R$ such that $u(x_0) = 1$,

(2) u is fuzzy convex set

$$(i.e.\ u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \quad \forall x, y \in R, \lambda \in [0,1]),$$

(3) u is upper semi-continuous on R,

(4) the $\overline{\{\, x \in R : u(x) > 0\}}$ is compact set.

The set of all fuzzy numbers is denoted by R_F. An alternative definition which yields the same R_F is given by [17].

Definition 2 (See [15, 19]). An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1],

(2) $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1],

(3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

The addition and scaler multiplication of fuzzy numbers in R_F are defined as follows:

(1) $(u \oplus v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

(2) $(\lambda \odot u)(r) = \begin{cases} (\lambda\underline{u}(r), \lambda\overline{u}(r)) & \lambda \geq 0, \\ (\lambda\overline{u}(r), \lambda\underline{u}(r)) & \lambda < 0. \end{cases}$

Definition 3 (See [1]). For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ the quantity $D(u,v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ is the distance between u and v.

The following properties are hold (See [6]):

(1) (R_F, D) is a complete metric space,

(2) $D(u \oplus w, v \oplus w) = D(u, v) \quad \forall\, u, v, w \in R_F$,

(3) $D(k \odot u, k \odot v) = |k|\, D(u, v) \quad \forall\, u, v \in R_F \quad \forall\, k \in R$,

(4) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e) \quad \forall\, u, v, w, e \in R_F$.

Theorem 1 (See [2, 3]).

(1) The pair $(R_F, \oplus)$ is a commutative semigroup with $\tilde{0} = \chi_0$ zero element.

(2) For fuzzy numbers which are not crisp, there is no opposite element (that is, $(R_F, \oplus)$ cannot be a group).

(3) For any $a, b \in R$ with $a, b \geq 0$ or $a, b \leq 0$ and for any $u \in R_F$, we have
$(a + b) \odot u = a \odot u \oplus b \odot u$.
For arbitrary $a, b \in R$, this property is not fulfilled.

(4) For any $\lambda, \mu \in R$ and $u \in R_F$, we have
$\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot u$.

(5) For any $\lambda \in R$ and $u, v \in R_F$, we have
$\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$.

(6) The function of $\|.\|_F : R_F \to R$ by $\|u\|_F = D(u, \tilde{0})$ has the usual properties of the norm, that is, $\|u\|_F = 0$ if and only if
$u = \tilde{o}, \|\lambda \odot u\|_F = |\lambda| \|u\|_F$ and $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F$.

(7) $|\|u\|_F - \|v\|_F| \leq D(u, v)$ and $D(u, v) \leq |u\|_F + \|v\|_F$ for any $u, v \in R_F$.

Definition 4 (See [17]). A fuzzy real number valued function $f : [a, b] \to R_F$ is said to be continuous in $x_0 \in [a, b]$, if for each $\varepsilon > 0$ there exist $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a, b]$, and denote the space of all such functions by $C_F[a, b]$.

Definition 5 (See[3, 13]). If $X = \{f : [a, b] \times [c, d] \to R_F |$ f is continuous$\}$, then X together with the metric

$$D^*(f, g) = \sup_{a \leq s, t \leq b} D(f(s, t), g(s, t))$$

is complete metric space.

Definition 6 (See[10, 13]). Let $f : [a, b] \times [c, d] \to R_F$, be a bounded mapping, then function $\omega_{[a,b]\times[c,d]}(f, .) : R_+ \cup \{0\} \to R_+$ defined by

$$\omega_{[a,b]\times[c,d]}(f, \delta) = \sup \left\{ D(f(x, y), f(s, t)) |\, x, y \in [a, b],\ s, t \in [c, d],\ \sqrt{(x - s)^2 + (y - t)^2} \leq \delta \right\}, \tag{2}$$

is called the modulus of oscillation of f on $[a, b] \times [c, d]$. In addition, if $f \in C_F([a, b] \times [c, d])$(i.e. $f : [a, b] \times [c, d] \to R_F$ is continuous on $[a, b] \times [c, d]$), then $\omega_{[a,b]\times[c,d]}(f, \delta)$ is called the modulus of continuity of f on $[a, b] \times [c, d]$.
Some properties of the modulus of continuity are given in below

Theorem 2 (See[6, 13]). The following properties hold:

(1) $D(f(x, y), f(s, t)) \leq \omega_{[a,b]\times[c,d]}(f, \sqrt{(x - s)^2 + (y - t)^2})$ for any $x, y \in [a, b]$, $s, t \in [c, d]$,

(2) $\omega_{[a,b]\times[c,d]}(f,\delta)$ is increasing function of δ,

(3) $\omega_{[a,b]\times[c,d]}(f,0)=0$,

(4) $\omega_{[a,b]\times[c,d]}(f,\delta_1+\delta_2)\leq\omega_{[a,b]\times[c,d]}(f,\delta_1)+\omega_{[a,b]\times[c,d]}(f,\delta_2)$ for any $\delta_1,\delta_2\geq 0$,

(5) $\omega_{[a,b]\times[c,d]}(f,n\delta)\leq n\omega_{[a,b]\times[c,d]}(f,\delta)$ for any $\delta\geq 0$ $n\in N$,

(6) $\omega_{[a,b]\times[c,d]}(f,\lambda\delta)\leq(\lambda+1)\omega_{[a,b]\times[c,d]}(f,\delta)$ for any $\delta,\lambda\geq 0$,

(7) if $[a,b]\times[c,d]\subseteq[e,f]\times[g,h]$ then $\omega_{[a,b]\times[c,d]}(f,\delta)\leq\omega_{[e,f]\times[g,h]}(f,\delta)$.

