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On Para-Kenmotsu Manifolds

Simeon Zamkovoy[a]

[a] *University of Sofia "St. Kl. Ohridski"*
Faculty of Mathematics and Informatics
Blvd. James Bourchier 5
1164 Sofia, Bulgaria

Abstract. In this paper we study para-Kenmotsu manifolds. We characterize this manifolds by tensor equations and study their properties. We are devoted to a study of $\eta-$Einstein manifolds. We show that a locally conformally flat para-Kenmotsu manifold is a space of constant negative sectional curvature -1 and we prove that if a para-Kenmotsu manifold is a space of constant $\varphi-$para-holomorphic sectional curvature H, then it is a space of constant sectional curvature and $H = -1$. Finally the object of the present paper is to study a 3-dimensional para-Kenmotsu manifold, satisfying certain curvature conditions. Among other, it is proved that any 3-dimensional para-Kenmotsu manifold with $\eta-$parallel Ricci tensor is of constant scalar curvature and any 3-dimensional para-Kenmotsu manifold satisfying cyclic Ricci tensor is a manifold of constant negative sectional curvature -1.

1. Introduction

In this paper we study a class of paracontact pseudo-Riemannian manifolds satisfying some special conditions. These manifolds are analogues to the Kenmotsu manifolds and they belong of the class $\mathbb{G}_6$ of the classification given in [8]. We characterize these manifolds by tensor equations and study their properties. From the definition by means of the tensor equations, it is easily verified that the structure is normal, but not quasi-para-Sasakian (and not para-Sasakian). We are devoted to a study of $\eta-$Einstein manifolds. We show that a locally conformally flat para-Kenmotsu manifold is a space of constant negative sectional curvature -1 and we prove that if a para-Kenmotsu manifold is a space of constant $\varphi-$para-holomorphic sectional curvature H, then it is a space of constant sectional curvature and $H = -1$. In the last section we study the 3-dimensional para-Kenmotsu manifolds. We prove that any 3-dimensional para-Kenmotsu manifold satisfying the condition $R(X,Y).Ric = 0$ is a manifold of constant negative sectional curvature, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of manifold (X, Y are tangent vectors). We study locally $\varphi-$symmetric para-Kenmotsu manifolds and obtain a necessary and sufficient condition 3-dimensional para-Kenmotsu manifold to be locally $\varphi-$symmetric. We obtain some interesting results about a 3-dimensional para-Kenmotsu manifolds with $\eta-$parallel Ricci tensor. We give a example for 3-dimensional para-Kenmotsu manifold with a scalar curvature equal to -6.

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Email address: zamkovoy@fmi.uni-sofia.bg (Simeon Zamkovoy)

2. Preliminaries

A (2n+1)-dimensional smooth manifold $M^{(2n+1)}$ has an *almost paracontact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying the following compatibility conditions

$$\begin{array}{ll} (i) & \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\ (ii) & \eta(\xi) = 1 \quad \varphi^2 = id - \eta \otimes \xi, \\ (iii) & \text{distribution } \mathbb{D} : p \in M \longrightarrow \mathbb{D}_p \subset T_pM : \\ & \mathbb{D}_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\} \text{ is called } \textit{paracontact} \\ & \textit{distribution} \text{ generated by } \eta. \end{array} \tag{1}$$

The tensor field φ induces an almost paracomplex structure [3] on each fibre on $\mathbb{D}$ and $(\mathbb{D}, \varphi, g_{|\mathbb{D}})$ is a $2n$-dimensional almost paracomplex distribution. Since g is non-degenerate metric on M and ξ is non-isotropic, the paracontact distribution $\mathbb{D}$ is non-degenerate.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism φ has rank $2n$, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$, (see [1, 2] for the almost contact case).

If a manifold $M^{(2n+1)}$ with (φ, ξ, η)-structure admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

then we say that $M^{(2n+1)}$ has an almost paracontact metric structure and g is called *compatible*. Any compatible metric g with a given almost paracontact structure is necessarily of signature $(n + 1, n)$.

Note that setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$.

Further, any almost paracontact structure admits a compatible metric.

Definition 2.1. *If $g(X, \varphi Y) = d\eta(X, Y)$ (where $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])$ then η is a paracontact form and the almost paracontact metric manifold $(M, \varphi, \eta, \xi, g)$ is said to be a* paracontact metric manifold.

