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On an Elementary Operator with 2-Isometric Operator Entries

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Abstract. A Hilbert space operator T is said to be a 2-isometric operator if $T^{*2}T^2 - 2T^*T + I = 0$. Let $d_{AB} \in B(B(H))$ denote either the generalized derivation $\delta_{AB} = L_A - R_B$ or the elementary operator $\Delta_{AB} = L_AR_B - I$, we show that if A and B^* are 2-isometric operators, then, for all complex λ, $(d_{AB} - \lambda)^{-1}(0) \subseteq (d^*_{AB} - \overline{\lambda})^{-1}(0)$, the ascent of $(d_{AB} - \lambda) \leq 1$, and d_{AB} is polaroid. Let $H(\sigma(d_{AB}))$ denote the space of functions which are analytic on $\sigma(d_{AB})$, and let $H_c(\sigma(d_{AB}))$ denote the space of $f \in H(\sigma(d_{AB}))$ which are non-constant on every connected component of $\sigma(d_{AB})$, it is proved that if A and B^* are 2-isometric operators, then $f(d_{AB})$ satisfies the generalized Weyl's theorem and $f(d^*_{AB})$ satisfies the generalized a-Weyl's theorem.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H. In [3] Agler obtained certain disconjugacy and Sturm-Lioville results for a subclass of the Toeplitz operators. These results were suggested by the study of operators $T \in B(H)$ which satisfy the equation,

$$T^{*2}T^2 - 2T^*T + I = 0.$$

Such T are natural generalizations of isometric operators ($T^*T = I$) and are called 2-isometric operators. It is known that an isometric operator is a 2-isometric operator. 2-isometric operators have been studied by many authors and they have many interesting properties (see [4, 5, 9, 11, 18]), for example, if $T \in B(H)$ is a 2-isometric operator, then $\sigma_p(T)$ for the point spectrum of T is a subset of the boundary $\partial\mathbb{D}$ of the unit disc $\mathbb{D}$ (in the complex plane $\mathbb{C}$), $\sigma(T)$ is the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ whenever T is not invertible, $\sigma(T) \subseteq \partial\mathbb{D}$ whenever T is invertible, and T is injective and has closed range.

For operators $A, B \in B(H)$, let $d_{AB} \in B(B(H))$ denote either the generalized derivation $\delta_{AB} = L_A - R_B$ or the elementary operator $\Delta_{AB} = L_AR_B - I$, where L_A and R_B are the left and right multiplication operators defined on $B(B(H))$ by $L_A(X) = AX$ and $R_B(X) = XB$ respectively. The following implications hold for a general bounded linear operator T on a Banach space $\mathcal{X}$, in particular for $T = d_{AB}$:

$$d^{-1}_{AB}(0) \perp R(d_{AB}) \Longrightarrow d^{-1}_{AB}(0) \cap R(d_{AB}) = 0 \Leftrightarrow asc(d_{AB}) \leq 1,$$

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where $asc(d_{AB})$ denotes the ascent of d_{AB}, $R(d_{AB})$ denotes the range of d_{AB} and $d_{AB}^{-1}(0)\perp R(d_{AB})$ denotes that the kernel of d_{AB} is orthogonal to the range of d_{AB} in the sense of G. Birkhoff. The range-kernel orthogonality of d_{AB} has been considered by a number of authors. A sufficient condition guaranteeing $d_{AB}^{-1}(0)\perp R(d_{AB})$ is that $d_{AB}^{-1}(0) \subseteq d_{AB}^{*-1}(0)$ [12]. The class of operators A, $B^* \in B(H)$ such that $d_{AB}^{-1}(0) \subseteq d_{AB}^{*-1}(0)$ is large, and includes in particular the class of hyponormal A and B^* [13]. If A, $B^* \in B(H)$ are hyponormal, then, for all complex λ, $(d_{AB} - \lambda)^{-1}(0) \subseteq (d_{AB}^* - \overline{\lambda})^{-1}(0)$ and the ascent of $(d_{AB} - \lambda) \leq 1$ [11].

