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Dual Orlicz Mixed Geominimal Surface Areas

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Abstract. The notion of L_p-geominimal surface area was originally introduced by Lutwak in 1996. Recently, Feng and Wang introduced the concept of L_p-dual mixed geominimal surface area based on L_p-dual mixed quermassintegrals. In this paper, based on dual Orlicz mixed quermassintegrals, we define the concept of dual Orlicz mixed geominimal surface area and establish some related inequalities for this new notion.

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the class of convex bodies containing the origin in their interiors, origin-symmetric convex bodies and whose centroid lie at the origin, we write $\mathcal{K}_o^n$, $\mathcal{K}_{os}^n$ and $\mathcal{K}_c^n$, respectively. Denote $\mathcal{S}_o^n$ by the set of star bodies (about the origin). By S^{n-1} we mean the unit sphere in $\mathbb{R}^n$. For the volume of the standard unit ball B, we write $\omega_n = V(B)$.

The notions of mixed volumes, dual mixed volumes, affine surface areas and geominimal surface areas are core contents in the classical Brunn-Minkowski theory. The classical Brunn-Minkowski theory is extended to the L_p-Brunn-Minkowski theory by Lutwak (see [19, 20]). In [19], Lutwak presented the L_p-mixed volumes. In [20], Lutwak showed the L_p-dual mixed volumes, L_p-affine surface areas and L_p-geominimal surface areas. In particular, Lutwak's L_p-geominimal surface areas can be stated as follows: For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \tag{1}$$

Here, $V_p(L,M)$ denotes the L_p-mixed volume of $L, M \in \mathcal{K}_o^n$ (see [20]). If $p = 1$ in (1), then $G_p(K)$ is just Petty's geominimal surface area $G(K)$ (see [24]).

Corresponding to Lutwak's L_p-geominimal surface areas, Wang and Qi ([28]) gave the notion of L_p-dual geominimal surface areas. For $K \in \mathcal{S}_o^n$, $p \geq 1$, the L_p-dual geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}(K) = \inf\{n\widetilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\}.$$

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Here, $\widetilde{V}_{-p}(L,M)$ denotes the L_p-dual mixed volume of $L,M \in \mathcal{S}_o^n$ (see (15)).

In 2005, Wang and Leng ([26]) extended Lutwak's L_p-dual mixed volumes to the L_p-dual mixed quermassintegrals. Combining with the L_p-dual mixed quermassintegrals, Feng and Wang ([8]) defined the L_p-dual mixed geominimal surface areas as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$ and real i satisfies $0 \leq i < n$, the L_p-dual mixed geominimal surface area, $\widetilde{G}_{-p,i}(K)$, of K is given by

$$\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(K) = \inf\{n\widetilde{W}_{-p,i}(K,Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\}. \tag{2}$$

Here, $\widetilde{W}_i(K)$ denotes the dual quermassintegrals of $K \in \mathcal{S}_o^n$ and $\widetilde{W}_{-p,i}(M,N)$ denotes the L_p-dual mixed quermassintegrals of $M,N \in \mathcal{S}_o^n$.

A further extension of the classical Brunn-Minkowski theory is the new Orlicz-Brunn-Minkowski theory. This new theory was launched originally by Lutwak, Yang and Zhang in their two farreaching articles (see [21, 22]). Whereafter, Xi, Jin and Leng (see [30]), Gardner, Hug and Weil (see [10]) respectively constructed a general framework for Orlicz-Brunn-Minkowski theory, which includes Orlicz addition and mixed volume, established the Orlicz-Brunn-Minkowski inequality and the Orlicz-Minkowski inequality. Meanwhile, a step towards the dual Orlicz-Brunn-Minkowski theory for star bodies has already been made by Zhu, Zhou and Xu (see [35], also see [11]). In a certain sense, the dual Orlicz-Brunn-Minkowski theory for star bodies is more subtle and thought-provoking than the Orlicz-Brunn-Minkowski theory for convex bodies.

Compared to the classical part and the L_p-Brunn-Minkowski theory, the Orlicz-Brunn-Minkowski theory (include its dual theory) has more general and abstract framework, just because of this, this theory received considerable attention and also motivated large numbers of researchers' interest, for example, see [1–6, 13, 14, 16, 17, 23, 27, 29, 31–34, 36–39].

