Filomat 32:14 (2018), 5071–5082
https://doi.org/10.2298/FIL1814071G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Phillips Operators Preserving Arbitrary Exponential Functions, e^{at}, e^{bt}

Vijay Gupta[a], Antonio-Jesús López-Moreno[b]

[a]*Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi-110078, India*
[b]*Departamento de Matemáticas, Universidad de Jaén, Campus Las Lagunillas, 23071-Jaén, Spain, ORCID: 0000-0001-6701-7550*

Abstract. In this paper we present a modification of the Phillips operators which reproduces the functions e^{at}, e^{bt}, $a, b \in \mathbb{R}$. We study the moments and basic approximation properties for these new operators. For this purpose we adapt several already known result to the particular features of the new sequence that we propose here.

1. Introduction

The Phillips operators [14] are defined by

$$S_n(f,x) = n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)\,dt + e^{-nx}f(0),$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

These operators preserve constant as well as linear functions. Some approximation results for the Phillips operators were discussed in [9, 10, 12, 17, 18]. Inspired by the work of Boyanov and Veselinov [6], and Holhoş [13], recently Acar et al. [3] modified the well known Szász-Mirakyan operators so as to preserve the function e^{2at} (see also [5] for an analogous modification of Baskakov-Szász-Stancu type operators) and Acar et al. [2] introduced the corresponding version of the Szász-Mirakyan operators with e^{at} and e^{2at} as preserved functions. In the same way, Gupta and Tachev [11] considered also Phillips type operators fixing e^{-t} and e^{At}, $A \in \mathbb{R}$, but not both together. Motivated by this last recent work but also by the rest of the cited ones, we propose here a sequence that summarizes many of the cases that, as mentioned, several authors studied in different papers. For this purpose, we define a modification of the Phillips operators so as to fix both e^{at} and e^{bt} for any two real numbers, different or not, $a, b \in \mathbb{R}$, and we analyze several aspects of its approximation behavior.

2010 *Mathematics Subject Classification*. Primary 41A25, 41A36
Keywords. Phillips operators, exponential functions, moments, quantitative results, weighted approximation, asymptotic expansion, Voronovskaja formula
Received: 26 March 2018; Revised: 12 August 2018; Accepted: 23 August 2018
Communicated by Dragan Djordjević
The second author is partially supported by Junta de Andalucía Research Project FQM-178, by Research Projects DGA (E-64), MTM2015-67006-P (Spain) and by FEDER founds.
Email addresses: vijaygupta2001@hotmail.com (Vijay Gupta), ajlopez@ujaen.es (Antonio-Jesús López-Moreno)

Let us first construct the new sequence. In order to preserve the functions e^{at} and e^{bt}, our starting point is the general sequence

$$P_n(f,x) = n\sum_{k=1}^{\infty} e^{-n\alpha_n(x)} \frac{(n\beta_n(x))^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^{k-1}}{(k-1)!} f(t)\,dt + e^{-n\alpha_n(x)} f(0), \tag{1}$$

$x \in [0,\infty)$. And now we will try to find suitable values for $\alpha_n(x)$ and $\beta_n(x)$ to reach the desired preservation behavior. By simple computation, we get

$$e^{ax} = e^{-n\alpha_n(x)} e^{n^2\beta_n(x)/(n-a)} = e^{n\left[\frac{n\beta_n(x)}{n-a} - \alpha_n(x)\right]}$$

and

$$e^{bx} = e^{-n\alpha_n(x)} e^{n^2\beta_n(x)/(n-b)} = e^{n\left[\frac{n\beta_n(x)}{n-b} - \alpha_n(x)\right]}.$$

Solving these two equations,

$$\alpha_n(x) = \frac{(n-(a+b))}{n} x, \qquad \beta_n(x) = \frac{(n-a)(n-b)}{n^2} x.$$

Now, substituting these values in (1), our modified Phillips operators take the following form

$$P_n(f,x) = n\sum_{k=1}^{\infty} e^{-(n-(a+b))x} \frac{\left(\frac{(n-a)(n-b)x}{n}\right)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^{k-1}}{(k-1)!} f(t)\,dt + e^{-(n-(a+b))x} f(0), \quad x \in [0,\infty). \tag{2}$$

It is obvious that these new operators are linear and positive whenever $n > a, b$ so that we will assume this condition from now on. Moreover, they preserve the functions e^{at} and e^{bt}, but due to that fact the preservation of constants will not be possible unless any of the parameters a or b vanishes.

We would like to stress the announced fact that this sequence include several interesting situations as particular cases. The Korovkin theorem for linear positive operators establishes that the number of independent preserved functions is two (the space of preserved functions has dimension two) but depending on the parameters a, b these functions can vary. In this way:

1. for $a = \lambda$ and $b = 2\lambda$ the space of preserved functions is spanned by $\varphi = e^{\lambda t}$ and $\varphi^2 = e^{2\lambda t}$ and, therefore, it is $\langle \varphi, \varphi^2 \rangle$ in the same line that we find in [2];
2. for $a \neq 0$ and $b = 0$ we have constants preservation and the preserved space is $\langle 1, e^{at} \rangle$ (obviously for $a = -1$, it is $\langle 1, e^{-t} \rangle$) and it is straightforward that we obtain both of the sequences of operators defined in [11] as a particular case;
3. for $a = b$ the space is $\langle e^{at}, e^{at} t \rangle$ as we will see in Lemma 2.3;
4. and, finally, for $a = b = 0$ we have $\langle 1, t \rangle$ and we recover the Phillips sequence inside our new family of operators.

Therefore, the explicit expressions for moments and mixed exponential/polynomial moments, and the convergence results in different spaces that we obtain in this work are valid for many examples that can be found in the more recent literature and offer in this way a more deep and unified perspective of their approximation properties.

Notice that throughout the paper, t denotes the identity map $t : [0,\infty) \ni x \mapsto t(x) = x \in [0,\infty)$ meanwhile x is a general fixed point of $[0,\infty)$. Therefore we will use t to write functional expressions and x for pointwise formulas. Moreover, for any operator $L : E_1 \subseteq \mathbb{R}^{[0,\infty)} \to E_2 \subseteq \mathbb{R}^{[0,\infty)}$ and $f \in E_1$, $L(f)$ or Lf stand for the image function for f and $L(f,x)$ is the evaluation of such a function at x.