Definition 7 (See[13, 23]). Let $f:[a,b]\times[c,d]\to R_F$, for $\Delta x: a=x_0<x_1<...<x_n=b$ and $\Delta y: c=y_0<y_1<...<y_n=d$, two partitions of the intervals $[a,b]$ and $[c,d]$, respectively. Let us consider the intermediates points $\xi_i\in[x_{i-1},x_i]$ and $\eta_i\in[y_{j-1},y_j]$, $i=1,...,n$; $j=1,...,n$, and $\delta:[a,b]\to R_+$ and $\sigma:[c,d]\to R_+$. The division $P_x=([x_{i-1},x_i];\xi_i); i=1,...,n$ and $P_y=([y_{j-1},y_j];\eta_j); j=1,...,n$, denoted shortly by $P_x=(\Delta^n,\xi)$ and $P_y=(\Delta^n,\eta)$ are said to be δ-fine and σ-fine respectively if $[x_{i-1},x_i]\subseteq(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i))$ and $[y_{j-1},y_j]\subseteq(\eta_j-\sigma(\eta_j),\eta_j+\sigma(\eta_j))$.

The function f is called two-dimensional Henstock integrable to $I\in R_F$ if for any $\varepsilon>0$, there are functions $\delta:[a,b]\to R_+$ and $\sigma:[c,d]\to R_+$ such that for any δ-fine and σ-fine division we have $D(\sum_{i=0}^{n}\sum_{j=0}^{n}(x_i-x_{i-1})(y_j-y_{j-1})\odot f(\xi_i,\eta_j),I)$, where Σ denotes the fuzzy summation. Then I is called the two -dimensional Henstock integral of f and denoted by $I(f)=(FH)\int_a^b(FH)\int_c^d f(s,t)dsdt$. If the above δ and σ are constant functions, then one recaptures the concept of Riemann integral. In this case $I\in R_F$will be called two-dimensional integral of f on $[a,b]\times[c,d]$ and will be denoted by $(FR)\int_a^b(FR)\int_c^d f(s,t)dsdt$.

In [13], the authors proved that if $f\in C_F([a,b]\times[c,d])$, its definite integral exists, and also,

$$\underline{(FR)\int_a^b(FR)\int_c^d f(s,t;r)dsdt}=\int_a^b\int_c^d \underline{f}(s,t,r)dsdt,$$

$$\overline{(FR)\int_a^b(FR)\int_c^d f(s,t;r)dsdt}=\int_a^b\int_c^d \overline{f}(s,t,r)dsdt.$$

Lemma 1 (See [13, 14]). If $f,g:[a,b]\times[c,d]\subseteq R\times R\to R_F$ are fuzzy continuous functions, then the function $F:[a,b]\times[c,d]\to R_+$ by $F(t)=D(f(s,t),g(s,t))$ is continuous on $A=[a,b]\times[c,d]$, and

$$D\left((FR)\int_a^b(FR)\int_c^d f(s,t)dsdt,(FR)\int_a^b(FR)\int_c^d g(s,t)dsdt\right)\leq\int_a^b\int_c^d D(f(s,t),g(s,t))dsdt. \quad (3)$$

3. Hybrid Bernstein polynomials and block pulse functions

Now, we recall some definitions and properties of hybrid Bernstein polynomials and block pulse functions. Then we generalize them to the fuzzy sets.

Definition 8 (See [16]). Block-pulse functions $\varphi_i(t), i=1,2,...,M$ on the interval $[0,1)$, are defined as

$$\varphi_i(t)=\begin{cases}1, & \frac{i-1}{M}\leq t<\frac{i}{M}\\ 0, & \text{otherwise.}\end{cases}$$

where M is an arbitrary positive integer number and $h=\frac{1}{M}$. Similar to one dimensional, a set of two dimensional(2D) block pulse functions $\varphi_{i,j}(t_1,t_2)$; $i=1,2,...,M_1$; $j=1,2...,M_2$ is defined in the region of $t_1\in[0,1)$ and $t_2\in[0,1)$ as:

$$\varphi_{i,j}(t_1,t_2)=\begin{cases}1, & \frac{i-1}{M_1}\leq t_1<\frac{i}{M_1}\quad\text{and}\quad\frac{j-1}{M_2}\leq t_2<\frac{j}{M_2}\\ 0 & \text{,otherwise,}\end{cases}$$

where M_1 and M_2 are arbitrary positive integers. Similar to one dimensional(1D), the 2D block pulse functions are disjointed with each other:

$$\varphi_{i,j}(t_1,t_2)\varphi_{p,q}(t_1,t_2) = \begin{cases} \varphi_{i,j}(t_1,t_2), & i = p \quad \text{and} \quad j = q \\ 0, & \text{otherwise.} \end{cases}$$