In this paper it is shown that if A and B^* are 2-isometric operators, then, for all complex λ, $(d_{AB}-\lambda)^{-1}(0) \subseteq (d_{AB}^* - \overline{\lambda})^{-1}(0)$ and $(d_{AB} - \lambda)^{-1}(0) \perp R(d_{AB})$. Furthermore, if λ is isolated in the spectrum of d_{AB}, $\lambda \in \mathrm{iso}\sigma(d_{AB})$, then the quasi-nilpotent part $H_0(d_{AB} - \lambda)$ of $d_{AB} - \lambda$ coincides with $(d_{AB} - \lambda)^{-1}(0)$; consequently, λ is a simple pole of the resolvent of d_{AB}. As the application of these properties, it is proved that if A and B^* are 2-isometric operators, then $f(d_{AB}^*)$ satisfies the generalized a-Weyl's theorem.

2. Some Results

Before stating main theorems, we need several preliminary results. Now we recall some definitions

Definition 2.1. *An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G.*

Definition 2.2. *An operator $T \in B(H)$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T.*

Lemma 2.3. *[10] Let T be a 2-isometric operator. Then T is polaroid.*

Lemma 2.4. *Let T be a 2-isometric operator, $\lambda \in \sigma_p(T)$ and*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \ \ on\ \ H = (T - \lambda)^{-1}(0) \oplus (T - \lambda)^{-1}(0)^{\perp}.$$

Then $T_{12} = 0$ and T_{22} is also a 2-isometric operator.

Proof. Let

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \ \ on\ \ H = (T - \lambda)^{-1}(0) \oplus (T - \lambda)^{-1}(0)^{\perp}.$$

Since T is a 2-isometric operator, by [9, Theorem 5] $T^*T - I \geq 0$. Then

$$T^*T - I = \begin{pmatrix} 0 & \overline{\lambda}T_{12} \\ \lambda T_{12}^* & T_{12}^*T_{12} + T_{22}^*T_{22} - I \end{pmatrix} \geq 0.$$

Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0, Z \geq 0$ and $Y = X^{\frac{1}{2}}WZ^{\frac{1}{2}}$ for some contration W. So we have $T_{12} = 0$, and T_{22} is a 2-isometric operator. □

Corollary 2.5. *Let T be a 2-isometric operator. Then $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$, where $\overline{\lambda}$ denotes the complex conjugate of λ.*

Proof. It is obvious from Lemma 2.4. □

Lemma 2.6. *If T is a 2-isometric operator, then it has Bishop's property (β).*

Proof. Let T be a 2-isometric operator and choose a positive number σ with $\|T^*T - I\| \leq \sigma$. By [5, Proposition 5.12 and Theorem 5.80], T has a Brownian unitary extension B of the form

$$B = \begin{pmatrix} V & \sigma E \\ 0 & U \end{pmatrix},$$

where V is an isometry operator, U is unitary, and E is a Hilbert space isomorphism onto $N(V^*)$. Let $f(z)$ be analytic on D. Let $(B - z)f(z) \to 0$ uniformly on each compact subsets of D. Then we can write

$$\begin{pmatrix} V - z & \sigma E \\ 0 & U - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (V - z)f_1(z) + \sigma E f_2(z) \\ (U - z)f_2(z) \end{pmatrix} \to 0.$$

Since V and U have Bishop's property (β), B has Bishop's property (β). T is the restriction of B to an invariant subspace, hence T has Bishop's property (β). □

Lemma 2.7. *[17] If A, B^* are reduced by each of its eigenspaces, polaroid and have Bishop's property (β), then $(d_{AB} - \lambda)^{-1}(0) \subseteq (d^*_{AB} - \overline{\lambda})^{-1}(0)$ for all $\lambda \in \mathbb{C}$.*

Theorem 2.8. *If A, B^* are 2-isometric operators, then $(d_{AB} - \lambda)^{-1}(0) \subseteq (d^*_{AB} - \overline{\lambda})^{-1}(0)$ for all $\lambda \in \mathbb{C}$.*

Proof. We can derive the result from Lemma 2.3, Corollary 2.5, Lemma 2.6 and Lemma 2.7. □

Lemma 2.9. *If A, B^* are 2-isometric operators, then $asc(d_{AB} - \lambda) \leq 1$ for all $\lambda \in \mathbb{C}$.*