2. Moments and auxiliary results

From the definition of P_n and S_n it is immediate that both sequences of operators are connected by means of the identity

$$P_n(f,x) = e^{\frac{abx}{n}} S_n(f,\beta x), \tag{3}$$

where for short we denote $\beta = \beta_n = \frac{(n-a)(n-b)}{n^2}$. In [12] Heilmann and Tachev obtain several results for the Phillips operators that can be transferred to P_n through identity (3). Thus, from [12, Lemma 2.1], the relation yields the following lemma.

Lemma 2.1. *For any $r \in \mathbb{N}_0 = \{0,1,2,\ldots\}$,*

$$P_n(t^r) = e^{\frac{abt}{n}} \sum_{j=0}^{r} \binom{r-1}{j-1} \frac{r!}{j!} n^{j-r} (\beta t)^j .$$

Here and from now on we adopt the usual convention that $\binom{i}{i} = 1$ for $i \in \mathbb{Z}$ and $\binom{i}{j} = 0$ for any $i, j \in \mathbb{Z}$ with $i < j$. In particular, it is straightforward that

$$P_n(1) = e^{\frac{abt}{n}}, \quad P_n(t) = e^{\frac{abt}{n}} \beta t, \quad P_n(t^2) = e^{\frac{abt}{n}} (\frac{2\beta t}{n} + \beta^2 t^2).$$

The same result of [12] also allows us to obtain the following expression for central moments.

Lemma 2.2. *For $m \in \mathbb{N}_0$,*

$$\mu_{n,m}(x) = e^{\frac{abx}{n}} \sum_{j=0}^{[\frac{m}{2}]} n^{j-m} (\beta x)^j \sum_{\nu=2j}^{m} \binom{m}{\nu} \binom{\nu-j-1}{j-1} \frac{\nu!}{j!} (n(\beta-1)x)^{m-\nu} ,$$

where we denote $\mu_{n,m}(x) = P_n((t-x)^m, x)$.

Proof. From (3) we have that

$$\mu_{n,m}(x) = e^{\frac{abx}{n}} S_n((t-x)^m, \beta x) = e^{\frac{abx}{n}} S_n((t - \beta x + (\beta-1)x)^m, \beta x)$$

and now binomial's formula and [12, Lemma 2.1] give us the result after properly arranging the order of the sums. □

This lemma yields the formulae

$$\begin{aligned} \mu_{n,0}(x) &= e^{\frac{abx}{n}}, \qquad \mu_{n,1}(x) = e^{\frac{abx}{n}} (\beta-1)x, \qquad \mu_{n,2}(x) = e^{\frac{abx}{n}} \left((\beta-1)^2 x^2 + \frac{2\beta x}{n} \right), \\ \mu_{n,3}(x) &= e^{\frac{abx}{n}} \left((\beta-1)^3 x^3 + \frac{6\beta(\beta-1)x^2}{n} + \frac{6\beta x}{n^2} \right), \\ \mu_{n,4}(x) &= e^{\frac{abx}{n}} \left((\beta-1)^4 x^4 + \frac{12(\beta-1)^2 \beta x^3}{n} + \frac{24(\beta-1)\beta x^2 + 12\beta^2 x^2}{n^2} + \frac{24\beta x}{n^3} \right). \end{aligned} \tag{4}$$

Since $\lim_{n\to\infty} n(\beta-1) = -(a+b)$, from the above lemma is is straightforward that $\mu_{n,m}(x) = O\left(n^{-[\frac{m+1}{2}]}\right)$ as it is usual for all classical sequences of linear positive operators and, moreover,

$$\lim_{n\to\infty} n^{[\frac{m+1}{2}]} \mu_{n,m}(x) = \begin{cases} \frac{m!}{(\frac{m}{2})!} x^{\frac{m}{2}}, & \text{if } m \text{ is even,} \\ \frac{m!}{(\frac{m-1}{2})!} x^{\frac{m-1}{2}} \left(-(a+b)x + \frac{m-1}{2} \right), & \text{if } m \text{ is odd.} \end{cases} \tag{5}$$

In particular,

$$\lim_{n\to\infty} \mu_{n,0}(x) = 1, \quad \lim_{n\to\infty} n\mu_{n,1}(x) = -(a+b)x, \quad \lim_{n\to\infty} n\mu_{n,2}(x) = 2x. \tag{6}$$

In this manner we see that it is possible to find suitable formulas for polynomial monomials and moments but let us go further and study closed expressions for non-polynomial functions and moments of the type $e^{ct}t^r$ and $e^{ct}(t-x)^m$.

Lemma 2.3. *For any $c \in \mathbb{R}$, $r \in \mathbb{N}_0$ and $n > c$, we have*

$$P_n\left(e^{ct}t^r\right) = e^{\frac{c(n-a-b)+ab}{n-c}t} \sum_{j=0}^{r} \frac{r!}{j!}\binom{r-1}{j-1} \frac{(n-a)^j(n-b)^j}{(n-c)^{j+r}} t^j.$$

Proof. Since $e^{(\theta+c)t} = \sum_{r=0}^{\infty} e^{ct}t^r \frac{\theta^r}{r!}$ we can use the idea of the moments generating functions to deduce that $P_n\left(e^{ct}t^r\right) = \frac{d^r}{d\theta^r}|_{\theta=0} P_n(e^{(\theta+c)t})$. It is a simple computation that

$$P_n(e^{\theta t}) = e^{\frac{\theta(n-a-b)+ab}{n-\theta}t} \tag{7}$$

and from here it is also immediate that

$$P_n(e^{(\theta+c)t}) = e^{\frac{c(n-a-b)+ab}{n-c}t} e^{\frac{(n-a)(n-b)}{n-c}\frac{\theta}{n-c-\theta}t}.$$