Also the functions are orthogonal with each other:

$$\int_0^1 \int_0^1 \varphi_{i,j}(t_1,t_2)\varphi_{p,q}(t_1,t_2)dt_1dt_2 = \begin{cases} h_1h_2, & i = p \quad \text{and} \quad j = q \\ 0, & \text{otherwise.} \end{cases}$$

Since each 2D block pulse function takes only one value in its subregion, the 2D block pulse functions can be expressed by two one dimensional block pulse functions

$$\varphi_{i,j}(t_1,t_2) = \varphi_i(t_1)\varphi_j(t_2)$$

where $\varphi_i(t_1)$ and $\varphi_j(t_2)$ are the one dimensional block pulse functions related to variables t_1 and t_2 , respectively.

Definition 9 (See[16, 22]). The Bernstein polynomials of degree n are defined on the interval $[0,1]$ as:

$$B_{n,p}(x) = \binom{n}{p} x^p(1-x)^{n-p} \quad ; p = 0,1,...,n. \tag{4}$$

There are $(n+1)$ Bernstein basis polynomials of degree n. For mathematical convenience, we usually set $B_{n,p}(x) = 0$ if $\quad p < 0 \quad or \quad p > n$.
It is obvious that $\forall x \in [0,1], B_{np}(x) \geq 0$; $\{B_{n0}(x), B_{n1}(x), ..., B_{nn}(x)\}$ are linearly independent algebraic polynomials of degree$\leq n$ and $\sum_{p=0}^{n} B_{np}(x) = 1, \forall n \in \mathcal{N}$.

Lemma 2 (See[20]). Let $x \in [0,1]$ then

$$\sum_{p=0}^{n} \mid x - \frac{p}{n} \mid B_{np}(x) \leq \frac{1}{2\sqrt{n}}$$

Definition 10 (See[16]). The set two variable of hybrid Bernstein block-pulse functions, $hBB_{ipjq}(x,y), i = 1,2,..,M_1, \quad j = 1,2,..,M_2; \quad p,q = 0,1,\ldots,n$; on interval $[0,1) \times [0,1)$ is defined as:

$$hBB_{ipjq}(x,y) = \begin{cases} B_{np}(M_1x - i + 1)B_{nq}(M_2y - j + 1), & \frac{(i-1)}{M_1} \leq x < \frac{i}{M_1} \quad , \frac{(j-1)}{M_2} \leq y < \frac{j}{M_2} \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

where $i = 1,2,...,M_1$ and $j = 1,2,...,M_2$ are the order of block-pulse functions, $p = 0,1,...,n$ is the order of Bernstein polynomials. M_1, M_2 denotes the number of subinterval of $[0,1]$.

3.1. *Function approximation*

For $f \in C_F([0,1] \times [0,1])$, let us consider a fuzzy hybrid polynomial of degree $2n$ as below

$$B_{pq}^F(f)(x,y) = \sum_{i=1}^{M_1}\sum_{p=0}^{n}\sum_{j=1}^{M_2}\sum_{q=0}^{n} c_{ipjq}hBB_{ipjq}(x,y), \tag{6}$$

where

$$c_{ipjq} = f(\frac{i_0-1}{M_1} + \frac{p}{nM_1}, \frac{j_0-1}{M_2} + \frac{q}{nM_2}) \tag{7}$$

Theorem 3. If $f \in C_F([0,1]\times[0,1])$ then

$$D^*(B^F_{pq}(f), f) \leq 2\omega(f, \frac{1}{\mu\sqrt{n}}), \tag{8}$$

where $\mu = min\{M_1, M_2\}$.

Proof. Let $(x,y) \in [0,1)\times[0,1)$, so there exists $i_0 \in \{1,2,...,M_1\}$ and $j_0 \in \{1,2,...,M_2\}$, such that $x \in [\frac{i_0-1}{M_1}, \frac{i_0}{M_1})$ and $y \in [\frac{j_0-1}{M_2}, \frac{j_0}{M_2})$. Then, from Eqs. (4)-(6), we observe that

$$D(B^F_{pq}(f)(x,y), f(x,y)) = D\left(\sum_{i=1}^{M_1}\sum_{p=0}^{n}\sum_{j=1}^{M_2}\sum_{q=0}^{n} c_{ipjq} hBB_{ipjq}(x,y), f(x,y)\right)$$

$$= D\left(\sum_{p=0}^{n}\sum_{q=0}^{n} f(\frac{i_0-1}{M_1}+\frac{p}{nM_1}, \frac{j_0-1}{M_2}+\frac{q}{nM_2}) B_{np}(M_1x-i_0+1)B_{nq}(M_2y-j_0+1), f(x,y)\right).$$