Proof. It is obvious from Theorem 2.8. □

Theorem 2.10. *If A, B^* are 2-isometric operators, then $H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)$ for all $\lambda \in iso\sigma(d_{AB})$.*

Proof. Evidently, A and B^* are reduced by each of its eigenspaces; $\sigma_p(A) \subseteq \partial\mathbb{D}$, $\sigma_p(B^*) \subseteq \partial\mathbb{D}$; eigenvectors of 2-isometric operators corresponding to distinct eigenvalues are orthogonal. Recall [1] that $\sigma(\delta_{AB}) = \{\lambda \in \sigma(A) - \sigma(B)\}$ and $\sigma(\Delta_{AB}) = \{\lambda \in \sigma(A)\sigma(B) - 1\}$. If $\lambda \in \text{iso}\sigma(d_{AB})$, then there exist finite sequences $\{\alpha_i\}_1^m$ and $\{\beta_i\}_1^m$ of isolated points in $\sigma(A)$ and $\sigma(B)$, respectively, such that $\lambda = \alpha_i - \beta_i$ if $\lambda \in \text{iso}\sigma(\delta_{AB})$ and $\lambda = \alpha_i\beta_i - 1$ if $\lambda \in \text{iso}\sigma(\Delta_{AB})$, for all $1 \leq i \leq m$. Let

$$M_1 = \oplus_{i=1}^m M_{1i},\ M_{1i} = (A - \alpha_i)^{-1}(0) \text{ and } M_2 = H \ominus M_1$$

and

$$N_1 = \oplus_{i=1}^m N_{1i},\ N_{1i} = (B - \beta_i)^{*-1}(0) \text{ and } N_2 = H \ominus N_1.$$

Then A and B have the representations

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } H = M_1 \oplus M_2,$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ on } H = N_1 \oplus N_2.$$

Since the spectrum of A_2 and B_2 don't contain isolated points, then $\lambda \notin \sigma(d_{A_kB_t})$ for all $1 \leq k, t \leq 2$ other than $k = t = 1$.

Let $X \in H_0(d_{AB} - \lambda)$, and let $X \in B(N_1 \oplus N_2, M_1 \oplus M_2)$ have the representation $X = [X_{kl}]_{k,l=1}^2$. Then

$$(d_{AB} - \lambda)^n X = \begin{pmatrix} * & * \\ * & (d_{A_2B_2} - \lambda)^n X_{22} \end{pmatrix}$$

(for some, as yet, non specified entries $*$). Since $\lim_{n\to\infty}\|(d_{AB}-\lambda)^nX\|^{\frac{1}{n}}=0$ implies $\lim_{n\to\infty}\|(d_{A_2B_2}-\lambda)^nX_{22}\|^{\frac{1}{n}}=0$, and since $d_{A_2B_2}-\lambda$ is invertible, we have $X_{22}=0$, and then

$$(d_{AB}-\lambda)^nX=\begin{pmatrix} * & (d_{A_1B_2}-\lambda)^nX_{12} \\ (d_{A_2B_1}-\lambda)^nX_{21} & 0 \end{pmatrix}$$

(for some, as yet, non specified entries $*$). Again, since $\lim_{n\to\infty}\|(d_{AB}-\lambda)^nX\|^{\frac{1}{n}}=0$ implies $\lim_{n\to\infty}\|(d_{A_1B_2}-\lambda)^nX_{12}\|^{\frac{1}{n}}=\lim_{n\to\infty}\|(d_{A_2B_1}-\lambda)^nX_{21}\|^{\frac{1}{n}}=0$, and since $d_{A_1B_2}-\lambda$ and $d_{A_2B_1}-\lambda$ are invertible, we have $X_{12}=0=X_{21}$. Hence, $(d_{AB}-\lambda)^nX=(d_{A_1B_1}-\lambda)^nX_{11}$. Let $X_{11}=[Y_{ij}]_{1\le i,j\le m}\in B(\oplus_{i=1}^m N_{1i},\oplus_{i=1}^m M_{1i})$. Then, for $1\le i,j\le m$,