Now if we combine the expansion for the exponential function $e^z = \sum_{j=0}^{\infty} z^j/j!$ with $\left(\frac{z}{1-z}\right)^j = \sum_{r=j}^{\infty}\binom{r-1}{j-1} z^r$ and we rearrange the order of the sums we obtain

$$e^{\frac{(n-a)(n-b)}{n-c}\frac{\theta}{n-c-\theta}t} = \sum_{r=0}^{\infty}\left(\sum_{j=0}^{r} \frac{1}{j!}\binom{r-1}{j-1}\frac{(n-a)^j(n-b)^j}{(n-c)^{j+r}}t^j\right)\theta^r$$

from which we deduce the result by differentiating with respect to θ as we indicated before. □

It is clear that Lemma 2.2 can be also deduced from Lemma 2.3.
In particular, for $c = a$ we have

$$P_n\left(e^{at}t^r\right) = e^{at}\sum_{j=0}^{r} \frac{r!}{j!}\binom{r-1}{j-1}\frac{(n-b)^j}{(n-a)^r}t^j,$$

which proves the fact announced by item (3) on page 5072, namely $P_n(e^{at}t) = e^{at}t$ for $a = b$. The analogous formula for $c = b$ also holds.
From this last lemma it is immediate that

$$P_n\left(e^{ct}\right) = e^{\frac{c(n-a-b)+ab}{n-c}t}$$

$$P_n\left(e^{ct}t\right) = e^{\frac{c(n-a-b)+ab}{n-c}t}\frac{(n-a)(n-b)}{(n-c)^2}t,$$

$$P_n\left(t^2e^{ct}\right) = e^{\frac{c(n-a-b)+ab}{n-c}t}\left[\left(\frac{(n-a)(n-b)}{(n-c)^2}\right)^2 t^2 + \frac{2(n-a)(n-b)}{(n-c)^3}t\right],$$

from which we also conclude that

$$P_n\left(e^{ct}(t-x)^2, x\right) = e^{\frac{c(n-a-b)+ab}{n-c}x}\left[\frac{2(n-a)(n-b)}{(n-c)^3}x + \frac{\left(c^2+n(a+b-2c)-ab\right)^2}{(n-c)^4}x^2\right]. \tag{8}$$

Furthermore, (8) means that $P_n\left(e^{ct}(t-x)^2, x\right) = O(n^{-1})$ for all $x \in [0,\infty)$. Actually we can extend this last property to check that these kind of moments also perform the usual behavior for polynomial moments in the sense that we show in the following result.

Corollary 2.4. *For $m \in \mathbb{N}_0$,*

$$P_n\left(e^{ct}(t-x)^m, x\right) = e^{cx}(1+x^m)O_x(n^{-\left[\frac{m+1}{2}\right]}).$$

Furthermore, for $a \leq c \leq b$ the infinitesimal expression is uniform on $x \in [0,\infty)$.

Proof. From Lemma 2.3 it is clear that we can write

$$P_n(e^{ct}(t-x)^m, x) = e^{\frac{c(n-a-b)+ab}{n-c}x} \sum_{i=0}^{m} q_i(x)(n-c)^{-i} \tag{9}$$

for certain polynomials q_i with degree m. Since P_n are linear positive operators, it a simple matter that a Schwartz type identity is satisfied and for any $f, g : [0,\infty) \to \mathbb{R}$ in the domain of P_n, we have that

$$\left|P_n(f \cdot g)\right| \leq \left(P_n(f^2)\right)^{\frac{1}{2}} \left(P_n(g^2)\right)^{\frac{1}{2}}.$$

Therefore, if we take $f = e^{ct}$, $g = (t-x)^m$, and we use Lemma 2.2 and (7), with $\theta = 2c$, it follows that

$$\left|P_n(e^{ct}(t-x)^m, x)\right| \leq \left(e^{\frac{2c(n-a-b)+ab}{n-2c}x}\right)^{\frac{1}{2}} O_x(n^{\frac{m}{2}}) = O_x(n^{\frac{m}{2}}).$$

But then, in (9), we have that $q_0(x) = \cdots = q_{\left[\frac{m-1}{2}\right]}(x) = 0$ and we finally obtain the result if we take into account that we can write

$$e^{\frac{c(n-a-b)+ab}{n-c}x} = e^{cx}e^{\frac{c(c-a-b)+ab}{n-c}x}.$$

Moreover, for $a \leq c \leq b$, in the last expression, the coefficient in the exponent of the second exponential satisfies $c(c-a-b)+ab \leq 0$ and therefore such an exponential is bounded on $[0,\infty)$ so that in this case the infinitesimal expression of the lemma is uniform on $[0,\infty)$. □

We finish this section including a technical result that will be useful in the rest of the paper.

Lemma 2.5. *Given $\lambda \in \mathbb{R}$, $c > 0$ and $a_0 < c$, for $N > 0$ big enough there exists a constant $K \geq 1$ such that*

$$\left\| \frac{e^{a_0 t} - e^{(a_0 + \frac{\lambda}{N})t}}{e^{ct}} \right\|_{[0,\infty)} \leq \frac{K}{e} \frac{|\lambda|}{(c-a_0)N - \lambda}.$$

Moreover, for $\lambda \geq 0$ we can take $K = 1$.

Proof. It is not difficult to check that $f = \frac{e^{a_0 t} - e^{(a_0+\frac{\lambda}{N})t}}{e^{ct}}$ fulfills $f(0) = 0$ and, for N big enough, $\lim_{x\to\infty} f(x) = 0$. Therefore, since f is not constant (for $\lambda \neq 0$), the function has to have a relative extreme in $(0,\infty)$ which, as a matter of fact, is unique and we can easily compute as $x_0 = \frac{N}{\lambda}\log\left(\frac{(c-a_0)N}{(c-a_0)N-\lambda}\right)$. Then

$$\|f\|_{[0,\infty)} = \left|f(x_0)\right| = \frac{|\lambda|}{(c-a_0)N-\lambda}\left(1 - \frac{\lambda}{(c-a_0)N}\right)^{\frac{N}{\lambda}(c-a_0)}.$$

But the last factor is equal to $g(\frac{\lambda}{(c-a_0)N})$ where $g = (1-t)^{\frac{1}{t}}$ which is a continuos decreasing function with $\lim_{x\to 0} g(x) = \frac{1}{e}$ so that we can easily finish the proof. □

3. Convergence analysis

In this section we will briefly study the approximation properties of the operators P_n calling attention to the fact that they present important differences depending on the values of the parameters a and b. First we analyze the uniform convergence on $[0,\infty)$ that cannot be guaranteed for all values of a, b. In the general situation, even in those cases when the uniform convergence fails, we can study weighted approximation properties and therefore that is our second analysis.