A paracontact metric manifold for which ξ is Killing is called a *K − paracontact manifold*. A paracontact structure on $M^{(2n+1)}$ naturally gives rise to an almost paracomplex structure on the product $M^{(2n+1)} \times \mathfrak{R}$. If this almost paracomplex structure is integrable, then the given paracontact metric manifold is said to be a *para-Sasakian*. Equivalently, (see [7]) a paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{3}$$

for all vector fields X and Y (where ∇ is the Livi-Civita connection of g).

Definition 2.2. *If $(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$, then the manifold $(M, \varphi, \eta, \xi, g)$ is said to be a* quasi-para-Sasakian manifold.

From *Definition* 2.2 (see [5]) we have

$$\nabla_X \xi = \varphi X. \tag{4}$$

Definition 2.3. *If $(\nabla_X \varphi)Y = \eta(Y)\varphi X + g(X, \varphi Y)\xi$, then the manifold $(M, \varphi, \eta, \xi, g)$ is said to be a* para-Kenmotsu manifold.

From *Definition* 2.3 (see [5]) we have

$$\nabla_X \xi = -X + \eta(X)\xi. \tag{5}$$

Definition 2.4. *A $(2n + 1)$-dimensional almost paracontact metric manifold is called normal if $N(x, y) - 2d\eta(x, y)\xi = 0$, where $N(x, y) = \varphi^2[x, y] + [\varphi x, \varphi y] - \varphi[\varphi x, y] - \varphi[x, \varphi y]$ is the Nijenhuis torsion tensor of φ (see [7]).*

In [8], it is proved that $(M, \varphi, \eta, \xi, g)$ is normal, since ξ is not a Killing vector field and the manifold is not quasi-para-Sasakian. Thus we have

Proposition 2.5. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold. Then $(M, \varphi, \eta, \xi, g)$ is normal but neither quasi-para-Sasakian nor para-Sasakian.*

Denoting by £ the Lie differentiation of g, we see

Proposition 2.6. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold. Then we have*

$$(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

$$(£_\xi g)(X, Y) = -2(g(X, Y) - \eta(X)\eta(Y)), \tag{7}$$

$$£_\xi \varphi = 0, \tag{8}$$

$$£_\xi \eta = 0, \tag{9}$$

where $X, Y \in T_pM$.

Since the proof of *Proposition* 2.6 follows by routine calculation, we shall omit it.
Denoting by R the curvature tensor of ∇, we have the following

Definition 2.7. *An almost paracontact structure (φ, ξ, η, g) is said to be* locally symmetric *if $(\nabla_W R)(X, Y, Z) = 0$, for all vector fields $W, X, Y, Z \in T_pM$.*

Definition 2.8. *An almost paracontact structure (φ, ξ, η, g) is said to be* locally $\varphi-$ symmetric *if $\varphi^2(\nabla_W R)(X, Y, Z) = 0$, for all vector fields W, X, Y, Z orthogonal to ξ.*

Finally, the sectional curvature $K(\xi, X) = \epsilon_X R(X, \xi, \xi, X)$, where $|X| = \epsilon_X = \pm 1$, of a plane section spanned by ξ and the vector X orthogonal to ξ is called *ξ-sectional curvature*, whereas the sectional curvature $K(X, \varphi X) = -R(X, \varphi X, \varphi X, X)$, where $|X| = -|\varphi X| = \pm 1$, of a plane section spanned by vectors X and φX orthogonal to ξ is called a *φ-para-holomorphic sectional curvature*.

3. Some properties of para-Kenmotsu manifolds

The following result is well-known from the theory of para-Sasakian manifolds: $K(X, \xi) = -1$ and if a para-Sasakian manifold is locally symmetric, then it is of constant negative sectional curvature -1([6]). On para-Kenmotsu manifolds we get

Proposition 3.1. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold. Then we have*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{10}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{11}$$

$$Ric(X, \xi) = -2n\eta(X), \tag{12}$$

$$K(X, \xi) = -1, \tag{13}$$

$$(\nabla_Z R)(X, Y, \xi) = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y, \tag{14}$$

where Ric is the Ricci tensor and $X, Y, Z \in T_pM$.