In this paper, we extend the L_p-dual mixed geominimal surface area (2) to the Orlicz version. Let Φ denote the set of convex functions $\phi : (0,\infty) \to \mathbb{R}$ which satisfies $\phi(1) = 1$, we define the dual Orlicz mixed geominimal surface areas as follows:

Definition 1.1. *For $K \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $0 \leq i < n$, the dual Orlicz mixed geominimal surface area, $\widetilde{G}_{\phi,i}(K)$, of K is given by*

$$\widetilde{G}_{\phi,i}(K) = \inf\{n\widetilde{W}_{\phi,i}(K,Q) : Q \in \mathcal{K}_c^n \text{ and } \widetilde{W}_i(Q^*) = \omega_n\}. \tag{3}$$

Here, $\widetilde{W}_{\phi,i}(K,Q)$ denotes the dual Orlicz mixed quermassintegrals of $K,Q \in \mathcal{S}_o^n$.

Obviously, let $\phi(t) = t^p$ with $p \geq 1$ in Definition 1.1, then (3) reduces to (2).

Remark 1.2. If $i = 0$ in Definition 1.1, then (3) gives the dual Orlicz geominimal surface area $\widetilde{G}_\phi(K)$ of $K \in \mathcal{S}_o^n$ which was defined by Ma and Wang (see [23]), but it should be noted that ϕ in (3) is different from ϕ in [23].

For the dual Orlicz mixed geominimal surface areas, we first obtain the infimum in the above definition (3) as follows.

Proposition 1.3. *If $K \in \mathcal{S}_o^n$, $\phi \in \Phi$, then there exists a body $L \in \mathcal{K}_c^n$, such that for any $0 \leq i < n$,*

$$\widetilde{G}_{\phi,i}(K) = n\widetilde{W}_{\phi,i}(K,L) \text{ and } \widetilde{W}_i(L^*) = \omega_n. \tag{4}$$

From Proposition 1.3, and together with the definition of dual Orlicz mixed quermassintegrals (see Section 2, (13)), we obtain the integral representation of dual Orlicz mixed geominimal surface area as follows: For $K \in \mathcal{S}_o^n$, $0 \leq i < n$ and $\phi \in \Phi$, there exists $L \in \mathcal{K}_c^n$ and $\widetilde{W}_i(L^*) = \omega_n$, such that

$$\widetilde{G}_{\phi,i}(K) = n\widetilde{W}_{\phi,i}(K,L) = \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\rho_K(u)^{n-i}dS(u). \tag{5}$$

As an application of Proposition 1.3, we give a lower bound of $\widetilde{G}_{\phi,i}(K)$.

Theorem 1.4. *If $K \in \mathcal{K}_c^n$, $\phi \in \Phi$ and $0 \leq i < n$, then*

$$\widetilde{G}_{\phi,i}(K) \geq n\widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(K)}{\omega_n}\right)^{\frac{1}{n-i}}\right), \tag{6}$$

with equality for $i = 0$ if and only if K is an ellipsoid centered at the origin, for $0 < i < n$ if and only if K is a ball centered at the origin.

In addition, by definition (3) we also get a supper bound for the dual Orlicz mixed geominimal surface area $\widetilde{G}_{\phi,i}(K)$.

Theorem 1.5. *If $K \in \mathcal{K}_c^n$, $\phi \in \Phi$ and $0 < i < n$, then*

$$\widetilde{G}_{\phi,i}(K) \leq \frac{n\omega_n^2}{\widetilde{W}_i(K^*)}\phi\bigg(\Big(\frac{\omega_n}{\widetilde{W}_i(K^*)}\Big)^{\frac{1}{n-i}}\bigg), \tag{7}$$

with equality when K is a ball centered at the origin.

Obviously, taking $\phi(t) = t^p$ with $p \geq 1$ in Theorems 1.4-1.5, then inequalities (6) and (7) reduce to the results for the L_p-dual mixed geominimal surface areas which were established respectively by Feng and Wang (see [8]).

If $Q = K$ in (3), then $\widetilde{W}_i(K^*) = \omega_n$. This together (7) with $\phi(1) = 1$ yields that

Corollary 1.6. *If $K \in \mathcal{K}_c^n$, $\phi \in \Phi$ and $0 < i < n$, then*

$$\widetilde{G}_{\phi,i}(K) \leq n\omega_n,$$

with equality when K is a ball centered at the origin.

From Corollary 1.6, we immediately get the Blaschke-Santaló type inequality for the dual Orlicz mixed geominimal surface areas as follows.