$$\begin{aligned}(\delta_{A_1B_1}-\lambda)^n(X_{11})=&((L_{A_1-\alpha_i}-R_{B_1-\beta_j})+(\alpha_i-\beta_j-\lambda))^n[Y_{ij}]_{1\le i,j\le m}\\ =&(\sum_{k=0}^{n}\binom{n}{k}(L_{A_1-\alpha_i}-R_{B_1-\beta_j})^k(\alpha_i-\beta_j-\lambda)^{n-k})[Y_{ij}]_{1\le i,j\le m}\end{aligned}$$

and

$$\begin{aligned}(\Delta_{A_1B_1}-\lambda)^n(X_{11})=&(L_{A_1-\alpha_i}R_{B_1}+\alpha_iR_{B_1-\beta_j}+\alpha_i\beta_j-1-\lambda)^n[Y_{ij}]_{1\le i,j\le m}\\ =&(\sum_{k=0}^{n}\binom{n}{k}(L_{A_1-\alpha_i}R_{B_1}+\alpha_iR_{B_1-\beta_j})^k(\alpha_i\beta_j-1-\lambda)^{n-k})[Y_{ij}]_{1\le i,j\le m}.\end{aligned}$$

Since $(A_1-\alpha_i)|M_{1i}=0=(B_1-\beta_i)|N_{1i}$, it follows that

$$(\delta_{A_1B_1}-\lambda)^n(X_{11})=(\alpha_i-\beta_j-\lambda)^n[Y_{ij}]_{1\le i,j\le m}$$

and

$$(\Delta_{A_1B_1}-\lambda)^n(X_{11})=(\alpha_i\beta_j-1-\lambda)^n[Y_{ij}]_{1\le i,j\le m}.$$

Recall, $\lim_{n\to\infty}\|(d_{A_1B_1}-\lambda)^nX_{11}\|^{\frac{1}{n}}=0$; hence $\lim_{n\to\infty}|\alpha_i-\beta_j-\lambda|\|Y_{ij}\|^{\frac{1}{n}}=0$ in the case in which $d=\delta$ and $\lim_{n\to\infty}|\alpha_i\beta_j-1-\lambda|\|Y_{ij}\|^{\frac{1}{n}}=0$ in the case in which $d=\Delta$. Thus $Y_{ij}=0$ for all i,j such that $i\ne j$. This implies that $X=X_{11}=\oplus_{i=1}^m Y_{ii}\in(d_{AB}-\lambda)^{-1}(0)$. Hence $H_0(d_{AB}-\lambda)\subset(d_{AB}-\lambda)^{-1}(0)$. Since the reverse inclusion holds for every operator, we must have $H_0(d_{AB}-\lambda)=(d_{AB}-\lambda)^{-1}(0)$. □

3. Weyl's Theorem

An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T)=\dim T^{-1}(0)<\infty$ and $\beta(T)=\dim H/R(T)<\infty$. Moreover if $i(T)=\alpha(T)-\beta(T)=0$, then T is called Weyl. The Weyl spectrum of T [15] is defined by $w(T):=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Weyl}\}$.

We consider the sets

$\Phi_+(H):=\{T\in B(H):R(T)\text{ is closed and }\alpha(T)<\infty\}$;
$\Phi_+^-(H):=\{T\in B(H):T\in\Phi_+(H)\text{ and }i(T)\le 0\}$.

And define

$$\sigma_{ea}(T):=\{\lambda\in\mathbb{C}:T-\lambda\notin\Phi_+^-(H)\};$$

$$\pi_{00}(T):=\{\lambda\in\operatorname{iso}\sigma(T):0<\alpha(T-\lambda)<\infty\};$$

$$\pi_{00}^a(T):=\{\lambda\in\operatorname{iso}\sigma_a(T):0<\alpha(T-\lambda)<\infty\}.$$

Following [16], we say that Weyl's theorem holds for T if $\sigma(T)\backslash w(T)=\pi_{00}(T)$, and that a-Weyl's theorem holds for T if $\sigma_a(T)\backslash\sigma_{ea}(T)=\pi_{00}^a(T)$, where $\sigma_a(T)$ is the approximate point spectrum of T.

More generally, Berkani investigated B-Fredholm theory and generalized Weyl's theorem as follows (see [6–8]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim T^{-1}_{[n]}(0) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{−Weyl}\}$. We say that generalized Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ where $E(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda)\}$. Note that, if generalized Weyl's theorem holds for T, then so does Weyl's theorem [7].

