The Sum of the Interior Angles in Geodesic and Translation Triangles of $\widetilde{SL_2(\mathbb{R})}$ Geometry

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Abstract. We study the interior angle sums of translation and geodesic triangles in the universal cover of real 2×2 matrices with unit determinant, as, a Thurston geometry denoted by P of $\widetilde{SL_2(\mathbb{R})}$ geometry. We prove that the angle sum $\sum_{i=1}^{3}(\alpha_i) \geq \pi$ for translation triangles and for geodesic triangles the angle sum can be larger, equal or less than π.

1. Introduction

In this paper we are interested in *geodesic and translation* triangles in $\widetilde{SL_2(\mathbb{R})}$ space that is one of the eight Thurston geometries [10, 18]. This twisted space can be derived from the 3-dimensional Lie group of all 2×2 real matrices with unit determinant. The space of left invariant Riemann metrics on the group $\widetilde{SL_2(\mathbb{R})}$ is 3-dimensional [2]. In Section 2 we describe the projective model of $\widetilde{SL_2(\mathbb{R})}$ and we shall use its standard Riemann metric obtained by pull back transform to the infinitesimal arc-length-square at the origin as usual. This coincides with the global projective metric belonging to the quadratic form (2). We also describe the isometry group of $\widetilde{SL_2(\mathbb{R})}$. In Section 3 we give an overview not only about geodesic, but also about translation curves summarizing previous results, which are related to the projective model of $\widetilde{SL_2(\mathbb{R})}$ suggested and introduced in [4]. Our main results will be presented in Section 4, namely the possible sum of the interior angles in a translation triangle must be greater or equal than π. However, in geodesic triangles this sum is less, greater or equal to π.

2. The projective model for $\widetilde{SL_2(\mathbb{R})}$

Real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with the unit determinant $ad - bc = 1$ constitute a Lie transformation group by the standard product operation, taken to act on row matrices as point coordinates

$$(z^0, z^1)\begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (w^0, w^1) \tag{1}$$

with

$$w = \frac{w^1}{w^0} = \frac{b + (z^1/z^0)\,a}{d + (z^1/z^0)\,c} = \frac{b + za}{d + zc},$$

$z = z^1/z^0$, on the complex projective line $\mathbb{C}^\infty$. This group is a 3-dimensional manifold, because of its 3 independent real coordinates and with its usual neighborhood topology [10, 18]. In order to model the above structure in the projective sphere $\mathcal{PS}^3$ and in the projective space $\mathcal{P}^3$ (see [4]), we introduce the new projective coordinates $(x^0 : x^1 : x^2 : x^3)$ where

$$a := x^0 + x^3, \quad b := x^1 + x^2, \quad c := -x^1 + x^2, \quad d := x^0 - x^3,$$

with positive, then the non-zero multiplicative equivalence as a projective freedom in $\mathcal{PS}^3$ and in $\mathcal{P}^3$, respectively. Then it follows that

$$0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 \tag{2}$$

describes the interior of the one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex, with the origin $E_0(1:0:0:0)$ and the ideal points of the axes $E_1^\infty(0:1:0:0)$, $E_2^\infty(0:0:1:0)$, $E_3^\infty(0:0:0:1)$. We consider the collineation group $\mathbf{G}_*$ that acts on the projective sphere $\mathcal{PS}^3$ and preserves the polarity, i. e. a scalar product of signature $(--++)$. This group leaves the one-sheeted hyperboloid solid $\mathcal{H}$ invariant. We have to choose an appropriate subgroup $\mathbf{G}$ of $\mathbf{G}_*$ as isometry group, then the universal covering group and space $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ will be the hyperboloid model of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ (see Figure 1 and [4]). Consider isometries
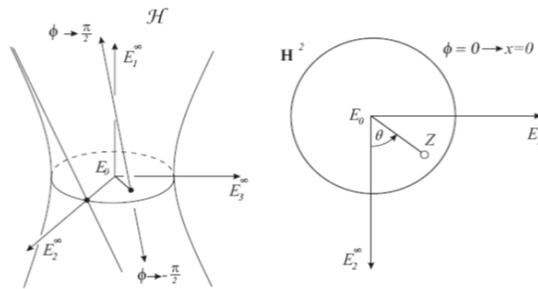

Figure 1: The hyperboloid model.

given by matrices

$$S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{3}$$

where $\phi \in [0; 2\pi)$. They constitute a one parameter group which we denote by $\mathbf{S}(\phi)$. The elements of $\mathbf{S}(\phi)$ are the so-called *fibre translations*. We obtain a unique fibre line to each $X(x^0 : x^1 : x^2 : x^3) \in \widetilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X. The coordinates of points, lying on the fibre line through X, can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$(x^0 : x^1 : x^2 : x^3) \xrightarrow{\mathbf{S}(\phi)} (x^0 \cos\phi - x^1 \sin\phi : x^0 \sin\phi + x^1 \cos\phi : \tag{4}$$
$$x^2 \cos\phi + x^3 \sin\phi : -x^2 \sin\phi + x^3 \cos\phi).$$

In (3) and (4) we can see the 2π-periodicity by ϕ. It admits us to introduce the extension to $\phi \in \mathbb{R}$, as real parameter, to obtain the universal covers $\widetilde{\mathcal{H}}$ and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, respectively, through the projective sphere $\mathcal{PS}^3$.

The elements of the isometry group of $SL_2(\mathbb{R})$ (and so by the above extension the isometries of $\widetilde{SL_2(\mathbb{R})}$) can be described by the matrix (a_i^j) (see [4, 5])

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix}, \tag{5}$$

where

$$\begin{cases} -(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 & = & -1, \\ -(a_2^0)^2 - (a_2^1)^2 + (a_2^2)^2 + (a_2^3)^2 & = & 1, \\ -a_0^0 a_2^0 - a_0^1 a_2^1 + a_0^2 a_2^2 + a_0^3 a_2^3 & = & 0, \\ -a_0^0 a_2^1 + a_0^1 a_2^0 - a_0^2 a_2^3 + a_0^3 a_2^2 & = & 0, \end{cases}$$

and we allow positive proportionality, of course, as projective freedom.

We define the *translation group* $\mathbf{G}_T$, as a subgroup of the isometry group of $SL_2(\mathbb{R})$, those isometries acting transitively on the points of $\mathcal{H}$ and by the above extension on the points of $\widetilde{\mathcal{H}}$. $\mathbf{G}_T$ maps the origin $E_0(1 : 0 : 0 : 0)$ onto $X(x^0 : x^1 : x^2 : x^3) \in \mathcal{H}$. These isometries and their inverses (up to a positive determinant factor) can be given by

$$\mathbf{T} = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}. \tag{6}$$

Horizontal intersection of the hyperboloid solid $\mathcal{H}$ with the plane $E_0 E_2^\infty E_3^\infty$ provides the *base plane* of the model $\widetilde{\mathcal{H}} = \widetilde{SL_2(\mathbb{R})}$.