For notational simplicity, let $X = M_1x - i_0 + 1$, $Y = M_2y - j_0 + 1$.Then

$$D(B^F_{pq}(f)(x,y), f(x,y)) = D\left(\sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) f(\frac{i_0-1}{M_1}+\frac{p}{nM_1}, \frac{j_0-1}{M_2}+\frac{q}{nM_2}), f(x,y)\right)$$

$$= D\left(\sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) f(\frac{i_0-1}{M_1}+\frac{p}{nM_1}, \frac{j_0-1}{M_2}+\frac{q}{nM_2}), \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) f(x,y)\right)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} D\left(B_{np}(X)B_{nq}(Y) f(\frac{i_0-1}{M_1}+\frac{p}{nM_1}, \frac{j_0-1}{M_2}+\frac{q}{nM_2}), B_{np}(X)B_{nq}(Y) f(x,y)\right)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) D\left(f(\frac{i_0-1}{M_1}+\frac{p}{nM_1}, \frac{j_0-1}{M_2}+\frac{q}{nM_2}), f(x,y)\right)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) \omega\left(f, \sqrt{(x-\frac{i_0-1}{M_1}-\frac{p}{nM_1})^2 + (y-\frac{j_0-1}{M_2}-\frac{q}{nM_2})^2}\right)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) \omega\left(f, \delta\frac{\sqrt{(x-\frac{i_0-1}{M_1}-\frac{p}{nM_1})^2 + (y-\frac{j_0-1}{M_2}-\frac{q}{nM_2})^2}}{\delta}\right)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) \left(1 + \frac{\sqrt{(x-\frac{i_0-1}{M_1}-\frac{p}{nM_1})^2 + (y-\frac{j_0-1}{M_2}-\frac{q}{nM_2})^2}}{\delta}\right)\omega(f,\delta)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) \left(1 + \frac{\mid x-\frac{i_0-1}{M_1}-\frac{p}{nM_1}\mid + \mid y-\frac{j_0-1}{M_2}-\frac{q}{nM_2}\mid}{\delta}\right)\omega(f,\delta)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) \left(1 + \frac{\mid X-\frac{p}{n}\mid}{M_1\delta} + \frac{\mid Y-\frac{q}{n}\mid}{M_2\delta}\right)\omega(f,\delta)$$

$$\leq \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y)\omega(f,\delta) + \sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y)\left(\frac{|X-\frac{p}{n}|}{M_1\delta} + \frac{|Y-\frac{q}{n}|}{M_2\delta}\right)\omega(f,\delta)$$

$$\leq \omega(f,\delta)\sum_{p=0}^{n}\sum_{q=0}^{n} B_{np}(X)B_{nq}(Y) + \left(\frac{1}{2M_1\delta\sqrt{n}} + \frac{1}{2M_2\delta\sqrt{n}}\right)\omega(f,\delta).$$

Putting $\delta = \frac{1}{\mu\sqrt{n}}$ and $\mu = min\{M_1, M_2\}$, we conclude that $D(B^F_{pq}(f)(x,y), f(x,y)) \leq 2\omega(f, \frac{1}{\mu\sqrt{n}})$. Therefore

$$D^*(B^F_{pq}(f), f) \leq 2\omega(f, \frac{1}{\mu\sqrt{n}}) \tag{9}$$

□

Remark 1. Theorem (3) demonstrates that $lim_{\mu,n\to\infty} D^*(B^F_{pq}(f), f) = 0$.

4. Convergence and numerical stability analysis

We consider the two dimensional nonlinear fuzzy Fredholm integral equation (2DNFFIE) (1), where $K(x,y,s,t)$ is a continuous positive crisp kernel defined on $[a,b]\times[c,d]\times[a,b]\times[c,d]$ and $G : R_F \to R_F$ is continuous and therefore uniformly continuous with respect to x, y. This property implies that there exists $M > 0$ such that

$$M = \max |K(x,y,s,t)|.$$

Now we prove the convergence of the proposed method.

Theorem 4. (See [11]). Let $K(x,y,s,t)$ is continuous and positive defined on $[a,b]\times[c,d]\times[a,b]\times[c,d]$, $f : [a,b]\times[c,d] \to R_F$ is a continuous fuzzy. Moreover assume that there exists $L > 0$ such that

$$D(G(u_1(x,y)), G(u_2(x,y))) \leq L.D(u_1(x,y), u_2(x,y)).$$

If $B = ML(b-a)(d-c) < 1$, then the fuzzy integral equation (1) has a unique solution and it can be obtained by the following successive approximation method

$$u_0(x,y) = f(x,y)$$

$$u_m(x,y) = f(x,y) \oplus (FR)\int_a^b (FR)\int_c^d K(x,y,s,t) \odot G(u_{m-1}(s,t))dsdt.$$

Moreover the sequence of successive approximation $(u_m)_{m\geq1}$ converges to the solution u^*, furthermore the following error bound holds

$$D^*(u^*, u_m) \leq \frac{M_0 B^{m+1}}{L(1-B)}, \quad where \quad M_0 = sup_{(s,t)\in[a,b]\times[c,d]} \parallel G(f(s,t)) \parallel \tag{10}$$