Proof. The equation (10) follows directly from (5), (6) and the definition of the curvature R. The equations (11), (12) and (13) are a consequence of (10). By virtue of (5) (6) and (10) we get (14):

$$(\nabla_Z R)(X, Y, \xi) = \nabla_Z(R(X, Y)\xi) - R(\nabla_Z X, Y)\xi - R(X, \nabla_Z Y)\xi - R(X, Y)\nabla_Z \xi =$$

$$= R(X, Y)Z + g(Y, Z)X - g(X, Z)Y.$$

□

Theorem 3.2. *If $(M, \varphi, \eta, \xi, g)$ is locally symmetric, then it is of constant negative sectional curvature -1.*

Proof. *Theorem* 3.2 follows from (14). □

We can generalize *Theorem* 3.2 slightly as follows:

Proposition 3.3. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold. If M is a semi-symmetric space, i.e., $R(X, Y)R = 0$, for any $X, Y \in T_pM$, then it is of constant negative sectional curvature -1.*

Proof. Let $X, Y \in \mathbb{D}$ and $g(X, Y) = 0$. Then, using (10) and (11) above, we obtain

$$(R(X,\xi)R)(X,Y)Y = R(X,\xi)R(X,Y)Y - R(R(X,\xi)X,Y)Y - R(X,R(X,\xi)Y)Y-$$
$$-R(X,Y)R(X,\xi)Y = R(X,Y,Y,X)\xi - R(X,Y,Y,\xi)X - g(X,X)R(\xi,Y)Y =$$
$$= (R(X,Y,Y,X) + g(X,X)g(Y,Y))\xi.$$

From the identity $R(X, Y)R = 0$, we get $R(X, Y, Y, X) = -g(X, X)g(Y, Y)$, which implies that $(M, \varphi, \eta, \xi, g)$ is of constant $\varphi-$para-holomorphic sectional curvature -1, and hence it is of constant sectional curvature -1. □

4. $\eta-$Einstein manifolds

An almost para-contact pseudo-Riemannian manifold is called $\eta-$*Einstein*, if the Ricci tensor *Ric* satisfies $Ric = a.d + b.\eta \otimes \eta$, where a, b are smooth scalar functions on M. If a para-Sasakian manifold is $\eta-$Einstein and $n > 1$, then a and b are constant (see [7]).

Proposition 4.1. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold. If M is an $\eta-$Einstein manifold, we have*

$$a + b = -2n, \tag{15}$$

$$Z(b) - 2b\eta(Z) = 0, \quad n > 1 \tag{16}$$

for any $Z \in T_pM$.

Proof. The equation (15) follows from $Ric(X, \xi) = -2n\eta(X)$ which is derived from (10). As M is an $\eta-$Einstein manifold, the scalar curvature *scal* is equal to $2n(a-1)$. We define the Ricci operator Q as follows: $g(QX, Y) = Ric(X, Y)$. By identity $Y(scal) = 2nY(a)$ and the trace of the map $[X \to (\nabla_X Q)Y]$, we have

$$Z(a) + \xi(b)\eta(Z) - 2nb\eta(Z) = nZ(a).$$

Setting $Z = \xi$, we get $\xi(b) = 2b$. Therefore we have $Z(b) - 2b\eta(Z) = 0$. □

Corollary 4.2. *If $(M, \varphi, \eta, \xi, g)$ is a para-Kenmotsu manifold and b =constant (or a =constant), then M is an Einstein one.*

5. Curvature tensor

At first we shall prove the following

Proposition 5.1. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold. Then we have the following identities*

$$R(X,Y)\varphi Z - \varphi R(X,Y)Z = g(Y,Z)\varphi X - g(X,Z)\varphi Y - g(Y,\varphi Z)X+, \tag{17}$$
$$+g(X,\varphi Z)Y,$$

$$R(\varphi X,\varphi Y)Z = -R(X,Y)Z - g(Y,Z)X + g(X,Z)Y+ \tag{18}$$
$$+g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y,$$

where $X, Y, Z \in T_pM$.