We generally introduce a so-called hyperboloid parametrization by [4] as follows

$$\begin{cases} x^0 = \cosh r \cos \phi, \\ x^1 = \cosh r \sin \phi, \\ x^2 = \sinh r \cos (\theta - \phi), \\ x^3 = \sinh r \sin (\theta - \phi), \end{cases} \tag{7}$$

where (r, θ) are the polar coordinates of the $\mathbf{H}^2$ base plane, and ϕ is the fibre coordinate. We note that

$$-x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

The inhomogeneous coordinates, which will play an important role in the later $\mathbb{E}^3$-visualization in $\widetilde{SL_2(\mathbb{R})}$, are given by

$$\begin{cases} x := \dfrac{x^1}{x^0} = \tan \phi, \\ y := \dfrac{x^2}{x^0} = \tanh r \dfrac{\cos (\theta - \phi)}{\cos \phi}, \\ z := \dfrac{x^3}{x^0} = \tanh r \dfrac{\sin (\theta - \phi)}{\cos \phi}. \end{cases} \tag{8}$$

3. Geodesic and translation curves

The infinitesimal arc-length-square of $\widetilde{SL_2(\mathbb{R})}$ can be derived by the standard pull back method. By $\mathbf{T}^{-1}$-action presented by (6) on differentials $(dx^0 : dx^1 : dx^2 : dx^3)$, we obtain the infinitesimal arc-length-square

at any point of $\widetilde{SL_2(\mathbb{R})}$ in coordinates (r, θ, ϕ):

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + \cosh^2 r \sinh^2 r (\mathrm{d}\theta)^2 + \left[(\mathrm{d}\phi) + \sinh^2 r (\mathrm{d}\theta)\right]^2.$$

Hence we get the symmetric metric tensor field g_{ij} on $\widetilde{SL_2(\mathbb{R})}$ by components:

$$g_{ij}^* := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\sinh^2 r + \cosh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \tag{9}$$

Remark 3.1. *Similarly to the above computations we obtain the metric tensor for coordinates (x, y, z):*

$$g_{ij} := \begin{pmatrix} \frac{1+y^2+z^2}{(-1-x^2+y^2+z^2)^2} & \frac{-xy-2z}{(-1-x^2+y^2+z^2)^2} & \frac{-xz+2y}{(-1-x^2+y^2+z^2)^2} \\ \frac{-xy-2z}{(-1-x^2+y^2+z^2)^2} & \frac{1+x^2+z^2}{(-1-x^2+y^2+z^2)^2} & \frac{yz}{(-1-x^2+y^2+z^2)^2} \\ \frac{-xz+2y}{(-1-x^2+y^2+z^2)^2} & \frac{yz}{(-1-x^2+y^2+z^2)^2} & \frac{1+x^2+y^2}{(-1-x^2+y^2+z^2)^2} \end{pmatrix}. \tag{10}$$

3.1. Geodesic curves

The geodesic curves of $\widetilde{SL_2(\mathbb{R})}$ are generally defined as having locally minimal arc length between any two of their (close enough) points.

By (10) the second order differential equation system of the $\widetilde{SL_2(\mathbb{R})}$ geodesic curve is the following:

$$\begin{cases} \ddot{r} = \sinh(2r)\dot{\theta}\dot{\phi} + \dfrac{1}{2}\left[\sinh(4r) - \sinh(2r)\right]\dot{\theta}\dot{\theta}, \\ \ddot{\theta} = \dfrac{2\dot{r}}{\sinh(2r)}\left[(3\cosh(2r) - 1)\dot{\theta} + 2\dot{\phi}\right], \\ \ddot{\phi} = 2\dot{r}\tanh(r)\left[2\sinh^2(r)\dot{\theta} + \dot{\phi}\right]. \end{cases} \tag{11}$$

We can assume, by the homogeneity, that the starting point of a geodesic curve is the origin $(1 : 0 : 0 : 0)$. Moreover,

$$\begin{cases} r(0) = 0, \\ \theta(0) = 0, \\ \phi(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \dot{r}(0) = \cos(\alpha), \\ \dot{\theta}(0) = -\sin(\alpha), \\ \dot{\phi}(0) = \sin(\alpha) \end{cases}$$

are the initial values in Table 1 for the solution of (11), and so the unit velocity will be achieved (see details in [1]). The solutions are parametrized by the arc-length s and the angle α from the initial condition: $(r(s, \alpha), \theta(s, \alpha), \phi(s, \alpha))$.

The parametrization of a geodesic curve in the hyperboloid model with the geographical sphere coordinates (λ, α), as longitude and altitude, $(-\pi < \lambda \leq \pi, \ -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2})$, and the arc-length parameter $s \geq 0$, has been determined in [1]. The Euclidean coordinates $X(s, \lambda, \alpha), Y(s, \lambda, \alpha), Z(s, \lambda, \alpha)$ of the geodesic curves can be determined by substituting the results of Table 1 (see also [1]) into formula (8) as follows

$$\begin{cases} X(s, \lambda, \alpha) = \tan(\phi(s, \alpha)), \\ Y(s, \lambda, \alpha) = \dfrac{\tanh(r(s, \alpha))}{\cos(\phi(s, \alpha))}\cos\left[\theta(s, \alpha) - \phi(s, \alpha) + \lambda\right], \\ Z(s, \lambda, \alpha) = \dfrac{\tanh(r(s, \alpha))}{\cos(\phi(s, \alpha))}\sin\left[\theta(s, \alpha) - \phi(s, \alpha) + \lambda\right]. \end{cases} \tag{12}$$

Definition 3.2. *The geodesic distance $d(P, Q)$ between points $P, Q \in \widetilde{SL_2(\mathbb{R})}$ is defined as the arc length of the geodesic curve from P to Q.*

Table 1: Geodesic curves.