We define $T \in SBF^-_+(H)$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim T^{-1}_{[n]}(0) = \dim T^{-1}(0) \cap R(T^n) < \infty$) and $i(T_{[n]}) \leq 0$ [8]. We define $\sigma_{SBF^-_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF^-_+(H)\}$. We say that generalized a-Weyl's theorem holds for T if $\sigma_a(T) \setminus \sigma_{SBF^-_+}(T) = E^a(T)$, where $E^a(T) := \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda)\}$. It's known from [7, 19] that if $T \in B(H)$ then we have
generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem $\Rightarrow$ Weyl's theorem;
generalized a-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem $\Rightarrow$ Weyl's theorem.

We know that Weyl's theorem holds for 2-isometric operators [18]. In this paper, we prove generalized Weyl's theorem for the elementary and the generalized derivation with 2-isometric operators as entries.

Recall that $T \in B(H)$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for short), if for every open neighborhood G of λ_0, the only analytic function $f : G \to H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in G$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Lemma 3.1. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then d_{AB} has SVEP.*

Proof. We can derive the result from Lemma 2.9. □

For an operator $T \in B(H)$, the analytic core $K(T - \lambda)$ of $T - \lambda$ is defined by $K(T - \lambda) = \{x \in H :$ there exists a sequence $\{x_n\} \subseteq H$ and $c > 0$ for which $x = x_0, (T - \lambda)x_{n+1} = x_n$ and $\|x_n\| \leq c^n\|x\|$ for all $n \in \mathbb{N}\}$. We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are generally non-closed hyperinvariant subspaces of $T - \lambda$ such that $N(T - \lambda)^n \subseteq H_0(T - \lambda)$ for all $n \in \mathbb{N}$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$; also, if $\lambda \in \text{iso}\sigma(T)$, then $H = H_0(T - \lambda) \dotplus K(T - \lambda)$, where $H_0(T - \lambda)$ and $K(T - \lambda)$ are closed.

Lemma 3.2. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then d_{AB} is polaroid.*

Proof. Let $\lambda \in \text{iso}\sigma(d_{AB})$. If A, B^* are 2-isometric operators, then $H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)$. By [2, Theorem 3.76] we have $H = H_0(d_{AB} - \lambda) \dotplus K(d_{AB} - \lambda)$. Thus d_{AB} is simply polaroid follows from the implications

$$\begin{aligned} & H = (d_{AB} - \lambda)^{-1}(0) \dotplus K(d_{AB} - \lambda) \\ \Rightarrow \quad & (d_{AB} - \lambda)H = 0 \dotplus (d_{AB} - \lambda)K(d_{AB} - \lambda) = K(d_{AB} - \lambda) \\ \Rightarrow \quad & H = (d_{AB} - \lambda)^{-1}(0) \dotplus R(d_{AB} - \lambda). \end{aligned}$$

□

Corollary 3.3. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then d_{AB} is isoloid and $R(d_{AB} - \lambda)$ is closed for all $\lambda \in \text{iso}\sigma(d_{AB})$,*

Theorem 3.4. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then generalized Weyl's theorem holds for d_{AB}.*

Proof. Since d_{AB} has SVEP, d_{AB} satisfies generalized Browder's theorem and generalized a-Browder's theorem. A sufficient condition for an operator d_{AB} satisfying generalized Browder's theorem to satisfy generalized Weyl's theorem is that d_{AB} is polaroid. By Lemma 3.2 generalized Weyl's theorem holds for d_{AB}. □

Theorem 3.5. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then generalized a-Weyl's theorem holds for d^*_{AB}.*

Proof. Since d_{AB} has SVEP and d_{AB} is polaroid, by [1, Theorem 3.10] generalized a-Weyl's theorem holds for d^*_{AB}. □

In the following, let $H(\sigma(d_{AB}))$ denote the space of functions which are analytic on $\sigma(d_{AB})$, and let $H_c(\sigma(d_{AB}))$ denote the space of $f \in H(\sigma(d_{AB}))$ which are non-constant on every connected component of $\sigma(d_{AB})$.