Proof. The equation (17) follows from the Ricci's identity:

$$\nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi = R(X,Y)\varphi Z - \varphi R(X,Y)Z.$$

We verify (18): By (17), we have

$$R(X,Y,\varphi Z,\varphi W) - g(\varphi R(X,Y)Z,\varphi W) =$$

$$= g(Y,Z)g(\varphi X,\varphi W) - g(X,Z)g(\varphi Y,\varphi W) -$$
$$-g(Y,\varphi Z)g(X,\varphi W) + g(X,\varphi Z)g(Y,\varphi W).$$

Using $\eta(R(X,Y)Z) = -\eta(X)g(Y,Z) + \eta(Y)g(X,Z)$, the above formula takes the form

$$R(\varphi Z,\varphi W,X,Y) = -R(Z,W,X,Y) - g(Y,Z)g(X,W) + g(X,Z)g(Y,W) -$$

$$-g(Y,\varphi Z)g(X,\varphi W) + g(X,\varphi Z)g(Y,\varphi W).$$

□

As an application of *Proposition* 5.1, we shall prove the following proposition.

Proposition 5.2. *Let (M,φ,η,ξ,g) be a para-Kenmotsu manifold of dimension greater than* 3*. If M is locally conformally flat, then M is a space of constant negative sectional curvature* −1*.*

Proof. Since M is conformally flat, the curvature tensor of M is written as

$$R(X,Y)Z = \frac{1}{2n-1}(Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY) - \tag{19}$$

$$+\frac{scal}{2n(2n-1)}(g(X,Z)Y - g(Y,Z)X).$$

We calculate $R(\xi,Y)\xi$ using the previous formula. Using (10) and

$$Ric(X,\xi) = -2n\eta(X),$$

we get

$$2nRic(Y,Z) = (scal + 2n)g(Y,Z) - (scal + 4n^2 + 2n)\eta(Y)\eta(Z). \tag{20}$$

By virtue of (17), (19) and (20), we have

$$(scal + 4n^2 + 2n)(g(Y,\varphi Z)X - g(X,\varphi Z)Y + g(X,Z)\varphi Y - g(Y,Z)\varphi X + \tag{21}$$

$$+g(X,\varphi Z)\eta(Y)\xi - g(Y,\varphi Z)\eta(X)\xi + \eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y) = 0.$$

Let $(e_1,...,e_n,\varphi e_1,...,\varphi e_n,\xi)$ be an orthonormal basis of T_pM. Setting $X = e_1, Y = e_2$ and $Z = \varphi e_2$ in (21), we see $scal = -2n(2n+1)$. Thus we have $Ric = -2ng$. *Proposition* 5.1 follows from (19). □

In a para-Sasakian manifold with constant $\varphi-$para-holomorphic sectional curvature, say H, the curvature tensor has a special feature (see [6]): The necessary and sufficient condition for a para-Sasakian manifold to have constant $\varphi-$para-holomorphic sectional curvature H is

$$4R(X,Y)Z = (H-3)(g(Y,Z)X - g(X,Z)Y) + (H+1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X +$$

$$+\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y + 2g(\varphi X,Y)\varphi Z).$$

In our case we have

Proposition 5.3. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold of dimension greater than 3. The necessary and sufficient condition for M to have constant $\varphi-$para-holomorphic sectional curvature H is*

$$4R(X,Y)Z = (H-3)(g(Y,Z)X - g(X,Z)Y) + (H+1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y + 2g(\varphi X, Y)\varphi Z), \tag{22}$$

where $X, Y, Z \in T_pM$.

Proof. For any vector fields $X, Y \in \mathbb{D}$, we have

$$R(X,\varphi X, X, \varphi X) = Hg^2(X,X) \tag{23}$$

By identity (17) we get

$$R(X,\varphi Y, X, \varphi Y) = R(X,\varphi Y, Y, \varphi X) - g^2(X,\varphi Y) + g^2(X,Y) - g(X,X)g(Y,Y), \tag{24}$$

$$R(X,\varphi X, Y, \varphi X) = R(X,\varphi X, X, \varphi Y). \tag{25}$$

Substituting $X + Y$ in (17) and using the Bianchi identity, we obtain

$$2R(X,\varphi X, X, \varphi Y) + 2R(Y,\varphi Y, Y, \varphi X) + 3R(X,\varphi Y, Y, \varphi X) - R(X,Y,X,Y) = H(2g^2(X,Y) + g(X,X)g(Y,Y) + 2g(X,Y)g(X,X) + 2g(X,Y)g(Y,Y)). \tag{26}$$

Replacing Y by $-Y$ in (26) and summing it to (26) we have

$$3R(X,\varphi Y, Y, \varphi X) - R(X,Y,X,Y) = H(2g^2(X,Y) + g(X,X)g(Y,Y)). \tag{27}$$