Let us denote by $C^*[0,\infty)$ the subspace of real-valued continuous functions which possess finite limit at infinity endowed with the uniform norm. The uniform convergence on noncompact domains for sequences of linear positive operators has been studied in several papers [6, 13]. In particular in [13] Holhoş uses the classical argument of Shisha and Mond [15] to obtain a quantitative estimation in terms of moduli of continuity. A rapid analysis shows that we can adapt the idea of Holhoş to be valid not only for test functions e^{-t}, e^{-2t}, as it appears in [13], but also, in general, for a^{At}, e^{2At}, with any $A<0$. For this purpose we consider the modulus of continuity

$$\omega^*(f,\delta)=\sup_{\substack{x_1,x_2\geq 0\\ \left|e^{Ax_1}-e^{Ax_2}\right|\leq\delta}}\left|f(x_1)-f(x_2)\right|$$

and then Theorem 2.1 of [13] can be rewritten as we show below.

Theorem A ([13, Theorem 2.1]). *Given $A<0$, if $L_n: C^*[0,\infty)\to C^*[0,\infty)$ is a sequence of linear positive operators with*

$$\|L_n(1)-1\|_{[0,\infty)}=\alpha_n,\qquad \|L_n(e^{At})-e^{At}\|_{[0,\infty)}=\beta_n,\qquad \|L_n(e^{2At})-e^{2At}\|_{[0,\infty)}=\gamma_n,$$

where α_n,β_n and γ_n tend to zero as n goes to infinity, then

$$\|L_nf-f\|_{[0,\infty)}\leq\alpha_n\|f\|_{[0,\infty)}+(2+\alpha_n)\omega^*(f,\sqrt{\alpha_n+2\beta_n+\gamma_n}),$$

for every function $f\in C^*[0,\infty)$.

With the aid of the result by Holhoş, we study the uniform convergence of P_n on $C^*[0,\infty)$.

Theorem 3.1.

1. *For $ab\neq 0$, P_n is not an approximation method in $\left(C^*[0,\infty),\|\cdot\|_{[0,\infty)}\right)$.*
2. *For $b=0$, we have that*

$$\|P_nf-f\|_{[0,\infty)}\leq 2\omega^*(f,\sqrt{\frac{c_a}{n-a}}),$$

for certain constant c_a that for $a<0$ can be taken as $c_a=|a|$ and for $a\geq 0$ as $c_a=3a+4$.

Proof. 1. From Lemma 1 we know that $P_n(1)=e^{\frac{abt}{n}}$ and therefore, for $ab\neq 0$,

$$\|P_n(1)-1\|_{[0,\infty)}=\|e^{\frac{abt}{n}}-1\|_{[0,\infty)}=\begin{cases}+\infty, & \text{if } ab>0,\\ 1, & \text{if } ab<0,\end{cases}$$

and we do not have uniform convergence for the constant function $1\in C^*[0,\infty)$.

2. In this case, $P_n(1)=1$ and then, in Theorem A, $\alpha_n=0$. On the other hand, from (7) we also have that

$$P_n(e^{\theta t})=e^{\frac{\theta(n-a-b)+ab}{n-\theta}t}=e^{\left(\theta+\frac{\theta^2-\theta(a+b)+ab}{n-\theta}\right)t}.\tag{10}$$

If $a < 0$, in Theorem A we can take $A = a$ and then $\beta_n = \|P_n(e^{at}) - e^{at}\|_{[0,\infty)} = 0$. To compute γ_n, from (10) we have

$$P_n(e^{2at}) = e^{\left(2a+\frac{2a^2-ab}{n-2a}\right)t}$$

and now, in Lemma 2.5 we take $c = 0$, $a_0 = 2a$, $\lambda = 2a^2 - ab$ and $N = n - 2a$ to obtain

$$\gamma_n = \|P_n(e^{2at}) - e^{2at}\|_{[0,\infty)} \leq \frac{1}{e}\frac{|a|}{n-a}.$$

If $a \geq 0$, we choose $A = -1$ and, again, in a similar way, combining Lemma 2.5 with (10),

$$\beta_n = \|P_n(e^{-t}) - e^{-t}\|_{[0,\infty)} \leq \frac{1}{e}\frac{a+1}{n-a}, \qquad \gamma_n = \|P_n(e^{-2t}) - e^{-2t}\|_{[0,\infty)} \leq \frac{1}{e}\frac{a+2}{n-a}.$$

□

This last result shows that not always the operators P_n present convergence for the uniform norm. However the situation is different if we consider weighted approximation. For an increasing weight function $\rho : [0,\infty) \to \mathbb{R}$, such that $\lim_{x\to\infty} \rho(x) = +\infty$ and $\rho(0) > 0$, we consider the spaces $B_\rho([0,\infty)) = \{f : [0,\infty) \to \mathbb{R} : |f(x)| \leq M_f \rho(x), x \geq 0\}$, $C_\rho([0,\infty)) = C([0,\infty)) \bigcap B_\rho([0,\infty))$ and $C^k_\rho([0,\infty)) = \{f \in C_\rho([0,\infty)) : \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k_f$ exists and it is finite$\}$ endowed with the norm $\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ (see [2]). The convergence of sequences of linear positive operators for this last norm in these spaces of functions has been analyzed in several papers. In [7] Bustamante et al. offer us a nice survey on the topic that, moreover, includes new results and many interesting comments. In particular in [7, §4.] the convergence for $\|\cdot\|_\rho$ in $C^k_\rho([0,\infty))$ is studied by means of several theorems that allow to determine whether a set of functions $\{\gamma_0, \gamma_1, \gamma_2\}$ is a Korovkin system for $(C^k_\rho([0,\infty)), \|\cdot\|_\rho)$; that is to say, the convergence for $\gamma_0, \gamma_1, \gamma_2$ implies the convergence for all function in $C^k_\rho([0,\infty))$. Since, many of the classical linear positive operators preserve linear functions, a typical choice for the first two functions of a Korovkin system is $\gamma_0 = 1$ and $\gamma_1 = t$; in [7, Theorem 4.5] the functions γ_2 that along with $1, t$ form a Korovkin system are characterized.