Corollary 1.7. *If $K \in \mathcal{K}_c^n$, $\phi \in \Phi$ and $0 < i < n$, then*

$$\widetilde{G}_{\phi,i}(K)\widetilde{G}_{\phi,i}(K^*) \leq (n\omega_n)^2,$$

with equality when K is a ball centered at the origin.

Further, using the integral representation of dual Orlicz mixed geominimal surface area (5), we obtain the following cyclic type inequality.

Theorem 1.8. *If $K \in \mathcal{S}_o^n$, $\phi \in \Phi$, $i, j, k \in \mathbb{R}$, and $0 < i < j < k < n$, then*

$$\widetilde{G}_{\phi,i}(K)^{k-j}\widetilde{G}_{\phi,k}(K)^{j-i} \geq \widetilde{G}_{\phi,j}(K)^{k-i}, \tag{8}$$

with equality if and only if K is a ball centered at the origin.

Finally, the following monotonic inequality for the dual Orlicz mixed geominimal surface areas is obtained.

Theorem 1.9. *Let $\phi_1, \phi_2 \in \Phi$ both are strictly increasing on $(0, \infty)$, $\phi_1 \leq \phi_2$ and $0 < i < n$. If $K \in \mathcal{S}_o^n$, then*

$$\phi_2^{-1}\Big(\frac{(n\widetilde{W}_i(K))^{n-i}}{\widetilde{G}_{\phi_2,i}(K)^{n-i}\phi_2(\widetilde{W}_i(K)^{-1})}\Big) \leq \phi_1^{-1}\Big(\frac{(n\widetilde{W}_i(K))^{n-i}}{\widetilde{G}_{\phi_1,i}(K)^{n-i}\phi_1(\widetilde{W}_i(K)^{-1})}\Big). \tag{9}$$

Let $\phi_1(t) = t^p, \phi_2(t) = t^q$ with $1 < p < q < \infty$ in Theorem 1.9, then inequality (9) yields the monotonic inequality for the L_p-dual mixed geominimal surface areas can be found in [8].

2. Background Materials

2.1 Support function, radial function and polar of convex bodies.

If $K \in \mathcal{K}^n$, then its support function, h_K=$h(K, \cdot)$: $\mathbb{R}^n \to (-\infty, +\infty)$, is defined by (see [9, 25])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

Define the radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, +\infty)$, of a compact star-shaped (about the origin) $K \in \mathbb{R}^n$ by (see [9])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L will be dilates (of one another)if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $M \in \mathcal{K}_o^n$, the polar body, M^*, of M is defined by (see [9, 25])

$$M^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1,\ y \in M\}. \tag{10}$$

Clearly, by definition (10), it follows that if $M \in \mathcal{K}_o^n$, then $(M^*)^* = M$ and

$$\rho_{M^*} = \frac{1}{h_M}. \tag{11}$$

2.2 Dual quermassintegrals and dual Orlicz mixed quermassintegrals.

For $K \in \mathcal{S}_o^n$ and any real i, define the dual quermassintegrals, $\widetilde{W}_i(K)$, of K by (see [18])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u).$$

For the dual quermassintegrals, Ghandehari ([12]) established the following Blaschke-Santal*ó* inequality:

Theorem 2.1 ([12]). *If $K \in \mathcal{K}_c^n$, real i satisfies $0 \leq i < n$, then*

$$\widetilde{W}_i(K)\widetilde{W}_i(K^*) \leq \omega_n^2, \tag{12}$$

with equality for $i = 0$ if and only if K is an ellipsoid centered at the origin, for $0 < i < n$ if and only if K is a ball centered at the origin.

Wang, Shi and Ye ([29]) introduced the notion of dual Orlicz mixed quermassintegrals. For $K, L \in \mathcal{S}_o^n$, real $i \neq n$, $\phi : (0, \infty) \to \mathbb{R}$, define the dual Orlicz mixed quermassintegrals, $\widetilde{W}_{\phi,i}(K, L)$, of K and L by

$$\widetilde{W}_{\phi,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \phi\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big) \rho_K(u)^{n-i} dS(u). \tag{13}$$

Taking $\phi(t) = t^p$ with $p \geq 1$ in (13), then the integral formula of L_p-dual mixed quermassintegrals can be obtained (see [26]):

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+p-i} \rho_L(u)^{-p} dS(u). \tag{14}$$

If $i = 0$, then (14) gives Lutwak's L_p-dual mixed volume by

$$\widetilde{V}_{-p}(K, L) = \widetilde{W}_{-p,0}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+p} \rho_L(u)^{-p} dS(u). \tag{15}$$

Further, a dual Orlicz-Minkowski inequality for the dual Orlicz mixed quermassintegrals was established by them (see [29]) as well.