Replacing Y by φY in (27) and from identities (28), (24) and (27), we get

$$4R(X,Y,X,Y) = (H-3)(g^2(X,Y) - g(X,X)g(Y,Y)) + (H+1)g^2(X,\varphi Y). \tag{28}$$

Let $X, Y, Z, W \in \mathbb{D}$, we calculate $R(X+Z, Y+W, X+Z, Y+W)$ and using (28) we see

$$4R(X,Y,Z,W) + 4R(X,W,Z,Y) = (H-3)(g(X,Y)g(Z,W) + g(X,W)g(Y,Z) - 2g(X,Z)g(Y,W)) + 3(H+1)(g(X,\varphi Y)g(Z,\varphi W) + g(X,\varphi W)g(Z,\varphi Y)). \tag{29}$$

and we have

$$-4R(X,Z,Y,W) - 4R(X,W,Y,Z) = -(H-3)(g(X,Z)g(Y,W) + g(X,W)g(Y,Z) - 2g(X,Y)g(Z,W)) - 3(H+1)(g(X,\varphi Z)g(Y,\varphi W) + g(X,\varphi W)g(Y,\varphi Z)). \tag{30}$$

Adding (29) to (30) we get by virtue of the Bianchi identity

$$4R(X,Y,Z,W) = (H-3)(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) + (H+1)(g(X,\varphi W)g(\varphi Y,Z) - g(X,\varphi Z)g(\varphi Y,W) + 2g(X,\varphi Y)g(Z,\varphi W)). \tag{31}$$

For any vector fields $X, Y, Z, W \in T_pM$ we have $\varphi X, \varphi Y, \varphi Z, \varphi W \in \mathbb{D}$, and using (31), (10), (17) and (18), we get (22).

Contracting the equation (22), we obtain

$$Ric = \frac{1}{2}(n(H-3) + H + 1)g - \frac{1}{2}(n+1)(H+1)\eta \otimes \eta. \tag{32}$$

Contracting the last equation, we obtain

$$scal = \frac{1}{2}(n(H-3) + H + 1)(2n+1) - \frac{1}{2}(n+1)(H+1). \tag{33}$$

On the other hand, from the second Bianchi identity and equations (32) and (33), we get

$$(n-1)X(H) + \xi(H)\eta(X) = 0. \tag{34}$$

Setting $X = \xi$ in (34), we get $\xi(H) = 0$ and hence (34) implies

$$X(H) = 0 \quad (n > 1).$$

Thus, the $\varphi-$para-holomorphic sectional curvature H is constant. □

Recall that, if a para-Sasakian manifold has a constant $\varphi-$para-holomorphic section curvature, then it is η-Einstein. Unlike this, we have the following theorem in our case:

Theorem 5.4. *Let $(M, \varphi, \eta, \xi, g)$ be a para-Kenmotsu manifold of dimension greater than 3. If M is a space of constant $\varphi-$para-holomorphic sectional curvature H, then M is a space of constant sectional curvature and $H = -1$.*

Proof. By virtue of *Proposition* 5.3, M is an $\eta-$Einstein manifold and

$$Ric = \frac{1}{2}(n(H-3) + H + 1)g - \frac{1}{2}(n+1)(H+1)\eta \otimes \eta.$$

Since the coefficients of Ric are constant on M, we have $H = -1$ by *Corollary* 4.2. □

6. 3-dimensional para-Kenmotsu manifolds

In a 3-dimensional pseudo-Riemannian manifold, we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \tag{35}$$
$$-\frac{scal}{2}(g(Y,Z)X - g(X,Z)Y).$$

Setting $Z = \xi$ in (31) and using (10) and (12), we have

$$\eta(Y)QX - \eta(X)QY = (\frac{scal}{2} + 1)(\eta(Y)X - \eta(X)Y). \tag{36}$$

Setting $Y = \xi$ in (36) and then using (12) (for n=1), we get

$$QX = \frac{1}{2}[(scal + 2)X - (scal + 6)\eta(X)\xi]$$

i.e.,

$$Ric(Y,Z) = \frac{(scal + 2)}{2}g(Y,Z) - \frac{(scal + 6)}{2}\eta(Y)\eta(Z). \tag{37}$$

Lemma 6.1. *A 3-dimensional para-Kenmotsu manifold is a manifold of constant negative sectional curvature if and only if the scalar curvature $scal = -6$.*

Proof. Using (37) in (35), we get

$$R(X,Y)Z = \frac{(scal + 4)}{2}(g(Y,Z)X - g(X,Z)Y) - \tag{38}$$
$$-\frac{(scal + 6)}{2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y)$$

and now the Lemma is obvious. □

Let us consider a 3-dimensional para-Kenmotsu manifold which satisfies the condition

$$R(X,Y).Ric = 0, \tag{39}$$

for any vector fields $X, Y \in T_pM$.