Theorem B ([7, Theorem 4.5]). *Consider a weight function ρ such that $\lim_{x\to\infty} 1/\rho(x) = \lim_{x\to\infty} x/\rho(x) = 0$. Given the function $\gamma_2 \in C^k_\rho([0,\infty))$, the following assertions are equivalent:*

(i) *$\{1, t, \gamma_2\}$ is a Korovkin system for $(C^k_\rho([0,\infty)), \|\cdot\|_\rho)$.*

(ii) *$\{1, t, \gamma_2\}$ is a Chebyshev systems on $[0,\infty)$ (that is to say, on every finite subinterval of $[0,\infty)$) and*

$$\lim_{x\to\infty} \gamma_2(x)/\rho(x) \neq 0.$$

Notice that the notation for the weight function in [7] is slightly different from the one that we use here since Bustamante considers the inverse weights but the translation is simple and direct. By means of this last result we can prove the following theorem for our operators.

Theorem 3.2. *If $a \leq b$ and $b > 0$, for any $r \in \mathbb{N}_0$ and $\rho_r = e^{bt}(1+t^r)$ we have that*

$$\|P_n f - f\|_{\rho_r} \to 0$$

for all $f \in C^k_{\rho_r}([0,\infty))$.

Proof. From Lemma 2.1 we know the explicit expression for both $P_n(1)$ and $P_n(t)$. Then, on the one hand, in Lemma 2.5 we take $a_0 = 0$, $c = b$, $\lambda = ab$ and $N = n$ to obtain

$$\|P_n(1) - 1\|_{\rho_0} = \left\|\frac{1 - e^{\frac{ab}{n}t}}{2e^{bt}}\right\|_{[0,\infty)} \leq \frac{1}{2}\frac{K}{e}\frac{|a|}{n-a}.$$

On the other hand, as $b > 0$, we can take ε with $0 < \varepsilon < b$ and applying again Lemma 2.5, now with $c = b - \varepsilon$, the definition of β of page 5073 and a simple bound for $\frac{t}{e^{\varepsilon t}}$, for n big enough we obtain

$$\|P_n(t) - t\|_{\rho_0} = \left\| \frac{t - e^{\frac{ab}{n}t}\beta t}{2e^{bt}} \right\|_{[0,\infty)} \leq \frac{1}{2} \left\| \left(\frac{1 - e^{\frac{ab}{n}t}}{e^{(b-\varepsilon)t}}\beta + \frac{1-\beta}{e^{(b-\varepsilon)t}} \right) \frac{t}{e^{\varepsilon t}} \right\|_{[0,\infty)}$$
$$\leq \frac{1}{2}\left(\frac{K}{e} \frac{|ab|}{(b-\varepsilon)n - ab} \frac{(n-a)(n-b)}{n^2} + \frac{(a+b)n - ab}{n^2} \right) \frac{1}{\varepsilon e}.$$

Accordingly, in the conditions of the theorem we always have convergence for 1 and t in the weighted norm $\|\cdot\|_{\rho_0}$. Of course, it is evident that we also have convergence for both functions for $\|\cdot\|_{\rho_r}$ for any $r = 1, 2, \ldots$

Let us study now the convergence for ρ_r with $\|\cdot\|_{\rho_r}$. From Lemma 2.3 it is not difficult to deduce that

$$\frac{P_n(\rho_r) - \rho_r}{\rho_r} = \frac{\sum_{j=0}^{r-1} \frac{r!}{j!}\binom{r-1}{j-1}\frac{(n-a)^j}{(n-b)^r} t^j + \left(\left(\frac{n-a}{n-b}\right)^r - 1 \right) t^r}{1 + t^r}$$

and then we conclude that for certain $K_1 > 0$

$$\|P_n(\rho_r) - \rho_r\|_{\rho_r} \leq \frac{K_1}{n}$$

and again we have convergence (as $P_n(e^{bt}) = e^{bt}$, the case $r = 0$ was anyway obvious).

Finally since the wronskian

$$W(1, t, \rho_r) = e^{bt}\left(r(r-1)t^{r-2} + 2brt^{r-1} + b^2(1 + t^r) \right)$$

is non vanishing for all cases, we have that, for $r \in \mathbb{N}_0$, $\{1, t, \rho_r\}$ is an extended Chebyshev system. It is evident that if we take $\rho = \rho_r$ all the conditions of Theorem B are fulfilled and we thus finish the proof. □

Remark 3.3. *Since our operators preserve e^{at} and e^{bt}, it could have been more natural to take $\{1, a^{at}, e^{bt}\}$ as Korovkin system because no additional computation is necessary to prove the convergence of these two functions. Actually, essentially with the same proof, we can also propose the following modification of Theorem B.*

Theorem $\tilde{\mathbf{B}}$. *Consider two functions $\gamma_0, \gamma_1 \in C^k_\rho([0,\infty))$ such that $\lim_{x\to\infty} \gamma_s(x)/\rho(x) = 0$, $s = 0, 1$. Given a third function $\gamma_2 \in C^k_\rho([0,\infty))$, the following assertions are equivalent:*

(ĩ) *$\{\gamma_0, \gamma_1, \gamma_2\}$ is a Korovkin system for $(C^k_\rho([0,\infty)), \|\cdot\|_\rho)$.*

(ĩi) *$\{\gamma_0, \gamma_1\}$ and $\{\gamma_0, \gamma_1, \gamma_2\}$ are Chebyshev systems on $[0,\infty)$ and $\lim_{x\to\infty} \gamma_2(x)/\rho(x) \neq 0$.*

The only important difference with respect to the proof by Bustamante we need now is that instead of the determinant that we find in the last line of the proof of [7, Theorem 4.5] we will have this other one

$$\begin{vmatrix} \gamma_0(x1) & \gamma_0(x_2) \\ \gamma_0(x_1) & \gamma_1(x_2) \end{vmatrix},$$

suitably multiplied by weight functions, which does not vanish due to the conditions included in **(ĩi)** *and then we reach the conclusion as in [7].*

Anyway, although this version allows to prove many of the cases of Theorem 3.2 with less computations, some others (for instance, $a = b$) finally need the test system $\{1, t, e^{bt}(1 + t^r)\}$ considered in the proof above and for this reason the initial version of Theorem B allows to study all possible situations in a more unified way.