4.1. *Presentation of the numerical method and its convergence analysis*

Now, we introduce the numerical method to find the approximate solution of the 2DNFFIE (1). For this porpose, we consider the uniform partition of $[a,b]\times[c,d]$ as $\Delta_x : a = t_0 < t_1 < ... < t_{M_1-1} < t_{M_1} = b$, $\Delta_y : c = s_0 < s_1 < ... < s_{M_2-1} < s_{M_2} = d$, with $t_j = t_0 + ih_1, s_j = s_0 + jh_2$ where $h_1 = \frac{(b-a)}{M_1}$ and $h_2 = \frac{(d-c)}{M_2}$. Then the following procedure gives the approximate solution of (1) as

$$z_0(x,y) = f(x,y)$$

$$z_m(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot B^F_{pq}(G(z_{m-1}(s,t)))dsdt. \tag{11}$$

The recursive relation (11) can be written in compact form as follows:

$$z_0(x,y) = f(x,y)$$

$$z_m(x,y) = f(x,y) \oplus \sum_{i=1}^{M_1}\sum_{p=0}^{n}\sum_{j=1}^{M_2}\sum_{q=0}^{n} G\left(z_{m-1}(\frac{i_0-1}{M_1} + \frac{p}{nM_1}, \frac{j_0-1}{M_2} + \frac{p}{nM_2})\right) I_{ipjq}(x,y), \tag{12}$$

where

$$I_{ipjq}(x,y) = \int_0^1 \int_0^1 K(x,y,s,t)hBB_{ipjq}(s,t)dsdt \tag{13}$$

Theorem 5. Assume equation (1) satisfies the following conditions

(1) $f : [0,1]\times[0,1] \to R_F$ is continuous.

(2) $K : [0,1]\times[0,1]\times[0,1]\times[0,1] \to R_+$ is continuous and there exists $M > 0$ such that $M = max_{0\le x,t\le 1;0\le s,y\le 1}|K(x,y,s,t)|$,

(3) $G : R_F \to R_F$ is continuous , in addition there exists $L > 0$ such that

$$D\left(G(u_1(x,y)), G(u_2(x,y))\right) \le L.D(u_1(x,y), u_2(x,y))\,,$$

where $\quad B = ML(b-a)(d-c) < 1.$

Then the iterative procedure (11) converges to the unique solution of Eq (1), u^*, and its error estimate is as follows

$$D^*(u^*, z_m) \le \frac{B}{L(1-B)}(M_0B^m + 2L\Gamma), \tag{14}$$

where

$$\Gamma = max_{(0\le p\le m-1)}\omega(z_p, \frac{1}{\mu\sqrt{n}}) \quad and \quad M_0 = sup_{(s,t)\in[0,1]\times[0,1]} \parallel G(f(s,t)) \parallel . \tag{15}$$

Proof. Since $u_1(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot G(u_0(s,t))dsdt$, we have

$$\begin{aligned}
&D(u_1(x,y), z_1(x,y)) = D(f(x,y), f(x,y)) \\
&+ D\Big((FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t)\odot G(u_0(s,t))dsdt, (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t)\odot B^F_{pq}(G(z_0(s,t)))dsdt\Big) \\
&\le \int_0^1\int_0^1 D\left(K(x,y,s,t)\odot G(f(s,t)), K(x,y,s,t)\odot B^F_{pq}(G(f(s,t)))\right)dsdt \\
&\le \int_0^1\int_0^1 |K(x,y,s,t)| D\left(G(f(s,t)), B^F_{pq}(G(f(s,t)))\right)dsdt \\
&\le M\int_0^1\int_0^1 D\left(G(f(s,t)), B^F_{pq}(G(f(s,t)))\right)dsdt \\
&\le M\int_0^1\int_0^1 2\omega(G(f), \frac{1}{\mu\sqrt{n}})dsdt \le 2ML\omega(f, \frac{1}{\mu\sqrt{n}}) = 2B\omega(z_0, \frac{1}{\mu\sqrt{n}}).
\end{aligned}$$

Now, since $u_2(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot G(u_1(s,t))dsdt$, we conclude that:

$$\begin{aligned}
&D(u_2(x,y), z_2(x,y)) = D(f(x,y), f(x,y)) \\
&+ D\left((FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot G(u_1(s,t))dsdt, (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot B^F_{pq}(G(z_1(s,t)))dsdt\right) \\
&\leq M\int_0^1\int_0^1 D\left(G(u_1(s,t)), B^F_{pq}(G(z_1(s,t)))\right)dsdt \\
&\leq M\left(\int_0^1\int_0^1 D\left(G(u_1(s,t)), (G(z_1(s,t)))\right)dsdt + \int_0^1\int_0^1 D\left(G(z_1(s,t)), B^F_{pq}(G(z_1(s,t)))\right)dsdt\right) \\
&\leq ML\int_0^1\int_0^1 D\left(u_1(s,t), z_1(s,t)\right)dsdt + M\int_0^1\int_0^1 D\left(G(z_1(s,t)), B^F_{pq}(G(z_1(s,t)))\right)dsdt \\
&\leq 2MLB\omega(f, \frac{1}{\mu\sqrt{n}}) + 2M\omega(G(z_1), \frac{1}{\mu\sqrt{n}}) \leq 2B^2\omega(f, \frac{1}{\mu\sqrt{n}}) + 2ML\omega(z_1, \frac{1}{\mu\sqrt{n}}) \\
&\leq 2B^2\omega(z_0, \frac{1}{\mu\sqrt{n}}) + 2B\omega(z_1, \frac{1}{\mu\sqrt{n}}).
\end{aligned}$$