Theorem 3.6. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then generalized Weyl's theorem holds for $f(d_{AB})$.*

Proof. Since d_{AB} has SVEP and d_{AB} is isoloid, we have that generalized Weyl's theorem holds for $f(d_{AB})$ by [20, Theorem 2.2] and Theorem 3.4. □

Corollary 3.7. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then Weyl's theorem holds for $f(d_{AB})$.*

A bounded linear operator $T \in B(H)$ is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Note that every a-isoloid operator is isoloid and the converse is not true in general.

Lemma 3.8. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then d^*_{AB} is a-isoloid.*

Proof. Let λ be an isolated point of $\sigma_a(d^*_{AB})$. Suppose that A, B^* are 2-isometric operators. By Lemma 3.1 and Lemma 3.2, we have that d_{AB} has SVEP and d^*_{AB} is isoloid. Hence, $\sigma_a(d^*_{AB}) = \sigma(d^*_{AB})$ by [14, Corollary 7]. We have that λ is an isolated point of $\sigma(d^*_{AB})$. Since d^*_{AB} is isoloid, we have that λ is an eigenvalue of d^*_{AB}. Hence, d^*_{AB} is a-isoloid. □

Theorem 3.9. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then generalized a-Weyl's theorem holds for $f(d^*_{AB})$.*

Proof. Suppose that A, B^* are 2-isometric operators. Then d_{AB} has SVEP and d^*_{AB} is a-isoloid by Lemma 3.8, we have that generalized a-Weyl's theorem holds for $f(d^*_{AB})$ by [20, Theorem 2.4] and Theorem 3.5. □

Corollary 3.10. *Let $A, B \in B(H)$. If A, B^* are 2-isometric operators, then a-Weyl's theorem holds for $f(d^*_{AB})$.*

References

[1] P. Aiena, E. Aponte, E. Balzan, Weyl type theorems for left and right polaroid operators, Integr. Equ. Oper. Theory 66 (2010) 1–20.
[2] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer Academic Publishers, London, 2004.
[3] J. Agler, A disconjugacy theorem for Toeplitz operators, Amer. J. Math. 112 (1990) 1–14.
[4] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. I, Integral Equ. Oper. Theory 21 (1995) 383–429.
[5] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. II, Integral Equ. Oper. Theory 23 (1995) 1–48.
[6] M. Berkani, A. Arroud, Generalized Weyl's theorem and hyponormal operators, J. Austra. Math. Soc. 76 (2004) 291–302.
[7] M. Berkani, J.J. Koliha, Weyl type theorems for bounded linear operators, Acta. Sci. Math.(Szeged) 69 (2003) 359–376.
[8] M. Berkani, M. Sarih, On semi B-Fredholm operators, Glasgow Math. J. 43 (2001) 457–465.
[9] M. Chō, S. Ôta, K. Tanahashi, A. Uchiyama, Spectral properties of m-isometric operators, Funct. Anal. Approx. Comput. 4 (2012) 33–39.
[10] B.P. Duggal, Tensor product of n-isometries, Linear Algebra Appl. 437 (2012) 307–318.
[11] B.P. Duggal, An elementary operator with log-hyponormal, p-hyponormal entries, Linear Algebra Appl. 437 (2008) 1109–1116.
[12] B. P. Duggal, Range-kernel orthogonality of derivations, Linear Algebra Appl. 304 (2000) 103–108.
[13] B. P. Duggal, A remark on generalised Putnam-Fuglede theorems, Proc. Amer. Math. Soc. 129 (2000) 83–87.
[14] J.K. Finch, The single valued extension property on a Banach space, Pacific. J. Math. 58 (1975) 61–69.
[15] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988.
[16] R.E. Harte, W.Y. Lee, Another note on Weyl's theorem, Trans. Amer. Math. Soc. 349 (1997) 2115–2124.
[17] F. Lombarkia, M. Amouch, Asymetric Fuglede Putnam's theorem for operators reduced by their eigenspaces, Filomat 31 (2017) 6409–6417.
[18] S.M. Patel, 2-isometry operators, Glasnik Mat. 37 (2002) 143–147.
[19] V. Rakočević, Operators obeying a-Weyl's theorem, Rev. Roumaine Math. Pures Appl. 10 (1989) 915–919.
[20] H. Zguitti, A note on generalized Weyl's theorem, J. Math. Anal. Appl. 316 (2006) 373–381.