In the case $a \leq b \leq 0$, for all weight function $\rho = e^{ct}$ with $c > 0$, from (10), since $c^2 - (a+b)c + ab > 0$, we have that

$$\left\| \frac{P_n\rho - \rho}{\rho} \right\|_{[0,\infty)} = \left\| e^{\frac{c^2-(a+b)c+ab}{n-c}t} - 1 \right\|_{[0,\infty)} = +\infty$$

and then $\|P_n(\rho) - \rho\|_\rho = +\infty$ for all n. Therefore P_n does not converge for any increasing exponential weight function and the problem makes no sense in this case.

Of course, in theorems 3.1 and 3.2, the role of a and b can be easily permuted.

4. Asymptotic behavior

To finish with the description of the approximation properties of the operators P_n we analyze the asymptotic behavior of the sequence. Although we showed several cases for which we do not have uniform or weighted convergence, from the definition of the operators it is straightforward that for any locally integrable function f of exponential growth (that is to say $\left|f\right| \leq Ke^{ct}$ for certain $K, c \geq 0$), $P_n(f,x)$ is defined for all $x \in [0,\infty)$ and, moreover, if f is continuous at x then $P_n(f,x) \to f(x)$ (actually, Lemma 4.1, for $q = 0$, could be considered a proof of this fact). As a consequence, for any continuous function of exponential growth we will always have pointwise convergence and now we prove that it is possible to obtain asymptotic expansions for that class. In particular we show the explicit expression for the expansions of order 1 (that is to say, a Voronovskaja formula) and order 2.

The classical result by Sikkema [16] allows to establish asymptotic expansions for functions of polynomial growth for a wide class of linear positive operators. We will make use of this result but in order to obtain expansions valid for functions of exponential growth we need to apply a localization argument so we first prove the following localization result.

Lemma 4.1. *Let q be an even number and let $f : [0,\infty) \to \mathbb{R}$ be a locally integrable function such that, for certain $K, c \geq 0$, $|f| \leq Ke^{ct}$ on $[0,\infty)$ and consider $x \in [0,\infty)$ such that f is differentiable of order q at x with $f^{(i)}(x) = 0$, $i = 0, \ldots, q$. Then*

$$P_n(f,x) = o(n^{-\frac{q}{2}}).$$

Proof. In the conditions of the lemma, the q order Taylor series expansion of f at x is

$$f = h(t-x)(t-x)^q,$$

where $h : [-x,\infty) \to \mathbb{R}$ is a function continuous at 0 with $h(0) = 0$. Therefore, for any $\varepsilon > 0$ we can find $\delta > 0$ such that for all $x_1 \in I_\delta = (x-\delta, x+\delta) \cap [0,\infty)$ we have $|h(x_1 - x)| \leq \varepsilon$. It is also obvious that using the bound that the statement of the lemma establishes for f we can find a constant $K_\varepsilon > 0$ and $r_1 > q$ such that $\left|f(x_1)\right| \leq K_\varepsilon e^{cx_1}(x_1 - x)^{r_1}$ for all $x_1 \in [0,\infty) - I_\delta$. Therefore, since P_n is positive,

$$\left|f\right| \leq \varepsilon(t-x)^q + K_\varepsilon e^{ct}(t-x)^{r_1} \Rightarrow \left|P_n f\right| \leq \varepsilon P_n((t-x)^q) + K_\varepsilon P_n\left(e^{ct}(t-x)^{r_1}\right).$$

But from Lemma 2.2, Corollary 2.4 and (5) it is immediate that

$$\lim_{n\to\infty} \left|n^{\frac{q}{2}} P_n(f,x)\right| \leq \varepsilon \frac{q!}{\left(\frac{q}{2}\right)!} x^{\frac{q}{2}}$$

and since ε is an arbitrary positive number we finish the proof. □

Theorem 4.2. *Let q be an even number and let $f : [0,\infty) \to \mathbb{R}$ be a locally integrable function such that, for certain $K, c \geq 0$, $|f| \leq Ke^{ct}$ on $[0,\infty)$ and consider $x \in [0,\infty)$ such that f is differentiable of order q at x. Then*

$$P_n(f,x) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} P_n\left((t-x)^i, x\right) + o(n^{-\frac{q}{2}}).$$

Proof. [16, Theorem 3] proves that the theorem holds for any $g : [0,\infty) \to \mathbb{R}$ with polynomial growth. For a suitable $\delta > 0$, let us consider a neighborhood of x, $I = (x-\delta, x+\delta) \cap [0,\infty)$ and take a locally integrable function $\tilde{g} : [0,\infty) \to \mathbb{R}$ such that $\tilde{g}$ is bounded on $[0,\infty)$ and $\tilde{g}|_I = f|_I$. $\tilde{g}$ is under the conditions of [16, Theorem 3] so that we have an asymptotic expansion for it. Moreover, $\tilde{g} - f$ is under the conditions of Lemma 4.1 and accordingly $P_n(\tilde{g} - f, x) = o(n^{-\frac{q}{2}})$. Now, since $\tilde{g}^{(i)}(x) = f^{(i)}(x)$, for $i = 0, \ldots, q$, the asymptotic expansion that [16, Theorem 3] yields for $\tilde{g}$ at x is actually valid for f and this ends the proof. □

As the expansion for $e^{\frac{abt}{n}}$ is known, Lemma 2.2, expressions (4) along with the last theorem allow to obtain explicit asymptotic expansions of any order. Thus, given f, locally integrable with $|f| \le Ke^{ct}$, for $q = 2$, if f is twice differentiable at x we have

$$P_n(f,x) = \underbrace{f(x) + x\left(abf(x) - (a+b)f'(x) + f''(x)\right)\frac{1}{n}}_{=E_{f,1,n}(x)} + o(n^{-1}), \tag{11}$$

or for $q = 4$ and f four times differentiable at x,

$$P_n(f,x) = E_{f,1,n}(x) + \frac{x}{2}\Big(a^2b^2xf(x) + 2ab(1-(a+b)x)f'(x) + \Big((2ab + (a+b)^2)x - 2(a+b)\Big)f''(x) + 2(1-(a+b)x)f'''(x) + xf^{(4)}(x)\Big)\frac{1}{n^2} + o(n^{-2}).$$