direction	parametrization of a geodesic curve
$0 \leq \alpha < \dfrac{\pi}{4}$ (**H^2 − like**)	$r(s, \alpha) = \text{arsinh}\left(\dfrac{\cos \alpha}{\sqrt{\cos 2\alpha}} \sinh(s \sqrt{\cos 2\alpha}) \right)$ $\theta(s, \alpha) = -\arctan\left(\dfrac{\sin \alpha}{\sqrt{\cos 2\alpha}} \tanh(s \sqrt{\cos 2\alpha}) \right)$ $\phi(s, \alpha) = 2s \sin \alpha + \theta(s, \alpha)$
$\alpha = \dfrac{\pi}{4}$ (light − like)	$r(s, \alpha) = \text{arsinh}\left(\dfrac{\sqrt{2}}{2} s \right)$ $\theta(s, \alpha) = -\arctan\left(\dfrac{\sqrt{2}}{2} s \right)$ $\phi(s, \alpha) = \sqrt{2}s + \theta(s, \alpha)$
$\dfrac{\pi}{4} < \alpha \leq \dfrac{\pi}{2}$ (fibre − like)	$r(s, \alpha) = \text{arsinh}\left(\dfrac{\cos \alpha}{\sqrt{-\cos 2\alpha}} \sin(s \sqrt{-\cos 2\alpha}) \right)$ $\theta(s, \alpha) = -\arctan\left(\dfrac{\sin \alpha}{\sqrt{-\cos 2\alpha}} \tan(s \sqrt{-\cos 2\alpha}) \right)$ $\phi(s, \alpha) = 2s \sin \alpha + \theta(s, \alpha)$

3.2. Translation curves

We recall some basic facts about translation curves in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ following [3, 5, 9]. For any point $X(x^0 : x^1 : x^2 : x^3) \in \mathcal{H}$ (and later also for points in $\widetilde{\mathcal{H}}$) the *translation map* from the origin $E_0(1 : 0 : 0 : 0)$ to X is defined by the *translation matrix* $\mathbf{T}$ and its inverse presented in (6).

Let us consider for a given vector $(q : u : v : w)$ a curve $C(t) = (x^0(t) : x^1(t) : x^2(t) : x^3(t))$, $t \geq 0$, in $\mathcal{H}$ starting at the origin: $C(0) = E_0(1 : 0 : 0 : 0)$ and such that

$$\dot{C}(0) = (\dot{x}^0(0) : \dot{x}^1(0) : \dot{x}^2(0) : \dot{x}^3(0)) = (q : u : v : w),$$

where $\dot{C}(t) = (\dot{x}^0(t) : \dot{x}^1(t) : \dot{x}^2(t) : \dot{x}^3(t))$ is the tangent vector at any point of the curve. For $t \geq 0$ there exists a matrix

$$\mathbf{T}(t) = \begin{pmatrix} x^0(t) & x^1(t) & x^2(t) & x^3(t) \\ -x^1(t) & x^0(t) & x^3(t) & -x^2(t) \\ x^2(t) & x^3(t) & x^0(t) & x^1(t) \\ x^3(t) & -x^2(t) & -x^1(t) & x^0(t) \end{pmatrix}$$

which defines the translation from $C(0)$ to $C(t)$:

$$C(0) \cdot \mathbf{T}(t) = C(t), \quad t \geq 0.$$

The t-parametrized family $\mathbf{T}(t)$ of translations is used in the following definition.

Definition 3.3. *The curve $C(t)$, $t \geq 0$, is said to be a* translation curve *if*

$$\dot{C}(0) \cdot \mathbf{T}(t) = \dot{C}(t), \quad t \geq 0.$$

The solution, depending on $(q : u : v : w)$ had been determined in [3], where it splits into three cases.

It was observed above that for any $X(x^0 : x^1 : x^2 : x^3) \in \widetilde{\mathcal{H}}$ there is a suitable transformation $\mathbf{T}^{-1}$, given by (6), which sent X to the origin E_0 along a translation curve.

Definition 3.4. *A* translation distance *$\rho(E_0, X)$ between the origin $E_0(1 : 0 : 0 : 0)$ and the point $X(1 : x : y : z)$ is the length of a translation curve connecting them.*

For a given translation curve $C = C(t)$ the initial unit tangent vector (u, v, w) (in Euclidean coordinates) at E_0 can be presented as

$$u = \sin \alpha, \quad v = \cos \alpha \cos \lambda, \quad w = \cos \alpha \sin \lambda, \tag{13}$$

for some $-\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$ and $-\pi < \lambda \leqslant \pi$. In $\widetilde{\mathcal{H}}$ this vector is of length square $-u^2 + v^2 + w^2 = \cos 2\alpha$. We always can assume that C is parametrized by the translation arc-length parameter $t = s \geqslant 0$. Then coordinates of a point $X(x, y, z)$ of C, such that the translation distance between E_0 and X equals s, depend on (λ, α, s) as geographic coordinates according to the above considered three cases as follows.

Table 2: Translation curves.

direction	parametrization of a translation curve
$0 \leq \alpha < \dfrac{\pi}{4}$ $(\mathbf{H}^2 - \text{like})$	$\begin{pmatrix} x(s, \alpha, \lambda) \\ y(s, \alpha, \lambda) \\ z(s, \alpha, \lambda) \end{pmatrix} = \dfrac{\tanh(s\sqrt{\cos 2\alpha})}{\sqrt{\cos 2\alpha}} \begin{pmatrix} \sin \alpha \\ \cos \alpha \cos \lambda \\ \cos \alpha \sin \lambda \end{pmatrix}$
$\alpha = \dfrac{\pi}{4}$ $(\text{light} - \text{like})$	$\begin{pmatrix} x(s, \alpha, \lambda) \\ y(s, \alpha, \lambda) \\ z(s, \alpha, \lambda) \end{pmatrix} = \dfrac{\sqrt{2}s}{2} \begin{pmatrix} 1 \\ \cos \lambda \\ \sin \lambda \end{pmatrix}$
$\dfrac{\pi}{4} < \alpha \leq \dfrac{\pi}{2}$ $(\text{fibre} - \text{like})$	$\begin{pmatrix} x(s, \alpha, \lambda) \\ y(s, \alpha, \lambda) \\ z(s, \alpha, \lambda) \end{pmatrix} = \dfrac{\tan(s\sqrt{-\cos 2\alpha})}{\sqrt{-\cos 2\alpha}} \begin{pmatrix} \sin \alpha \\ \cos \alpha \cos \lambda \\ \cos \alpha \sin \lambda \end{pmatrix}$

4. Geodesic and translation triangles

4.1. Geodesic triangles

We consider 3 points A_1, A_2, A_3 in the projective model of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ space (see Section 2). The *geodesic segments* a_k between the points A_i and A_j ($i < j$, $i, j, k \in \{1, 2, 3\}, k \neq i, j$ are called sides of the *geodesic triangle* T_g with vertices A_1, A_2, A_3.