Theorem 2.2 ([29]). *Let $K, L \in \mathcal{S}_o^n$, real $i \neq n$, $\phi : (0, \infty) \to \mathbb{R}$, and $F(t) = \phi(t^{-\frac{1}{n-i}})$, $t > 0$. If F is convex, then*

$$\widetilde{W}_{\phi,i}(K, L) \geq \widetilde{W}_i(K) \phi\bigg(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\bigg). \tag{16}$$

When F is strictly convex, equality holds if and only if K and L are dilates.

The following proposition shows that the sequence of dual Orlicz mixed quermassintegrals is continuous.

Lemma 2.3 ([23]). *Suppose f_i, f are strictly positive and continuous functions on S^{n-1}; $\phi_j, \phi \in \Phi$; μ_k, μ are Borel probability measures on S^{n-1}; $i, j, k \in \mathbb{N}$. If $f_i \to f$ pointwise, $\phi_j \to \phi$ uniformly, and $\mu_k \to \mu$ weakly, then as $i, j, k \to \infty$,*

$$\int_{S^{n-1}} \phi_j(f_i)d\mu_k \to \int_{S^{n-1}} \phi(f)d\mu.$$

By Lemma 2.3, it can conclude that

Lemma 2.4. *Suppose $K, K_i, L, L_j \in \mathcal{S}_o^n$, and $\phi, \phi_k \in \Phi$ ($i, j, k \in \mathbb{N}$). If $K_i \to K$, $L_j \to L$ and $\phi_k \to \phi$, then for real $l \neq n$,*

$$\lim_{i,j,k\to\infty} \widetilde{W}_{\phi_k,l}(K_i, L_j) = \widetilde{W}_{\phi,l}(K, L).$$

Combining with the definition (3) of dual Orlicz mixed geominimal surface area, the following proposition can be viewed as an immediate consequence of Lemma 2.4.

Lemma 2.5. *If $\phi \in \Phi$, then the functional $\widetilde{G}_{\phi,i} : \mathcal{S}_o^n \to (0, \infty)$ is continuous.*

3. Proofs of Results

In this section, we will complete the proofs of our main Theorems. First, by the next Lemmas, we will show that the infimum in the definition (3) of dual Orlicz mixed geominimal surface area $\widetilde{G}_{\phi,i}(K)$ can be attained.

Lemma 3.1. *Let C^n denote the set of compact convex subsets of n-dimensional Euclidean space $\mathbb{R}^n$, and suppose $K_i \in \mathcal{K}_o^n$ ($i \in \mathbb{N}$) such that $K_i \to L \in C^n$. If the sequence $\widetilde{W}_j(K_i^*)$ ($j \in \mathbb{R}$) is bounded, then $L \in \mathcal{K}_o^n$.*

Proof. Since the sequence $\widetilde{W}_j(K_i^*)$ is bounded, so choose $\lambda_0 > 0$, such that $\widetilde{W}_j(K_i^*) \leq \lambda_0$ for all $i \in \mathbb{N}$. For a compact set L, there exists a real $r_0 > 0$, such that $L \subset r_0 B$, where B is a unit ball centered at the origin. Note that $K_i \to L$, the real r_0 can be chosen so that $K_i \subset r_0 B$ for all $i \in \mathbb{N}$.

Now, we introduce the real r_i, which may be denoted by

$$r_i = r(K_i) = \min_{u \in S^{n-1}} h(K_i, u) = h(K_i, u_i),$$

where $u_i \in S^{n-1}$. By (11), $\rho(K_i^*, u_i) = h(K_i, u_i)^{-1} = \frac{1}{r_i}$, it is easy checked that K_i^* contains the point $\frac{1}{r_i}u_i$. Moreover, $K_i \subset r_0 B$, one gets $\rho(K_i^*, \cdot) = h(K_i, \cdot)^{-1} \geq \frac{1}{r_0}$. Thus

$$\left(\frac{1}{r_0}\right)^{n-j-1} \omega_n \frac{1}{r_i} \leq \widetilde{W}_j(K_i^*) \leq \lambda_0,$$

therefore, we have

$$r_i \geq \omega_n r_0^{1-n+j} \lambda_0^{-1}.$$

Thus, the ball, centered at the origin, of radius $\omega_n r_0^{1-n+j}\lambda_0^{-1}$ is contained in each K_i, it infers that this ball is contained in L as well. This and $L \in C^n$ yield $L \in \mathcal{K}_o^n$. □