With our last result we try to offer a quantitative estimate of the Voronovskaja formula (11) in the line of many papers that deal with the problem of expressing the remainder term of the asymptotic formulae in terms of moduli of continuity. Here we have to tackle the question of the non-compactness of the domain of the operators P_n and therefore again we need to consider weighted norms and moduli with the purpose of establishing results for the wide class of functions for which the sequence is an approximation method. This kind of problem has been studied by several authors for concrete sequences of operators (see for instance [1]) and also for general sequences in the case of polynomial weights [4]. In [19], Gupta et al. extend already known results in the topic to the case of functions of exponential growth in the interval $[0,\infty)$ establishing quantitative expressions in terms of the modulus of continuity with exponential weight defined as

$$\omega_1(f,\delta,a) = \sup_{h\le\delta, 0\le x<\infty} |f(x) - f(x+h)|e^{-ax}.$$

They also consider the spaces $\mathrm{Lip}(\alpha, a)$, $0 < \alpha \le 1$, that consist of all function such that $\omega_1(f,\delta,a) \le M\delta^\alpha$ for all $\delta < 1$.

Theorem C ([19, Theorem 1.1]). *Let E be a subspace of $C[0,\infty)$ which contains all continuous functions with exponential growth and let $L_n : E \to C[0,\infty)$ be a sequence of linear positive operators preserving the linear functions. We suppose that for each constant $A > 0$ and fixed $x \in [0,\infty)$ the operators L_n satisfy*

$$L_n\left((t-x)^2e^{At}, x\right) \le C(A,x)\cdot\mu_{n,2}^L(x) \tag{$*$}$$

where $C(A,x)$ is some function depending on A and x, and we denote $\mu_{n,2}^L(x) = L_n\left((t-x)^2, x\right)$.

If in addition $f \in C^2[0,\infty) \cap E$ and $f'' \in \mathrm{Lip}(\alpha, A)$, $0 < \alpha \le 1$, then we have, for $x \in [0,\infty)$,

$$\left|L_n(f,x) - f(x) - \frac{1}{2}f''(x)\mu_{n,2}^L(x)\right| \le \left[e^{2Ax} + \frac{C(A,x)}{2} + \frac{\sqrt{C(2A,x)}}{2}\right]\cdot\mu_{n,2}^L(x)\cdot\omega_1\left(f'', \sqrt{\frac{\mu_{n,4}^L(x)}{\mu_{n,2}^L(x)}}, A\right).$$

We would like to make a couple of remarks about this theorem. First, in the inequality that appears in [19], we find e^{Ax} but from the proof contained in that paper it follows that it should be read e^{2Ax} instead and thus we correct here this missprint including the proper coefficient in the inequality of the theorem. Second, although it is assumed that $f'' \in \mathrm{Lip}(\alpha, A)$, the theorem also holds for any function for which $\omega_1(f'', h, A) \xrightarrow[h \to 0]{} 0$.

We can see that in Theorem C it is supposed that the operators of the sequence preserve linear functions. However this restriction is not essential and the original proof [19] remains valid for a general sequence of linear positive operators if we replace inside the absolute value in the left hand side of the final inequality $-f(x)$ with the terms $-\mu^L_{n,0}(x)f(x) - \mu^L_{n,1}(x)f'(x)$ that in the case of linear preservation simplify to the inequality showed above.

We know that P_n fixes e^{at} and e^{bt} but on the contrary, from Lemma 2.2, it is immediate that no polynomial is preserved. Therefore this last remark is important here. Anyway, once considered the slight modification proposed in the preceding paragraph, the crucial step is to prove that condition $(*)$ holds.

From Lemma 2.2 and (8),

$$\mu^{P_n}_{n,2}(x) = P_n\left((t-x)^2, x\right) = e^{\frac{abx}{n}}\Bigg[\underbrace{2\frac{(n-a)(n-b)}{n^3}}_{=A_n} x + \underbrace{\left(\frac{-(a+b)n + ab}{n^2}\right)^2}_{=B_n} x^2\Bigg],$$

$$P_n\left(e^{ct}(t-x)^2, x\right) = e^{\frac{c(n-a-b)+ab}{n-c}x}\Bigg[\underbrace{\frac{2(n-a)(n-b)}{(n-c)^3}}_{=\tilde{A}_n} x + \underbrace{\frac{\left(c^2 + n(a+b-2c) - ab\right)^2}{(n-c)^4}}_{=\tilde{B}_n} x^2\Bigg].$$

Let us see that we can prove $(*)$ for every $c > 0$.

For any $d_1, d_2 \in \mathbb{R}$, if we consider the constant $K_{d_1,d_2} = \max\{1, |d_2 - d_1 + 1|\}$, it is immediate that $\left|\frac{n-d_1}{n-d_2}\right| \leq K_{d_1,d_2}$ for $n \geq d_2 + 1$.

Using the notation that we have just introduced, for $a + b \neq 0$ it is easy that

$$\tilde{A}_n \leq K^3_{0,c} A_n \quad \text{and} \quad \tilde{B}_n \leq \left(\frac{a+b-2c}{a+b}\right)^2 K^2_{\frac{ab-c^2}{a+b-2c}, \frac{ab}{a+b}} K^4_{0,c} B_n. \tag{12}$$

Therefore we can take $C(a,b,c,x) = \max\{K^3_{0,c}, \left(\frac{a+b-2c}{a+b}\right)^2 K^2_{\frac{ab-c^2}{a+b-2c}, \frac{ab}{a+b}} K^4_{0,c}\} e^{cx}$ for $n \geq N(a,b,c) = \max\{a, b, c, \frac{ab}{a+b}\} + 1$ (in the special case $a + b - 2c = 0$ it is simple to check that $\tilde{B}_n \leq \frac{\left((a-b)K_{0,c}\right)^4}{2^4(ab)^2} B_n$ for $n > \frac{2ab}{a+b}$ and then we accordingly modify the definition of $C(a,b,c,x)$ and $N(a,b,c)$ for this case).