In Riemannian geometries the metric tensor (10) is used to define the angle θ between two geodesic curves. If their tangent vectors in their common point are $\mathbf{u}$ and $\mathbf{v}$ and g_{ij} are the components of the metric tensor then

$$\cos(\theta) = \frac{u^i g_{ij} v^j}{\sqrt{u^i g_{ij} u^j \ v^i g_{ij} v^j}} \tag{14}$$

It is clear by the above definition of the angles and by the metric tensor (3.2), that the angles are the same as the Euclidean ones at the origin of the projective model of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ geometry.

We note here that the angle of two intersecting geodesic curves depend on the orientation of the tangent vectors. We will consider the *interior angles* of the triangles that are denoted at the vertex A_i by ω_i ($i \in \{1, 2, 3\}$).

4.1.1. Fibre-like right angled triangles

A geodesic triangle is called fibre-like if one of its edges lies on a fibre line. In this section we study the right angled fibre-like triangles. We can assume without loss of generality that the vertices A_1, A_2, A_3 of a fibre-like right angled triangle (see Figure 2) T_g have the following coordinates:

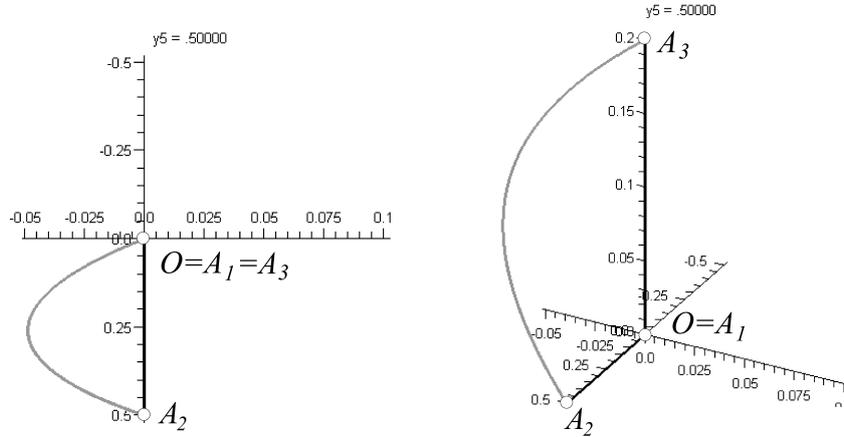$$A_1 = (1 : 0 : 0 : 0), \ A_2 = (1 : 0 : y^2 : 0), \ A_3 = (1 : x^3 : 0 : 0) \tag{15}$$

Figure 2: Fibre-like geodesic triangle

The geodesic segment A_1A_2 lies on the y axis, the geodesic segment A_1A_3 lies on the x axis (see Table 1) and its angle is $\omega_1 = \frac{\pi}{2}$ in the $\widetilde{SL_2(\mathbb{R})}$ space (this angle is in Euclidean sense also $\frac{\pi}{2}$ since $A_1 = E_0$).

In order to determine the further interior angles of fibre-like geodesic triangle $A_1A_2A_3$ we define *translations* $\mathbf{T}_{A_i}$, ($i \in \{2, 3\}$) as elements of the isometry group of $SL_2(\mathbb{R})$, (see 6) that maps the origin E_0 onto A_i. E.g. the isometry $\mathbf{T}_{A_2}$ and its inverse (up to a positive determinant factor) can be given by:

$$\mathbf{T}_{A_2} = \begin{pmatrix} 1 & 0 & y^2 & 0 \\ 0 & 1 & 0 & -y^2 \\ y^2 & 0 & 1 & 0 \\ 0 & -y^2 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}_{A_2}^{-1} = \begin{pmatrix} 1 & 0 & -y^2 & 0 \\ 0 & 1 & 0 & y^2 \\ -y^2 & 0 & 1 & 0 \\ 0 & y^2 & 0 & 1 \end{pmatrix}, \tag{16}$$

and the images $\mathbf{T}_{A_2}(A_i)$ of the vertices A_i ($i \in \{1, 2, 3\}$ are the following (see also Figure 3):

$$\mathbf{T}_{A_2}^{-1}(A_1) = A_1^2 = (1 : 0 : -y^2 : 0), \quad \mathbf{T}_{A_2}^{-1} = A_2^2(A_2) = E_0 = (1 : 0 : 0 : 0),$$
$$\mathbf{T}_{A_2}^{-1}(A_3) = A_3^2 = (1 : x^3 : -y^2 : x^3y^2), \tag{17}$$

Similarly to the above cases we obtain:

$$\mathbf{T}_{A_3}^{-1}(A_1) = A_1^3 = (1 : -x^3 : 0 : 0), \quad \mathbf{T}_{A_3}^{-1}(A_2) = A_2^3 = (1 : -x^3 : y^2 : -x^3y^2),$$
$$\mathbf{T}_{A_3}^{-1}(A_3) = A_3^3 = E_0 = (1 : 0 : 0 : 0), \tag{18}$$

Our aim is to determine angle sum $\sum_{i=1}^{3}(\omega_i)$ of the interior angles of the above right angled fibre-like geodesic triangle $A_1A_2A_3$. We have seen that $\omega_1 = \frac{\pi}{2}$ and the angle of geodesic curves with common point at the origin E_0 is the same as the Euclidean one therefore it can be determined by usual Euclidean sense. Moreover, the translations $\mathbf{T}_{A_2}$ and $\mathbf{T}_{A_3}$ are isometries in $\widetilde{SL_2(\mathbb{R})}$ geometry thus ω_2 is equal to the angle $(g(A_2^2, A_1^2)g(A_2^2, A_3^2))\angle$ (see Figure 4) where $g(A_2^2, A_1^2)$, $g(A_2^2, A_3^2)$ are the oriented geodesic curves and ω_3 is equal to the angle $(g(A_3^3A_1^3)g(A_3^3A_2^3))\angle$ ($E_0 = A_2^2 = A_3^3$ see 17, 18).

The parametrization of a geodesic curve in the model is given by the geographical sphere coordinates (λ, α), as longitude and altitude, $(-\pi < \lambda \le \pi, \ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2})$, and the arc-length parameter $s \ge 0$ (see Table 1 and (3.4)).