Lemma 3.1 ensures $K_i \in \mathcal{K}_c^n$ and $K_i \to L$, we have $L \in \mathcal{K}_c^n$.

The following Lemma is the well-known Jensen's inequality (see [15]), which can be stated as follows.

Lemma 3.2 ([15]). *If μ is a probability measure on a space X and $h : X \to I \subset \mathbb{R}$ is a μ-integrable function, here I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(h(x))d\mu(x) \geq \phi\Big(\int_X h(x)d\mu(x)\Big). \tag{17}$$

If ϕ is strictly convex, equality holds if and only if $h(x)$ is a constant for μ-almost all $x \in X$.

Proof of Proposition 1.3. By the definition (3) of $\widetilde{G}_{\phi,i}(K)$, there is a sequence $Q_j \in \mathcal{K}_c^n$ ($j \in \mathbb{N}$), which satisfies $\widetilde{W}_i(Q_j^*) = \omega_n$ and $\widetilde{W}_{\phi,i}(K,B) \geq \widetilde{W}_{\phi,i}(K,Q_j)$ for any $0 \leq i < n$ and $j \in \mathbb{N}$, thus $n\widetilde{W}_{\phi,i}(K,Q_j) \to \widetilde{G}_{\phi,i}(K)$.

To see that $Q_j \in \mathcal{K}_c^n$ ($j \in \mathbb{N}$) are uniformly bounded, let

$$R_j = R(Q_j) = \rho(Q_j, u_j) = \max\{\rho(Q_j, u) : u \in S^{n-1}\},$$

where u_j is any of the points in S^{n-1} at which the maximum is attained.

Let $r(K) = \max\{\lambda > 0 : \lambda B \subset K\}$, then $r(K)B \subset K$. Hence

$$\widetilde{W}_i(r(K)B) = r(K)^{n-i}\omega_n \leq \widetilde{W}_i(K). \tag{18}$$

But by the Jensen's inequality (17), we get

$$\begin{aligned}
\widetilde{W}_{\phi,i}(K,B) &\geq \widetilde{W}_{\phi,i}(K,Q_j)\\
&= \frac{1}{n}\int_{S^{n-1}} \phi\Big(\frac{\rho_K(u)}{\rho_{Q_j}(u)}\Big)\rho_K(u)^{n-i}dS(u)\\
&= \widetilde{W}_i(K)\cdot\frac{1}{n\widetilde{W}_i(K)}\int_{S^{n-1}} \phi\Big(\frac{\rho_K(u)}{\rho_{Q_j}(u)}\Big)\rho_K(u)^{n-i}dS(u)\\
&\geq \widetilde{W}_i(K)\cdot\phi\Big(\frac{1}{n\widetilde{W}_i(K)}\int_{S^{n-1}} \rho_K(u)^{n-i+1}\rho_{Q_j}(u)^{-1}dS(u)\Big)\\
&\geq \widetilde{W}_i(K)\cdot\phi\Big(\frac{r(K)}{n\widetilde{W}_i(K)R_j}\int_{S^{n-1}} \rho_K(u)^{n-i}dS(u)\Big)\\
&\geq \widetilde{W}_i(K)\cdot\phi\Big(\frac{r(K)}{R_j}\Big).
\end{aligned}$$

This and (18) yield

$$r(K)^{n-i}\omega_n\phi\Big(\frac{r(K)}{R_j}\Big) \leq \widetilde{W}_i(K)\cdot\phi\Big(\frac{r(K)}{R_j}\Big) \leq \widetilde{W}_{\phi,i}(K,Q_j) \leq \widetilde{W}_{\phi,i}(K,B) < +\infty.$$