For $a + b = 0$, the inequality in (12) for $\tilde{A}_n$ and A_n also holds and in a similar way

$$\tilde{B}_n \leq 2c^2 K^3_{0,c} K^2_{\frac{c^2-ab}{2c}, c} K_{c,b} A_n$$

and therefore we can take $C(a,b,c,x) = \max\{K^3_{0,c}, 2c^2 K^3_{0,c} K^2_{\frac{c^2-ab}{2c}, c} K_{c,b}\}(1+x)e^{cx}$ for $n \geq N(a,b,c) = \max\{a,b,c\} + 1$.

Thus, in all cases,

$$P_n\left(e^{ct}(t-x)^2, x\right) \leq C(a,b,c,x)\mu^{P_n}_{n,2}(x), \qquad \text{for } n \geq N(a,b,c).$$

This all, together with (4), makes possible to apply Theorem C to obtain a quantitative version of Voronovskaja formula (11).

Theorem 4.3. *Let $f \in C[0,\infty)$ be such that $|f| \le Ke^{ct}$ for certain $K, c \ge 0$. If $f \in C^2[0,\infty) \bigcap E$ and $f'' \in Lip(\alpha, c), 0 < \alpha \le 1$, then, with $C(a,b,c,x)$ and $N(a,b,c)$ as given before, we have for $n > N(a,b,c)$ and $x \in [0,\infty)$ that*

$$\left|P_n(f,x) - e^{\frac{abx}{n}}\left[f(x) + (\beta-1)xf'(x) + \left((\beta-1)^2x^2 + \frac{2\beta x}{n}\right)\frac{f''(x)}{2}\right]\right| \le \left(e^{2cx} + \frac{C(a,b,c,x)}{2} + \frac{\sqrt{C(a,b,2c,x)}}{2}\right)\cdot \mu_{n,2}(x)\cdot \omega_1\left(f'', \sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}, c\right).$$

Of course, in the same way that we did to obtain the explicit expression for $E_{f,1,n}(x)$ in Voronovskaja formula (11), we can use the expansion for $e^{\frac{abx}{n}}$ to rewrite the last inequality in order to give rise to a more precise description of the error for (11). If we call $R_{f,n}(x)$ to the right hand side of the inequality of the last theorem, it is easy that

$$\left|P_n(f,x) - E_{f,1,n}(x)\right| \le R_{f,n}(x) + \left(e^{\frac{abx}{n}} - 1 - \frac{abx}{n}\right)\left|f(x)\right| + \left(e^{\frac{abx}{n}} - 1\right)\left|\mu_{n,1}(x)f'(x) + \mu_{n,2}(x)\frac{f''(x)}{2}\right| + \left|\frac{ab}{n^2}xf'(x) + \left((\beta-1)^2x^2 + \frac{2(\beta-1)}{n}x\right)\frac{f''(x)}{2}\right|.$$

References

[1] T. Acar, Asymptotic formulas for generalized Szász-Mirakyan operators, Appl. Math. Comput. 263 (2015) 233–239.

[2] T. Acar, A. Aral, D. Cárdenas-Morales, P. Garrancho, Szász-Mirakyan type operators which fix exponentials, Results Math. 72 no. 3 (2017) 1393–1404.

[3] T. Acar, A. Aral, H. Gonska, On Szász–Mirakyan operators preserving $e^{2ax}, a > 0$, Mediterr. J. Math. 14 no. 1 Art. 6 (2017), doi:10.1007/s00009-016-0804-7.

[4] T. Acar, A. Aral, I. Rasa, The new forms of Voronovskaya's theorem in weighted spaces, Positivity 20(1) (2016) 25–40.

[5] M. Bodur, Ö. G. Yilmaz, A. Aral, Approximation by Baskakov-Szász-Stancu operators preserving exponential functions, Constructive Mathematical Analysis 1(1) (2018), 1–8.

[6] B. D. Boyanov, V. M. Veselinov, A note on the approximation of functions in an infinite interval by linear positive operators, Bull. Math. Soc. Sci. Math. Roum. 14(62) (1970) 9–13.

[7] J. Bustamante, L. Morales de la Cruz, Positive linear operators and continuous functions on unbounded intervals, Jaen J. Approx. 1 no. 2 (2009) 145–173.

[8] Z. Ditzian, On global inverse theorems of Szász and Baskakov operators, Can. J. Math. 31(2) (1979) 255–263.

[9] Z. Finta, V. Gupta, Direct and inverse estimates for Phillips type operators, J. Math Anal Appl. 303(2) (2005) 627–642.

[10] V. Gupta, A note on modified Phillips operators, Southeast Asian Bulletin of Mathematics 34 (2010) 847–851.

[11] V. Gupta, G. Tachev, On approximation properties of Phillips operators preserving exponential functions, Mediterr. J. Math. 14 no. 4 Art. 177 (2017).

[12] M. Heilmann, G. Tachev, Commutativity, direct and strong converse results for Phillips operators, East J. Approx. 17(3) (2011) 299–317.

[13] A. Holhoş, The rate of approximation of functions in an infinite interval by positive linear operators, Stud. Univ. "Babes-Bolyai", Math. (2) (2010) 133–142.

[14] R. S. Phillips, An inversion formula for Laplace transforms and semi-groups of linear operators, Annals Math. 59 (1954) 325–356.

[15] O. Shisha, B. Mond, The degree of convergence of sequences of linear positive operators, Proc. Nat. Acad. Sci. 60 (1968) 1196–1200.

[16] P. C. Sikkema, On some linear positive operators, Indagationes Mathematicae (Proceedings) 73 (1970) 327–337.

[17] G. Tachev, A global inverse theorem for combinations of Phillips operators, Mediterr. J. Math. 13(5) (2016) 2709–2719.

[18] G. Tachev, Pointwise estimate for linear combinations of Phillips operators, J. Classical Analysis 8(1) (2016) 41–51.

[19] G. Tachev, V. Gupta, A. Aral, Voronovskaja's theorem for functions with exponential growth, Georgian Math. J., to appear.