We denote the oriented unit tangent vector of the oriented geodesic curves $g(E_0, A_i^j)$ with $\mathbf{t}_i^j$ where $(i, j) = (2, 3), (3, 2), (1, 2), (1, 3)$. The Euclidean coordinates of $\mathbf{t}_i^j$ are:

$$\mathbf{t}_i^j = (\sin(\alpha_i^j), \cos(\alpha_i^j)\cos(\lambda_i^j), \cos(\alpha_i^j)\sin(\lambda_i^j)). \tag{19}$$
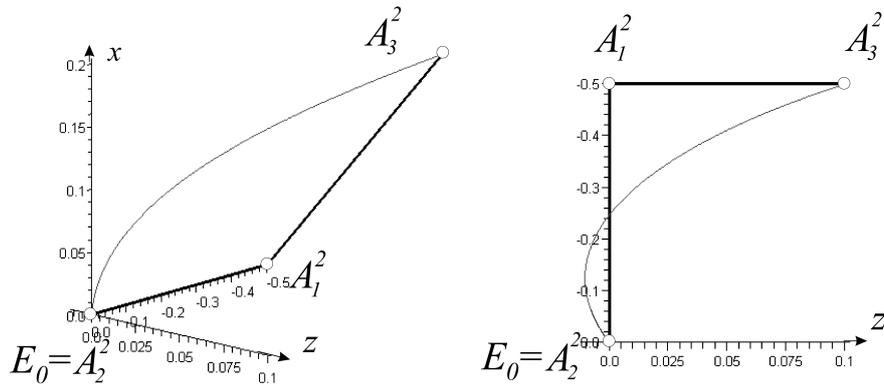
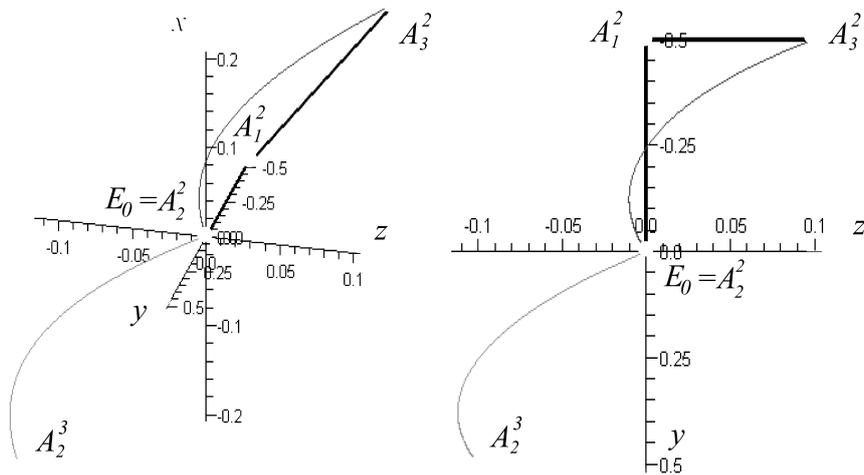Figure 3: Translated image $A_1^2 A_2^2 A_3^3$ of the fibre-like geodesic triangle $A_1 A_2 A_3$ by translation $\mathbf{T}_{A_2}$

Figure 4: The translated points A_2^3 and A_3^2 are antipodal related to the origin E_0

Lemma 4.1. *The sum of the interior angles of a fibre-like right angled geodesic triangle is greater or equal to π.*

Proof: It is clear, that $\mathbf{t}_1^2 = (0, -1, 0)$ and $\mathbf{t}_1^3 = (-1, 0, 0)$. Moreover, the points A_2^3 and A_3^2 are antipodal related to the origin E_0 therefore the equation $|\alpha_2^3| = |\alpha_3^2|$ holds (i.e. the angle between the vector $\mathbf{t}_2^3$ and $[y, z]$ plane are equal to the angle between the vector $\mathbf{t}_3^2$ and the $[y, z]$ plane). That means, that $\omega_3 = \frac{\pi}{2} - |\alpha_2^3| = \frac{\pi}{2} - |\alpha_3^2|$.

The vector $\mathbf{t}_1^2$ lies in the $[y, z]$ plane therefore the angle ω_2 is greater or equal than $\alpha_2^3 = \alpha_3^2$. Finally we obtain, that

$$\sum_{i=1}^3 (\omega_i) = \frac{\pi}{2} + \frac{\pi}{2} - |\alpha_2^3| + \omega_2 \geq \pi. \quad \square$$

In Table 3 we summarize some numerical data of geodesic triangles for given parameters:

Table 3: Fibre-like right angled geodesic triangles for $x^3 = 1/5$

y^2	$\|\alpha_2^3\| = \|\alpha_3^2\|$	$d(A_2A_3)$	ω_2	ω_3	$\sum_{i=1}^{3}(\omega_i)$
$\to 0$	$\to \pi/2$	$\to$ $\arctan(1/5) \approx$ ≈ 0.1974	$\to \pi/2$	$\to 0$	$\to \pi$
1/1000	1.5657	0.1974	1.5658	0.0051	3.1417
1/3	0.4993	0.3970	0.3560	1.0715	3.1806
1/2	0.3170	0.5809	0.3560	1.2538	3.1806
3/4	0.1630	0.9891	0.2043	1.4078	3.1829
999/1000	0.0299	3.8032	0.0422	1.5409	3.1540
$\to 1$	$\to 0$	$\to \infty$	$\to 0$	$\to \pi/2$	$\to \pi$

4.1.2. Hyperbolic-like right angled geodesic triangles

A geodesic triangle is hyperbolic-like if its vertices lie in the base plane (i.e. $[y, z]$ coordinate plane) of the model. In this section we analyze the interior angle sum of the right angled hyperbolic-like triangles. We can assume without loss of generality that the vertices A_1, A_2, A_3 of a hyperbolic-like right angled triangle (see Figure 5) T_g have the following coordinates:

$$A_1 = (1 : 0 : 0 : 0), \ A_2 = (1 : 0 : y^2 : 0), \ A_3 = (1 : 0 : 0 : z^3) \tag{20}$$

The geodesic segment A_1A_2 lies on the y axis, the geodesic segment A_1A_3 lies on the z axis and its angle is
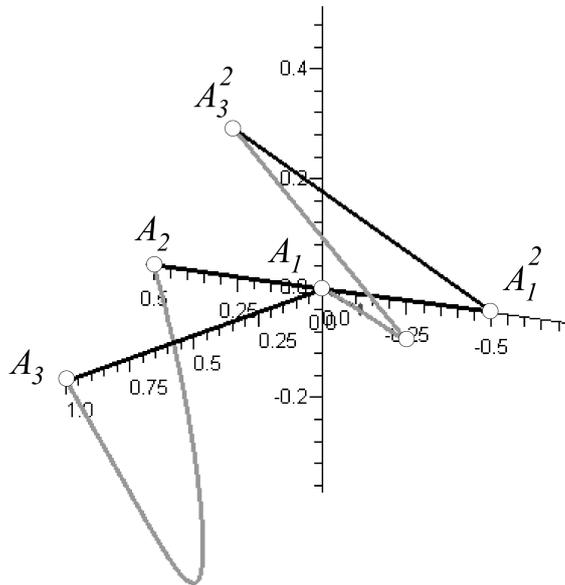

Figure 5: Hyperbolic-like geodesic triangle $A_1A_2A_3$ and its translated copy $A_1^2A_2^2A_3^2$.