By induction, for $m \geq 3$, we have:

$$D(u_m(x,y), z_m(x,y)) \leq 2B^m\omega(z_0, \frac{1}{\mu\sqrt{n}}) + 2B^{m-1}\omega(z_1, \frac{1}{\mu\sqrt{n}}) + ... + 2B\omega(z_{m-1}, \frac{1}{\mu\sqrt{n}}). \tag{16}$$

Thus

$$D(u_m(x,y), z_m(x,y)) \leq (2B^m + 2B^{m-1} + ... + 2B)\Gamma = 2B\Gamma\frac{1-B^m}{1-B} \qquad , \tag{17}$$

for each $x, y \in [0,1]$. Since $B < 1$, according to $\frac{1-B^m}{1-B} \leq \frac{1}{1-B}$ for each $m \in N$, we get:

$$D^*(u_m, z_m) \leq \frac{2B\Gamma}{1-B} \qquad . \tag{18}$$

Considering inequalities (10,18), we have

$$D^*(u^*, z_m) \leq D^*(u^*, u_m) + D^*(u_m, z_m) \tag{19}$$

$$\leq \frac{M_0B^{m+1}}{L(1-B)} + \frac{2B\Gamma}{1-B} = \frac{B}{L(1-B)}(M_0B^m + 2L\Gamma), \tag{20}$$

□

which completes the proof of theorem (5).

Remark 2. Since$B < 1$, it is easy to show that

$$lim_{\mu,m,n\to\infty}D^*(u^*, z_m) = 0. \tag{21}$$

Thus, the proposed method is convergent.

4.2. Numerical stability analysis

For the presented iterative numerical method, we study the stability of the first iteration. So, in order to investigate the numerical stability of the iterative algorithm (11) with respect to a small perturbation in the starting approximation, we consider another starting approximation $v_0 \in C_F([0,1]\times[0,1])$ such that there exists $\varepsilon > 0$ for which $D(u_0(x,y), v_0(x,y)) < \varepsilon, \forall(x,y) \in [0,1]\times[0,1]$. The obtained sequence of successive approximation is

$$v_0(x,y) = f(x,y),$$

$$v_m(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot G(v_{m-1}(s,t))dsdt. \tag{22}$$

and using the same iterative method, the terms of the produced sequence are

$$\overline{v_m(x,y)} = f(x,y) \oplus \sum_{i=1}^{M_1}\sum_{p=0}^{n}\sum_{j=1}^{M_2}\sum_{q=0}^{n} G(\overline{v_{m-1}}(x,y))I_{ipjq}(x,y). \tag{23}$$

As in [7, 12], we give the following definition and derive the following numerical stability result.

Definition 11(See [7, 12]). We say that the algorithm of approximation applied to integral equation (1) is numerically stable with respect to the choice of the first iteration iff there exist two constants K_1, K_2 ,which are independent by $h = \frac{1}{\mu}$, and a function $\beta(h)$, such that $lim_{h\to 0}\beta(h) = 0$ and

$$D(z_m(x,y), \overline{v_m}(x,y)) < K_1\varepsilon + K_2\beta(h), \quad \forall m \in N; \quad x,y \in [0,1]. \tag{24}$$

Theorem 6. With assumptions of Theorem (4), and $B = ML(b-a)(d-c) < 1$, the proposed method (11) is numerically stable with respect to the choice of the first iteration.

Proof. Similarly as Theorem (5), it follows that

$$D^*(v_m, \overline{v_m}) \leq \frac{2B\overline{\Gamma}}{1-B},$$

where $\overline{\Gamma} = max_{(0\leq p\leq m-1)}\omega(\overline{v_p}, \frac{1}{\mu\sqrt{n}})$.