According to the uniform boundness of $Q_j \in \mathcal{K}_c^n$ ($j \in \mathbb{N}$), the Blaschke's selection theorem ensures the sequence Q_j has a subsequence, for convenience, which will also be denoted by Q_j. Note that the compact convex set $L \in C^n$ such that $Q_j \to L$. Since $\widetilde{W}_i(Q_j^*) = \omega_n$, by Lemma 3.1, we conclude that $L \in \mathcal{K}_c^n$. From $Q_j \to L$ and $\widetilde{W}_i(Q_j^*) = \omega_n$, it can infer that $Q_j^* \to L^*$ and $\widetilde{W}_i(L^*) = \omega_n$. By Lemma 2.4, we see that L will serve as the desired body. □

Proof of Theorem 1.4. Since ϕ is convex, thus for $0 \leq i < n$, $F(t) = \phi(t^{-\frac{1}{n-i}})$ is convex. From this, by Proposition 1.3 we know that for $K \in \mathcal{K}_c^n$, there exists for $L \in \mathcal{K}_c^n$, such that

$$\widetilde{G}_{\phi,i}(K) = n\widetilde{W}_{\phi,i}(K,L) \text{ and } \widetilde{W}_i(L^*) = \omega_n.$$

This together dual Orlicz-Minkowski inequality (16) with Blaschke-Santaló inequality (12) gives

$$\begin{aligned}
\widetilde{G}_{\phi,i}(K) &\geq n\widetilde{W}_i(K)\phi\bigg(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\bigg) \text{ and } \widetilde{W}_i(L^*) = \omega_n\\
&\geq n\widetilde{W}_i(K)\phi\bigg(\Big(\frac{\widetilde{W}_i(K)}{\omega_n}\Big)^{\frac{1}{n-i}}\bigg).
\end{aligned}$$

This gives inequality (6).

According to the equality conditions of (16) and (12), it follows that equality holds in (6) with equality for $i = 0$ if and only if K is an ellipsoid centered at the origin, for $0 < i < n$ if and only if K is a ball centered at the origin. □

Proof of Theorem 1.5. By definition (3) of $\widetilde{G}_{\phi,i}(K)$, and together with the Blaschke-Santal*ó* inequality (12), we obtain

$$\begin{aligned}\widetilde{G}_{\phi,i}(K) &= \inf\Big\{n\widetilde{W}_{\phi,i}(K,Q) : Q \in \mathcal{K}_c^n \ \text{ and } \ \widetilde{W}_i(Q^*) = \omega_n\Big\} \\ &\leq n\widetilde{W}_{\phi,i}(K,Q) \ \ and \ \ \widetilde{W}_i(Q^*) = \omega_n.\end{aligned} \tag{19}$$

Since $K \in \mathcal{K}_c^n$, let $Q = \lambda K$ in (19), thus by (13) and inequality (12) we have

$$\widetilde{G}_{\phi,i}(K) \leq n\widetilde{W}_{\phi,i}(K,\lambda K) = n\widetilde{W}_i(K)\phi\Big(\frac{1}{\lambda}\Big) \leq \frac{n\omega_n^2}{\widetilde{W}_i(K^*)}\phi\Big(\frac{1}{\lambda}\Big).$$

But $\widetilde{W}_i(Q^*) = \omega_n$ and $Q = \lambda K$, then $\lambda = \Big(\frac{\widetilde{W}_i(K^*)}{\omega_n}\Big)^{\frac{1}{n-i}}$. Therefore,

$$\widetilde{G}_{\phi,i}(K) \leq \frac{n\omega_n^2}{\widetilde{W}_i(K^*)}\phi\bigg(\Big(\frac{\omega_n}{\widetilde{W}_i(K^*)}\Big)^{\frac{1}{n-i}}\bigg).$$

This gives (7).

If K is a ball, it can infer that equality holds in (7). Thus, equality holds in (7) when K is a ball centered at the origin. □

In the end, we proved the cyclic and monotonic type inequalities of dual Orlicz mixed geominimal surface areas.