$\omega_1 = \frac{\pi}{2}$ in the $\widetilde{\mathrm{SL_2(\mathbb{R})}}$ space (this angle is in Euclidean sense also $\frac{\pi}{2}$ since $A_1 = E_0$).

In order to determine the further interior angles of fibre-like geodesic triangle $A_1A_2A_3$ similarly to the fibre-like case we define a *translations* $\mathbf{T}_{A_i}$, (see 6) that maps the origin E_0 onto A_i. E.g. the isometry $\mathbf{T}_{A_3}$ and

Table 4: Hyperbolic-like geodesic triangles for $y^2 = 1/2$

z^3	$|\alpha_2^3| = |\alpha_3^2|$	$d(A_2A_3)$	ω_1	ω_2	$\sum_{i=1}^3 (\omega_i)$
$\to 0$	$\to 0$	$\to$ arctanh$(1/2) \approx$ ≈ 0.5493	$\to 0$	$\to \pi/2$	$\to \pi$
$1/10$	0.0811	0.5638	0.1334	1.2830	2.9872
$1/3$	0.2103	0.6994	0.3613	0.7170	2.6491
$\frac{999}{1000}$	0.0649	4.0707	0.5817	0.0913	2.2438
$\frac{(10^6-1)}{10^6}$	0.0330	7.5174	0.0467	0.6112	2.2288

their inverses (up to a positive determinant factor) can be given by:

$$\mathbf{T}_{A_3} = \begin{pmatrix} 1 & 0 & 0 & z^3 \\ 0 & 1 & z^3 & 0 \\ 0 & z^3 & 1 & 0 \\ z^3 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}_{A_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -z^3 \\ 0 & 1 & -z^3 & 0 \\ 0 & -z^3 & 1 & 0 \\ -z^3 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

We get similarly to the above cases that the images $\mathbf{T}_{A_j}^{-1}(A_i)$ of the vertices A_i ($i \in \{1,2,3\}$, $j \in \{2,3\}$) are the following (see also Figure 5):

$$\mathbf{T}_{A_2}^{-1}(A_1) = A_1^2 = (1:0:-y^2:0), \quad \mathbf{T}_{A_2}^{-1} = A_2^2(A_2) = E_0 = (1:0:0:0),$$
$$\mathbf{T}_{A_2}^{-1}(A_3) = A_3^2 = (1:y^2z^3:-y^2:z^3), \tag{22}$$

$$\mathbf{T}_{A_3}^{-1}(A_1) = A_1^3 = (1:-z^3:0:0), \quad \mathbf{T}_{A_3}^{-1}(A_2) = A_2^3 = (1:-y^2z^3:y^2:-z^3),$$
$$\mathbf{T}_{A_3}^{-1}(A_3) = A_3^3 = E_0 = (1:0:0:0), \tag{23}$$

We study similarly to the above fibre-like case the sum $\sum_{i=1}^3 (\omega_i)$ of the interior angles of the above right angled hyperbolic-like geodesic triangle $A_1A_2A_3$.

It is clear, that the angle of geodesic curves with common point at the origin E_0 is the same as the Euclidean one therefore it can be determined by usual Euclidean sense. The translations $\mathbf{T}_{A_2}$ and $\mathbf{T}_{A_3}$ preserve the measure of angles ω_i ($i \in \{2,3\}$ therefore $\omega_2 = (g(A_2^2, A_1^2)g(A_2^2, A_3^2))\angle$ (see Figure 5) and $\omega_3 = (g(A_3^3A_1^3)g(A_3^3A_2^3))\angle$.

Similarly to the fibre-like case the Euclidean coordinates of the oriented unit tangent vector $\mathbf{t}_i^j$ of the oriented geodesic curves $g(E_0, A_i^j)$ $((i,j) = (2,3), (3,2), (1,2), (1,3))$ is given by (4.6). Finally, we get the following lemma:

Lemma 4.2. *The interior angle sums of hyperbolic-like geodesic triangles can be less or equal to π.*

Conjecture 4.3. *The sum of the interior angles of any hyperbolic-like right angled geodesic triangle is less or equal to π.*

In Table 4 we summarize some numerical data of geodesic triangles for given parameters:

4.1.3. Geodesic triangles with interior angle sum π

In the above sections we discussed the fibre- and hyperbolic-like geodesic triangles and proved that there are right angled geodesic triangles whose angle sum $\sum_{i=1}^3 (\omega_i)$ is greater or equal to π, less or equal to π, but $\sum_{i=1}^3 (\omega_i) = \pi$ is realized if one of the vertices of a geodesic triangle $A_1A_2A_3$ tends to the infinity (see Table 3-4). We prove the following

Lemma 4.4. *There is geodesic triangle $A_1A_2A_3$ with interior angle sum π where its vertices are proper (i.e. $A_i \in$ SL$_2(\mathbb{R})$).*

Proof: We consider a hyperbolic-like geodesic right angled triangle with vertices $A_1 = E_0 = (1 : 0 : 0 : 0)$, $A_2 = (1 : 0 : y^2 : 0)$, $A_3^h = (1 : 0 : 0 : z^3)$ and a fibre-like right angled geodesic triangle with vertices $A_1 = E_0 = (1 : 0 : 0 : 0)$, $A_2 = (1 : 0 : y^2 : 0)$, $A_3^f = (1 : x^3 : 0 : 0)$ $(0 < y^2, z^3 < 1, 0 < x^3 < \infty)$. We consider the straight line segment (in Euclidean sense) $A_3^f A_3^h \subset$ SL$_2(\mathbb{R})$.

We consider a geodesic right angled triangle $A_1A_2A_3(t)$ where $A_3 \in A_3^f A_3^h$, $(t \in [0,1])$. A_3 is moving on the segment $A_3^h A_3^f$ and if $t = 0$ then $A_3(0) = A_3^h$, if $t = 1$ then $A_3(1) = A_3^f$.