From Definition 1, we obtain

$$D(z_m(x,y), \overline{v_m}(x,y)) \leq D(z_m(x,y), u_m(x,y)) + D(u_m(x,y), v_m(x,y)) + D(v_m(x,y), \overline{v_m}(x,y))$$

$$\leq \frac{2B}{1-B}\Gamma + \frac{2B}{1-B}\overline{\Gamma} + D(u_m(x,y), v_m(x,y)).$$

Since $D(u_0(x,y), v_0(x,y)) < \varepsilon$ for all $(x,y) \in [0,1]\times[0,1]$, we have

$$\begin{aligned} &D(u_1(x,y), v_1(x,y)) \\ &\leq D\left((FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t)\odot G(u_0(s,t))dsdt, (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t)\odot G(v_0(s,t)))dsdt\right) \\ &< ML\varepsilon < \varepsilon. \end{aligned}$$

By induction, we have $D(u_m(x,y), v_m(x,y)) < B^m\varepsilon < \varepsilon$, thus

$$D(z_m(x,y), \overline{v_m}(x,y)) \leq \frac{2B}{(1-B)}(\Gamma + \overline{\Gamma}) + \varepsilon$$

$$= K_1\varepsilon + K_2\beta(h),$$

with $K_1 = 1$, $K_2 = \frac{2B}{1-B}$, and $\beta(h) = \Gamma + \overline{\Gamma}$. □

So, we proved the numerical stability of the proposed method.

5. Numerical examples

To illustrate the efficiency of the presented method in the previous Section, we give two examples. Also, we compare the numerical solution obtained by using the proposed method with the exact solution.

Example 1(See [11]). Consider the following nonlinear fuzzy Fredholm integral equations of the second kind:

$$u(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 K(x,y,s,t) \odot G(u(s,t))dsdt, \tag{25}$$

where

$$f(x,y) = (\underline{f}(x,y,\alpha), \overline{f}(x,y,\alpha)) \tag{26}$$

$$\underline{f}(x,y,\alpha) = (2+\alpha)xy - \frac{1}{16}(2+\alpha)^2 xy \tag{27}$$

$$\overline{f}(x,y,\alpha) = (4-\alpha)xy - \frac{1}{16}(-4+\alpha)^2 xy \tag{28}$$

$$K(x,y,s,t) = xyst \tag{29}$$

$$G(\beta) = \beta^2, \tag{30}$$

the exact solution is given by

$$u(x,y) = (\underline{u}(x,y,\alpha), \overline{u}(x,y,\alpha)) \tag{31}$$

$$\underline{u}(x,y,\alpha) = (2+\alpha)xy \tag{32}$$

$$\overline{u}(x,y,\alpha) = (4-\alpha)xy. \tag{33}$$

To obtain numerical solution, we use the proposed method in Section 4. So, to compare the numerical and exact solutions, one can see Table 1. Also for comparing the result of proposed method in Section 4 and the method of [11], see Table 1.

Table 1. The accuracy on the level sets for Example 1 in $x = 0.5$, $y = 0.5$

	proposed method n=5, m=10		proposed method n=7, m=20		method of [11] n=50, m=20	
r-level	$\lvert\underline{u}-\underline{z}_m\rvert$	$\lvert\overline{u}-\overline{z}_m\rvert$	$\lvert\underline{u}-\underline{z}_m\rvert$	$\lvert\overline{u}-\overline{z}_m\rvert$	$\lvert\underline{u}-\underline{z}_m\rvert$	$\lvert\overline{u}-\overline{z}_m\rvert$
0.0	0	1e-4	0	0	4e-10	7e-10
0.2	0	1e-4	0	0	3e-11	3e-11
0.4	0	5e-5	0	0	8e-10	4e-12
0.6	1e-6	2e-5	0	0	8e-11	9e-13
0.8	3e-6	1e-5	0	0	1e-10	5e-12
1.0	7e-6	7e-6	0	0	5e-11	8e-11

Example 2. Consider the following nonlinear fuzzy Fredholm integral equations of the second kind:

$$u(x,y) = f(x,y) \oplus (FR)\int_1^2 (FR)\int_1^2 K(x,y,s,t) \odot G(u(s,t))dsdt, \tag{34}$$

where

$$f(x,y) = (\underline{f}(x,y,\alpha), \overline{f}(x,y,\alpha)) \tag{35}$$

$$\underline{f}(x,y,\alpha) = \alpha xy - \frac{\alpha^2}{64}(x^2+y^2-2) \tag{36}$$

$$\overline{f}(x,y,\alpha) = (2-\alpha)xy - \frac{(2-\alpha)^2}{64}(x^2+y^2-2) \tag{37}$$

$$K(x,y,s,t) = (x^2+y^2-2)st/4 \tag{38}$$

$$G(\beta) = \beta^2, \tag{39}$$

the exact solution is given by

$$u(x,y) = (\underline{u}(x,y,\alpha), \overline{u}(x,y,\alpha)) \tag{40}$$

$$\underline{u}(x,y,\alpha) = \alpha xy \tag{41}$$

$$\overline{u}(x,y,\alpha) = (2-\alpha)xy. \tag{42}$$

To obtain numerical solution, we use the proposed method in Section 4. So, to compare the numerical and exact solutions, one can see Table 2.