Theorem 6.2. *A 3-dimensional para-Kenmotsu manifold $(M, \varphi, \eta, \xi, g)$ satisfying the condition $R(X,Y).Ric = 0$ is a manifold of constant negative sectional curvature -1.*

Proof. From (39), we have

$$Ric(R(X,Y)U,V) + Ric(U,R(X,Y)V) = 0. \tag{40}$$

Setting $X = \xi$ and using (11)

$$\eta(U)Ric(Y,V) - g(Y,U)Ric(\xi,V) + \eta(V)Ric(U,\xi) - g(Y,V)Ric(\xi,U) = 0. \tag{41}$$

Using (12) in (41), we have

$$\eta(U)Ric(Y,V) + 2g(Y,U)\eta(V) + \eta(V)Ric(Y,U) + 2g(Y,V)\eta(U) = 0. \tag{42}$$

Taking the trace in (42), we get

$$Ric(\xi,V) + 8\eta(V) + scal\eta(V) = 0. \tag{43}$$

Using (12) in (43), we obtain

$$(scal + 6)\eta(V) = 0.$$

This gives $scal = -6$ (since $\eta(V) \neq 0$), which implies, by *Lemma* 6.1, that the manifold is of constant negative sectional curvature -1. □

Theorem 6.3. *A 3-dimensional para-Kenmotsu manifold $(M, \varphi, \eta, \xi, g)$ is locally $\varphi-$symmetric if and only if the scalar curvature scal is constant.*

Proof. Differentiating (38) covariantly with respect to W we get

$$\begin{aligned}(\nabla_W R)(X,Y,Z) = {} & \frac{W(scal)}{2}(g(Y,Z)X - g(X,Z)Y) - \\ & -\frac{W(scal)}{2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) - \\ & -\frac{(scal+6)}{2}(g(Y,Z)(\nabla_W\eta)X\xi - g(X,Z)(\nabla_W\eta)Y\xi + g(Y,Z)\eta(X)\nabla_W\xi - \\ & -g(X,Z)\eta(Y)\nabla_W\xi + (\nabla_W\eta)Y\eta(Z)X + \eta(Y)(\nabla_W\eta)ZX - \\ & -(\nabla_W\eta)X\eta(Z)Y - \eta(X)(\nabla_W\eta)ZY).\end{aligned} \tag{44}$$

Taking X, Y, Z, W orthogonal to ξ and using (5) and (6), we get from the above

$$\begin{aligned}(\nabla_W R)(X,Y,Z) = {} & \frac{W(scal)}{2}(g(Y,Z)X - g(X,Z)Y) - \\ & -\frac{(scal+6)}{2}(-g(Y,Z)g(X,W)\xi + g(X,Z)g(Y,W)\xi).\end{aligned} \tag{45}$$

From (45) it follows that

$$\varphi^2(\nabla_W R)(X,Y,Z) = \frac{W(scal)}{2}(g(Y,Z)X - g(X,Z)Y). \tag{46}$$

□

Definition 6.4. *The Ricci tensor Ric of a para-Kenmotsu manifold M is called* $\eta-$parallel *if it satisfies* $(\nabla_X Ric)(\varphi Y, \varphi Z) = 0$ *for all vector fields X, Y and Z.*

The notation for Ricci-$\eta-$parallelity for Sasakian manifolds was introduce in [4].

