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Bounds of Some Relative Operator Entropies in a General Form

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Abstract. In this paper, we determine the bounds of the generalized relative operator entropy in general. In particular, we identify the bounds of the parametric extension of the Shannon entropy and of the generalized Tsallis relative operator entropy. Moreover, we improve the upper bound of the relative operator entropy in some sense.

1. Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If A is a positive invertible operator, we call it strictly positive operator. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we say $A \leq B$ if $B - A \geq 0$.

A relative operator entropy of strictly positive operators A and B was introduced in the noncommutative information theory by Fujii and Kamei [9] by

$$S(A|B) = A^{\frac{1}{2}} \ln(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

Furuta [8] defined the operator Shannon entropy by

$$S_p(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p \ln(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

where $p \in [0,1]$ and A, B are strictly positive operators on a Hilbert space $\mathcal{H}$ and proved parametric extensions of the Shannon inequality and its reverse one in Hilbert space operators, see also [14]. Moreover, the generalized Tsallis relative operator entropy was introduced and then several operator inequalities were derived by Yanaghi et. al [18]. They generalized the definition of the Tsallis relative operator entropy. For two strictly positive operator A and B, $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$, $k \in \mathbb{Z}$ the generalized Tsallis relative operator entropy was defined by

$$\tilde{T}_{\mu,k,\lambda}(A, B) := \frac{A\#_{\mu+k\lambda}B - A\#_{\mu+(k-1)\lambda}B}{\lambda},$$

where $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ is the operator α-geometric mean. In particular, for $\lambda \neq 0$ we have

$$\tilde{T}_{0,1,\lambda}(A, B) = \frac{A\#_\lambda B - A\#_0 B}{\lambda} = \frac{A\#_\lambda B - A}{\lambda} = T_\lambda(A, B),$$

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where $T_\lambda(A,B)$ is the Tsallis relative operator entropy, cf. [10, 19].

Drogomir found in [2, 3] some bounds for the following differences

$$\frac{m\ln m}{M-m}(MA-B)+\frac{M\ln M}{M-m}(B-mA)-S_1(A|B) \tag{1}$$

$$S(A|B)-\frac{\ln m}{M-m}(MA-B)-\frac{\ln M}{M-m}(B-mA), \tag{2}$$

where A, B are two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M > 0$ with $m < M$. We generalized Dragomir's results in [16] and identified the upper and lower bounds for the difference

$$\frac{m^q\ln m}{M-m}(MA-B)+\frac{M^q\ln M}{M-m}(B-mA)-S_q(A|B), \tag{3}$$

where A and B are two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [0, e^{\frac{2q-1}{q(1-q)}}]$ with $m < M$ and $0 < q \leq 1$ and the difference

$$T_\lambda(A|B)-\frac{m^\lambda-1}{\lambda(M-m)}(MA-B)-\frac{M^\lambda-1}{\lambda(M-m)}(B-mA), \tag{4}$$

where A and B are two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M > 0$ with $m < M$ and $0 < \lambda \leq 1$. In particular, when $q \to 1^-$ in (3), we get (1) and when $\lambda \to 0^+$ in (4), we get (2). Inspired by this idea, we present the upper and lower bounds for the difference of the general form

$$S_q^f(A|B)-\frac{m^q f(m)}{M-m}(MA-B)-\frac{M^q f(M)}{M-m}(B-mA), \tag{5}$$

where $f : (0,\infty) \to \mathbb{R}$ is a twice differentiable function and A and B are two strictly positive operators such that $mA \leq B \leq MA$ for some m, M in the concave domain of the function $t^q f(t)$ with $m < M$ and $q \in \mathbb{R}$. We emphasis a geometric framework revealing a close link behind this idea. The concavity of the function $t^q f(t)$ means geometrically that the points of the graph of the restriction of $t^q f(t)$ on $[m, M]$ are on the chord joining the end points $(m, m^q f(m))$ and $(M, M^q f(M))$, that is,

$$m^q f(m)+\frac{M^q f(M)-m^q f(m)}{M-m}(t-m) \leq t^q f(t) \tag{6}$$

for all $t \in [m, M]$. By rewriting the left hand side of (6) we obtain

$$\frac{M^q f(M)}{M-m}(t-m)+\frac{m^q f(m)}{M-m}(M-t) \leq t^q f(t)$$

for all $t \in [m, M]$ and taking the perspective in the sense of Corollary 2.2 we get the difference (5) is positive. Indeed, the term $\frac{m^q f(m)}{M-m}(MA-B)+\frac{M^q f(M)}{M-m}(B-mA)$ appeared in the difference (5) is the perspective of the line joining the points $(m, m^q f(m))$ and $(M, M^q f(M))$.