Proof of Theorem 1.8. For the sake of convenience, in integral formula (5) let

$$f_{\phi,i}(K,L) = \phi\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big)\rho_K(u)^{n-i}.$$

Then by (5) and H*ö*lder's integral inequality (see [7]), we get for $0 < i < j < k < n$,

$$\begin{aligned}\widetilde{G}_{\phi,i}(K)^{\frac{k-j}{k-i}}\widetilde{G}_{\phi,k}(K)^{\frac{j-i}{k-i}} &= \Big[\int_{S^{n-1}} f_{\phi,i}(K,L)dS(u)\Big]^{\frac{k-j}{k-i}}\Big[\int_{S^{n-1}} f_{\phi,k}(K,L)dS(u)\Big]^{\frac{j-i}{k-i}} \\ &= \bigg\{\int_{S^{n-1}}\Big\{\Big[f_{\phi,i}(K,L)\Big]^{\frac{k-j}{k-i}}\Big\}^{\frac{k-i}{k-j}}dS(u)\bigg\}^{\frac{k-j}{k-i}} \times \bigg\{\int_{S^{n-1}}\Big\{\Big[f_{\phi,k}(K,L)\Big]^{\frac{j-i}{k-i}}\Big\}^{\frac{k-i}{j-i}}dS(u)\bigg\}^{\frac{j-i}{k-i}} \\ &\geq \int_{S^{n-1}}\Big[f_{\phi,i}(K,L)\Big]^{\frac{k-j}{k-i}}\Big[f_{\phi,k}(K,L)\Big]^{\frac{j-i}{k-i}}dS(u) \\ &= \int_{S^{n-1}} f_{\phi,j}(K,L)dS(u) = \widetilde{G}_{\phi,j}(K).\end{aligned}$$

Thus, we obtain

$$\widetilde{G}_{\phi,i}(K)^{k-j}\widetilde{G}_{\phi,k}(K)^{j-i} \geq \widetilde{G}_{\phi,j}(K)^{k-i}.$$

This yields inequality (8).

According to the equality condition of H*ö*lder's integral inequality, the equality in (8) holds if and only if K is a ball centered at the origin. □

Proof of Theorem 1.9. By the definition (3) of $\widetilde{G}_{\phi,i}(K)$, and note that $\phi_1 \leq \phi_2 \in \Phi$, then

$$\phi_1\Big(\frac{(n\widetilde{W}_i(K))^{n-i}}{\widetilde{G}_{\phi_2,i}(K)^{n-i}\phi_2(\widetilde{W}_i(K)^{-1})}\Big)$$

$$
\begin{aligned}
&= \phi_1\Big(\frac{(n\widetilde{W}_i(K))^{n-i}\phi_2(\widetilde{W}_i(K)^{-1})^{-1}}{\inf\{n^{n-i}\widetilde{W}_{\phi_2,i}(K,Q)^{n-i} : Q \in \mathcal{K}_c^n \text{ and } \widetilde{W}_i(Q^*) = \omega_n\}}\Big)\\
&\le \phi_1\Big(\frac{(n\widetilde{W}_i(K))^{n-i}\phi_1(\widetilde{W}_i(K)^{-1})^{-1}}{\inf\{n^{n-i}\widetilde{W}_{\phi_1,i}(K,Q)^{n-i} : Q \in \mathcal{K}_c^n \text{ and } \widetilde{W}_i(Q^*) = \omega_n\}}\Big)\\
&\le \phi_2\Big(\frac{(n\widetilde{W}_i(K))^{n-i}\phi_1(\widetilde{W}_i(K)^{-1})^{-1}}{\inf\{n^{n-i}\widetilde{W}_{\phi_1,i}(K,Q)^{n-i} : Q \in \mathcal{K}_c^n \text{ and } \widetilde{W}_i(Q^*) = \omega_n\}}\Big)\\
&= \phi_2\Big(\frac{(n\widetilde{W}_i(K))^{n-i}}{\widetilde{G}_{\phi_1,i}(K)^{n-i}\phi_1(\widetilde{W}_i(K)^{-1})}\Big).
\end{aligned}
$$

Notice that $\phi_1 \le \phi_2 \in \Phi$ are strictly increasing on $(0,\infty)$, there are inverse functions ϕ_1^{-1}, ϕ_2^{-1} and $\phi_1^{-1} \ge \phi_2^{-1}$ are strictly increasing on $(0,\infty)$ as well. Then, we have

$$
\phi_2^{-1}\Big(\frac{(n\widetilde{W}_i(K))^{n-i}}{\widetilde{G}_{\phi_2,i}(K)^{n-i}\phi_2(\widetilde{W}_i(K)^{-1})}\Big) \le \phi_1^{-1}\Big(\frac{(n\widetilde{W}_i(K))^{n-i}}{\widetilde{G}_{\phi_1,i}(K)^{n-i}\phi_1(\widetilde{W}_i(K)^{-1})}\Big).
$$

This gives inequality (9). □

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