Similarly to the above cases the interior angles of the geodesic triangle $A_1A_2 A_3(t)$ are denoted by $\omega_i(t)$ $(i \in \{1,2,3\})$. The angle sum $\sum_{i=1}^3(\omega_i(0)) < \pi$ and $\sum_{i=1}^3(\omega_i(1)) > \pi$. Moreover the angles $\omega_i(t)$ change continuously if the parameter t run in the interval $[0,1]$. Therefore there is a $t_E \in (0,1)$ where $\sum_{i=1}^3(\omega_i(t_E)) = \pi$.
□

We obtain by the Lemmas of this Section the following

Theorem 4.5. *The sum of the interior angles of a geodesic triangle of $\widetilde{\text{SL}_2(\mathbb{R})}$ space can be greater, less or equal to π.*

4.2. Translation triangles

We consider 3 points A_1, A_2, A_3 in the projective model of $\widetilde{\text{SL}_2(\mathbb{R})}$ space (see Section 2). The *part of the translation curve a_k* between the points A_i and A_j $(i < j,\ i, j, k \in \{1,2,3\}, k \neq i, j$ are called sides of the *translation triangle T_t* with vertices A_1, A_2, A_3. We have seen in the Section 2 by the equations of the translation curves (see Table 2) that the translation curves are straight lines in the projective model. It is easy to see that the images of the translation curves by translations $\mathbf{T}^{-1}$ are also straight lines because the translation is a collineation (see 6).

Considering a translation triangle $A_1A_2A_3$ we can assume by the homogeneity of the $\widetilde{\text{SL}_2(\mathbb{R})}$ geometry that one of its vertex coincide with the origin $A_1 = E_0 = (1 : 0 : 0 : 0)$ and the other two vertices are $A_2(1 : x^2 : y^2 : z^2)$ and $A_3(1 : x^3 : y^3 : z^3)$.

We will consider the *interior angles* of translation triangles that are denoted at the vertex A_i by ω_i $(i \in \{1,2,3\})$. We note here that the angle of two intersecting translation curves depend on the orientation of their tangent vectors.

Similarly to the geodesic cases, in order to determine the interior angle sum $\sum_{i=1}^3(\omega_i)$ of translation triangle $A_1A_2A_3$ we define a *translations* $\mathbf{T}_{A_i}$, $(i \in \{2,3\})$ (see 6) that maps the origin E_0 onto A_i.

We have seen that ω_1 and the angle of geodesic curves with common point at the origin E_0 is the same as the Euclidean one therefore it can be determined by usual Euclidean sense.

The parametrization of a translation curve in the model is given by the geographical sphere coordinates (λ, α), as longitude and altitude, $(-\pi < \lambda \leq \pi, \ -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2})$, and the arc-length parameter $s \geq 0$ (see Table 2).

We denote the oriented unit tangent vector of the oriented translation curves (E_0, A_i^j) with $\mathbf{t}_i^j$ where $(i, j) = (2, 3), (3, 2), (1, 2), (1, 3)$. The Euclidean coordinates of $\mathbf{t}_i^j$ are the same as the Euclidean coordinates of points A_i^j.

It was observed that the neighborhood of the origin behaves like the Euclidean space so that the angle of two oriented tangent vectors with the origin as base point seems real size in the model and it can be determined by usual formula of the Euclidean geometry (see 10). Now we translate both vertices A_i by translations $\mathbf{T}_{A_i}^{-1}$ to the origin to determine the other two interior angle of the translation triangle $A_1A_2A_3$:

$$A_2 = (1 : x^2 : y^2 : z^2), \ \mathbf{T}_{A_2}^{-1}(A_2) = A_2^2 = (1 : 0 : 0 : 0), \mathbf{T}_{A_2}^{-1}(A_1) = A_1^2 =$$

$$= (1 : -x^2 : -y^2 : -z^2), \ \mathbf{T}_{A_2}^{-1}(A_3) = A_3^2 = \Big(1 : \frac{x^2 - x^3 - y^2z^3 + y^3z^2}{-x^2x^3 + y^2y^3 + z^2z^3 - 1} :$$

$$\frac{-x^2z^3 + x^3z^2 + y^2 - y^3}{-x^2x^3 + y^2y^3 + z^2z^3 - 1} : \frac{x^2y^3 - x^3y^2 + z^2 - z^3}{-x^2x^3 + y^2y^3 + z^2z^3 - 1}\Big) \tag{24}$$

$$A_3 = (1 : x^3 : y^3 : z^3), \ \mathbf{T}_{A_3}^{-1}(A_3) = A_3^3 = (1 : 0 : 0 : 0), \ \mathbf{T}_{A_3}^{-1}(A_1) = A_1^3 =$$

$$= (1 : -x^3 : -y^3 : -z^3), \ \mathbf{T}_{A_3}^{-1}(A_2) = A_2^3 = \left(1 : \frac{x^3 - x^2 - y^3 z^2 + y^2 z^3}{-x^2 x^3 + y^2 y^3 + z^2 z^3 - 1} : \right. \quad (25)$$

$$\left. \frac{-x^3 z^2 + x^2 z^3 + y^3 - y^2}{-x^2 x^3 + y^2 y^3 + z^2 z^3 - 1} : \frac{x^3 y^2 - x^2 y^3 + z^3 - z^2}{-x^2 x^3 + y^2 y^3 + z^2 z^3 - 1} \right).$$

It is easy to see that the point pairs $(A_2, \mathbf{T}_{A_2}^{-1}(A_1))$, $(A_3, \mathbf{T}_{A_3}^{-1}(A_1))$ and $(\mathbf{T}_{A_2}^{-1}(A_3), \mathbf{T}_{A_3}^{-1}(A_2))$ are antipodal. In Figure 6 there can be seen the translated triangles in the hyperboloid model. Also during the translation, the plane containing the triangle *twists, i.e.* the translated plane does not coincides generally with the original plane.
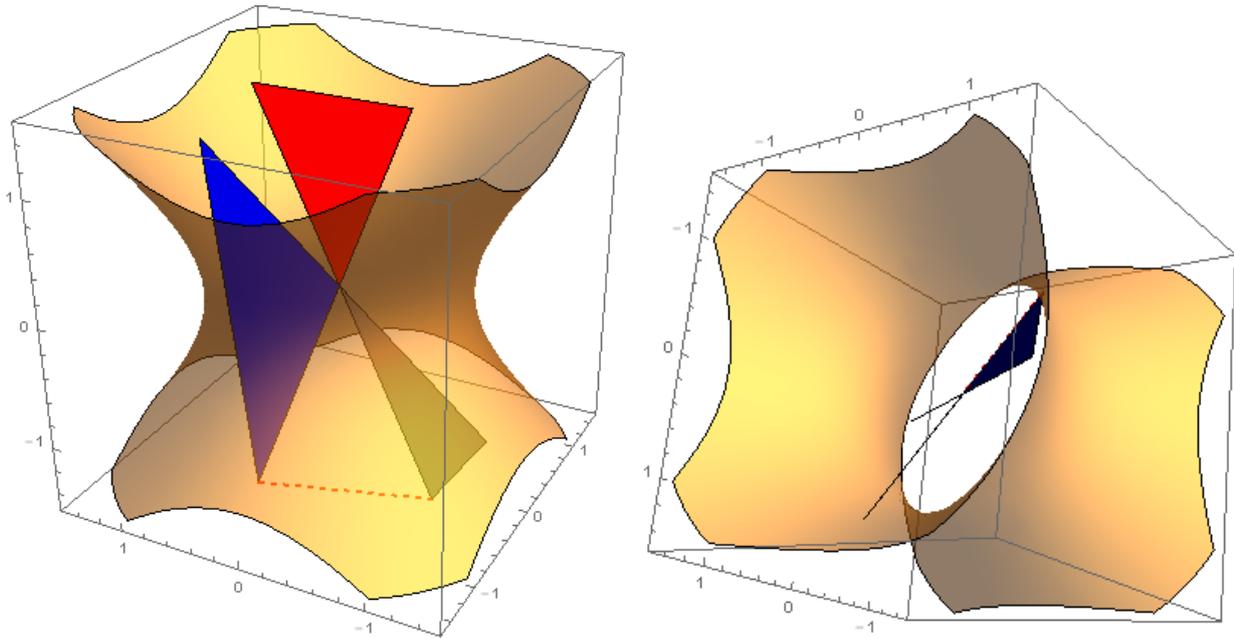