Proposition 6.5. *If a 3-dimensional para-Kenmotsu manifold* $(M, \varphi, \eta, \xi, g)$ *has* $\eta-$ *parallel Ricci tensor, then the scalar curvature scal is constant and* $(M, \varphi, \eta, \xi, g)$ *is locally* $\varphi-$*symmetric.*

Proof. From (37), we get, by virtue of (2) and $\eta \circ \varphi = 0$,

$$Ric(\varphi X, \varphi Y) = -\frac{(scal+2)}{2}(g(X,Y) - \eta(X)\eta(Y)). \tag{47}$$

Differentiating (47) covariantly along Z, we get

$$(\nabla_Z Ric)(\varphi X, \varphi Y) = -\frac{Z(scal)}{2}(g(X,Y) - \eta(X)\eta(Y)) + \tag{48}$$

$$+\frac{(scal+2)}{2}(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y).$$

By using $(\nabla_X Ric)(\varphi Y, \varphi Z) = 0$ and (48), we get

$$-Z(scal)(g(X,Y) - \eta(X)\eta(Y)) + \tag{49}$$

$$+(scal+2)(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y) = 0.$$

Taking the trace in (49), we get $Z(scal) = 0$, for all Z. From *Theorem* 6.3 we have that $(M, \varphi, \eta, \xi, g)$ is locally $\varphi-$symmetric. □

Let us suppose that a 3-dimensional para-Kenmotsu manifold satisfies the cyclic Ricci tensor. Then we have

$$(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y) = 0. \tag{50}$$

We have the following

Theorem 6.6. *If a 3-dimensional para-Kenmotsu manifold* $(M, \varphi, \eta, \xi, g)$ *satisfies the condition* (50), *then the manifold is a manifold of constant negative sectional curvature* -1.

Proof. Taking the trace in (50), we obtain

$$X(scal) = 0, \tag{51}$$

for any vector field X. From (37), we have

$$(\nabla_Z Ric)(X,Y) = \frac{Z(scal)}{2}(g(X,Y) - \eta(X)\eta(Y)) - \tag{52}$$

$$-\frac{(scal+6)}{2}(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y).$$

Now using (51) and (52), we have

$$(\nabla_Z Ric)(X,Y) = -\frac{(scal+6)}{2}(\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y) \tag{53}$$

By virtue of (53), we get from (50) that

$$(scal+6)(\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z + \eta(Y)(\nabla_Z \eta)X + \tag{54}$$

$$+\eta(X)(\nabla_Z\eta)Y + \eta(Z)(\nabla_Y\eta)X + \eta(X)(\nabla_Y\eta)Z) = 0.$$

Taking the trace in (54), we obtain

$$(scal + 6)\eta(X) = 0, \tag{55}$$

which implies that $scal = -6$. The rest of the proof follows immediately from this (again see the proof of *Lemma* 6.1). □

Unlike the case when the dimension is greater than 3, when the dimension is equal to 3 the right-hand parenthesis in (21) is trivially equal to 0 and thus nothing follows (from (21)) about the scalar curvature. We shall give an example of a 3-dimensional para-Kenmotsu manifold, which has scalar curvature equal to -6.

Example 6.7. *Let L be a 3-dimensional real connected Lie group and* $\mathfrak{g}$ *be its Lie algebra with a basis* $\{E_1, E_2, E_3\}$ *of left invariant vector fields (see [8]), by the following commutators:*

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1 + \beta E_2, \quad [E_2, E_3] = \beta E_1 + E_2, \tag{56}$$

where $\beta \neq 0$.

We define an almost paracontact structure (φ, ξ, η) *and a pseudo-Riemannian metric* g *in the following way:*

$$\begin{aligned} &\varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0 \\ &\xi = E_3, \quad \eta(E_3) = 1, \quad \eta(E_1) = \eta(E_2) = 0, \\ &g(E_1, E_1) = g(E_3, E_3) = -g(E_2, E_2) = 1, \\ &\quad g(E_i, E_j) = 0, \quad i \neq j \in \{1, 2, 3\}. \end{aligned}$$

Then $(L, \varphi, \xi, \eta, g)$ *is a 3-dimensional almost paracontact metric manifold. Since the metric* g *is left invariant the Koszul equality becomes*

$$\begin{aligned} &\nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = E_1, \\ &\nabla_{E_2}E_1 = 0, \quad \nabla_{E_2}E_2 = E_3, \quad \nabla_{E_2}E_3 = E_2, \\ &\nabla_{E_3}E_1 = -\beta E_2, \quad \nabla_{E_3}E_2 = -\beta E_1, \quad \nabla_{E_3}E_3 = 0. \end{aligned}$$

It is not hard to see that

$Ric(X, Y) = -2g(X, Y)$ *and* $R(X, Y)Z = -(g(Y, Z)X - g(X, Z)Y)$.

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