Table 2. The accuracy on the level sets for Example 2 in $x = 0.5$, $y = 0.5$

	n=3, m=5		n=4, m=10		n=5, m=15	
r-level	$\lvert\underline{u}-\underline{z}_m\rvert$	$\lvert\overline{u}-\overline{z}_m\rvert$	$\lvert\underline{u}-\underline{z}_m\rvert$	$\lvert\overline{u}-\overline{z}_m\rvert$	$\lvert\underline{u}-\underline{z}_m\rvert$	$\lvert\overline{u}-\overline{z}_m\rvert$
0.0	0	7e-4	0	5e-6	0	0
0.2	0	3e-4	0	2e-6	0	0
0.4	0	5e4	0	1e-6	0	0
0.6	1e-6	6e-6	0	0	0	0
0.8	1e-6	2e-5	0	0	0	0
1.0	6e-6	6e-6	0	4e-5	0	0

6. Conclusions

In this paper we proposed a new approach for solving nonlinear 2DFFIE using HBB method. To this end, we applied the HBB for approximation of the unique solution of 2DNFFIE. The efficiency and simplicity of the proposed method illustrated by introducing some numerical examples with known exact solutions. The main advantage of this method is low cost of setting up the equations without using any projection method and any integration.

References

[1] G. A. Anastassiou, Fuzzy mathematics, Approximation theory, Springer-Verlag Berlin Heidelberg, (2010).

[2] G.A. Anastassiou, S.G. Gal, On a fuzzy trigonometric approximation theorem of Weirstrass-type, Jornal of Fuzzy Mathematics, Los Angeles, **9** (2001) 701-708.

[3] A.M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, Information Sciences, **178** (2008) 1279-1292.

[4] M. Baghmisheh, R. Ezzati, Numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using hybrid ofblock-pulse functions and Taylor series, Advances in Difference Equations, **5** (2015) 22–34.

[5] K. Balachandran, K. Kanagarajan, Existence of solutions of general nonlinear fuzzy Volterra-Fredholm integral equations, J. Appl. Math.Stoch. Anal, **3** (2005) 333–343.

[6] B. Bede, SG. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, Fuzzy Sets Syst, **145** (2004) 359–380.

[7] A.M. Bica, C. Popescu, Approximating the solution of nonlinear Hammerstein fuzzy integral equations, Fuzzy Sets Syst,**95** (2014) 1–17.

[8] A.M. Bica, C. Popescu, Numerical solutions of the nonlinear fuzzy Hammerstein-Volterra delay integral equations, Inf.Sci, **95** (2013) 236–255.

[9] R. Ezzati, S. Ziari, Numerical solution of nonlinear fuzzy Fredholm integral equations using iterative method, Appl.Math. Comput, **225** (2013) 33–42.

[10] R. Ezzati and S. Ziari, Numerical Solution of Two-Dimensional Fuzzy Fredholm Integral Equations of the Second Kind Using Fuzzy Bivariate Bernstein Polynomials, International Journal of Fuzzy Systems, **15**, (2013) 84–89.

[11] R. Ezzati, S.M. Sadatrasoul, Iterative method for numerical solution of two-dimensional nonlinear fuzzy integral equations, fuzzy set and system, **280** (2015) 91–106 .

[12] R. Ezzati, S.M. Sadatrasoul, Numerical Solution of Two-Dimensional Nonlinear Hammerstein Fuzzy Integral Equations Based on Optimal Fuzzy Quadrature Formula, Journal of Computational and Applied Mathematics, **292** (2016) 430–446.

[13] R. Ezzati, S.M. Sadatrasoul, Quadrature rules and iterative method for numerical solution of two-dimensional fuzzy integral equations, Astract and Applied Analsis. Special Issue (2014) 1–18.
[14] SG. Gal, Approximation theory in fuzzy setting, Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton (2000) 617-666.
[15] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems **18** (1986) 31-43.
[16] Z.H. Jung, W. Schanfelberger, Block-Pulse Functions and Their Applications in Control Systems, Springer, Berlin,(1992).
[17] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, **24** (1987) 301-317.
[18] F. Mirzaee, Numerical solution of Fredholm fuzzy integral equations of the second kind using hybrid of block-pulse-functions and Taylor series, Ain Shams Eng, **114**, (2014) 631–636.
[19] M. Friedman, M. Ma, A. Kandel, Numerical solutions of fuzzy differential and integral equations, Fuzzy Sets and Systems **106** (1999) 35-48.
[20] T.J. Rivlin, An introdction to the approoximation of function, Blaisdell Publishing , New York, (1969).
[21] P.V. Subrahmanyam, S.K. Sudarsanam, A note on fuzzy Volterraintegral equations, Fuzzy Sets Syst, **95** (1996) 237–240.
[22] G. Tachev, Pointwise approximation by Bernstein polynomials,Bull. Aust. Math. Soc, **85** (2012) 353–358.
[23] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions, Fuzzy Sets System, **120** (2001) 523–532.
[24] C. Wu, S. Song, H. Wang, On the basic solutions to the generalized fuzzy integral equation, Fuzzy Sets Syst, **95**(1998) 255–260.
[25] M. zarrini, S. torkman, The application of hybrid orthonormal Bernstein and block-pulse functions in finding numerical solution of Fredholm fuzzy integral equations, International journal of mathematical and Computational sciences, **8** (2014) 1074–1078.