Figure 6: Translations of the original (red) translation triangle.

Lemma 4.6. *Let $\sigma = A_1 A_2 A_3$ be a plane in Euclidean sense trough the origin ($A_1 = E_0$) and be its Euclidean normal vector $\mathbf{v}$. Then σ is invariant for $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ translation $E_0 Q$ where $Q \in \sigma$ if and only if $\mathbf{v}$ is light–like (see ...).*

Proof: The Euclidean equation of the plane σ is:

$$\sigma : \ (y^2 z^3 - y^3 z^2)x + (x^3 z^2 - x^2 z^3)y + (x^2 y^3 - x^3 y^2)z = 0 \quad (26)$$

The inhomogeneous coordinates of $\mathbf{T}_{A_2}^{-1}(A_3)$ satisfies the (26) equation if and only if it is on σ:

$$\frac{1}{-x_1 x_2 + y_1 y_2 + z_1 z_2 - 1} \cdot \Big((x_1 - x_2)(y_1 z_2 - y_2 z_1) - (y_1 z_2 - y_2 z_1)^2 +$$

$$(y_1 - y_2)(x_2 z_1 - x_1 z_2) + +(x_2 z_1 - x_1 z_2)^2 + (z_1 - z_2)(x_1 y_2 - x_2 y_1) +$$

$$(x_1 y_2 - x_2 y_1)^2 \Big) = -(y_1 z_2 - y_2 z_1)^2 + (x_2 z_1 - x_1 z_2)^2 + (x_1 y_2 - x_2 y_1)^2 =$$

$$= -\mathbf{v}_1^2 + \mathbf{v}_2^2 + \mathbf{v}_3^2 = 0 \Leftrightarrow \mathbf{v} \text{ is light} - \text{like. } \square$$

Now we can claim the following theorem:

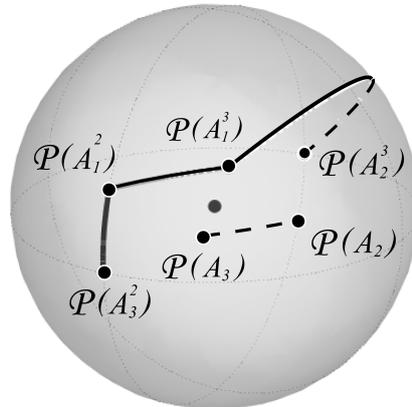**Theorem 4.7.** *The sum of the interior angles of the translation triangle is greater or equal to π.*

Figure 7: Antipodal point pairs of the translation triangle projected onto the unit sphere

Proof: The translations $\mathbf{T}_{A_2}^{-1}$ and $\mathbf{T}_{A_3}^{-1}$ are isometries in $\widetilde{\mathbf{SL}_2(\mathbb{R})}$ geometry thus ω_2 is equal to the angle $((A_2^2 A_1^2), (A_2^2 A_3^2))\angle$ (see Figure 7) of the oriented translation segments (Euclidean segments as well) $(A_2^2 A_1^2)$, $(A_2^2 A_3^2)$ and ω_3 is equal to the angle $((A_3^3 A_1^3), (A_3^3 A_2^3))\angle$ of the oriented translation segments (Euclidean segments as well) $(A_3^3 A_1^3)$ and $(A_3^3 A_2^3)$ $(E_0 = A_2^2 = A_3^3$ see 17, 18).

To get the angles ω_i we apply the $\mathcal{P}$ projection from the origin onto the unit sphere around E_0 to the vertexes A_2, A_3 and A_i^j, $(i, j) = (1, 2), (1, 3), (2, 3), (3, 2)$. The measure of angle ω_i $(i \in \{1, 2, 3\})$ is equal to the spherical distance of the corresponding projected points on the unit sphere (see Figure 7). Due to the antipodality $\omega_1 = \mathcal{P}(A_2)A_1\mathcal{P}(A_3)\angle = \mathcal{P}(A_1^2)A_1\mathcal{P}(A_1^3)\angle$, therefore their corresponding spherical distances are equal, as well (see Figure 7). Now, the sum of the interior angles can be considered as three consecutive arc $(\mathcal{P}(A_3^2)\mathcal{P}(A_1^2))$, $(\mathcal{P}(A_1^2)\mathcal{P}(A_1^3))$, $(\mathcal{P}(A_1^3)\mathcal{P}(A_2^3))$ and $\mathcal{P}(A_3^2)$ is antipodal to $\mathcal{P}(A_2^3)$.

Since the triangle inequality holds on the sphere, the sum of these arc lengths is greater or equal to the half of the circumference of the main circle on the unit sphere *i.e.* π.

By Lemma 4.5 we obtain that the three consecutive arcs $(\mathcal{P}(A_3^2)\mathcal{P}(A_1^2))$, $(\mathcal{P}(A_1^2)\mathcal{P}(A_1^3))$, $(\mathcal{P}(A_1^3)\mathcal{P}(A_2^3))$ lie on a main circle of the unit sphere if and only if the normal vector of the plane containing the triangle is light–like thus in this case the sum of the interior angles of the translation triangle is π. □

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