In this paper, we develop the notion of the relative operator entropy and determine the bounds of the generalized relative operator entropy in a general form. In particular, we identify the bounds of the parametric extension of the Shannon entropy, the generalized Tsallis relative operator entropy, and the Tsallis relative operator entropy. We also derive the bounds of the perspective of a twice differentiable function and as a consequence of this result we improve the upper bound of the relative operator entropy in some sense.

2. Relative operator entropies in a general form

The notion of the operator perspective function was introduced in [7] by Effros for two commuting operators. We considered a fully non-commutative perspective of the one variable function f in [6] by setting

$$P_f(A,B) = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$$

and the generalized perspective of two variables (associated with f and h) by

$$P_{f\Delta h}(A,B) = h(A)^{1/2} f(h(A)^{-1/2}Bh(A)^{-1/2})h(A)^{1/2},$$

where A is a strictly positive operator and B is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and the spectra of the operators $A^{-1/2}BA^{-1/2}$ and $h(A)^{-1/2}Bh(A)^{-1/2}$ lie in the domain of the function f. The main results of [7] are generalized in [6] for the non-commutative perspective and the necessary and sufficient conditions for joint convexity (resp. concavity) of the perspective and generalized perspective functions are established. For some recent results on this subject we refer the readers to see [12, 15, 17]. Kubo and Ando [11] discussed the axiomatic theory for connections and established the existence of an affine order isomorphism between the class of connections and the class of positive operator monotone functions. This affine order isomorphism which has a perspective form was only considered for the class of positive operator monotone functions.

Example 2.1. *The perspective of the functions t^α, $\ln t$, $t^p \ln t$, $\frac{t^\lambda - 1}{\lambda}$, $\frac{t^{\mu+k\lambda} - t^{\mu+(k-1)\lambda}}{\lambda}$ is the operator α-geometric mean, the relative operator entropy, the operator Shannon entropy, the Tsallis relative operator entropy, and the generalized Tsallis relative operator entropy, respectively.*

Lemma 2.2. *Let r, s, k be real valued and continuous functions on the closed interval $\mathbb{I}$. If $r(t) \le s(t) \le k(t)$ for $t \in \mathbb{I}$, then*

$$P_r(A,B) \le P_s(A,B) \le P_k(A,B),$$

for every strictly positive operator A and every self-adjoint operator B such that the spectrum of the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ lies in $\mathbb{I}$.

Proof. By using the assumption one can ensure that

$$r(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \le s(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \le k(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$$

for the strictly positive operator A and the self-adjoint operator B. By multiplying $A^{\frac{1}{2}}$ from both sides we obtain the desired inequalities. □

Definition 2.3. *Let $f : (0,\infty) \to \mathbb{R}$ be a twice differentiable function and $q \in \mathbb{R}$. For two strictly positive operators A and B, we consider the generalized f-relative operator entropy by setting*

$$S_q^f(A|B) := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

In particular, when we put $q = 0$, we reach $S_0^f(A|B) = P_f(A,B)$ and when we consider $f(t) = \ln t$, we have $S_q^f(A|B) = S_q(A|B)$.

Define $\Lambda(t) := t^q f(t)$ for $q \in \mathbb{R}$, where $f : (0,\infty) \to \mathbb{R}$ is a twice differentiable function and consider

$$\mathbb{H}_q := \{t \ge 0 : \Lambda''(t) \le 0\}.$$

Lemma 2.4. *The function $\Lambda(t) = t^q f(t)$ is concave on $\mathbb{H}_q$ for $q \in \mathbb{R}$.*

It is worth mentioning that $P_\Lambda(A,B) = S_q^f(A|B)$. In particular, when we take $f(t) = \ln t$, then a simple calculation indicates that the concavity domain of the function $\Lambda(t) = t^q \ln t$ is $\mathbb{H}_q = [e^{\frac{2q-1}{q(1-q)}}, \infty)$ for $0 \le q < 1$. It should be noted that when $q \to 0^+$ one can obtain $\mathbb{H}_{0^+} = (0,\infty)$. This fact confirms the concavity domain of the function $\ln t$ is $(0,\infty)$. Moreover, if $q \to 1^-$, then $\mathbb{H}_{1^-} = \emptyset$. This shows the concavity domain of the function $t \ln t$ is empty.

For the sake of simplified writing throughout this paper, we define

$$r(u) := \min\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} - \left|\frac{u - \frac{M+m}{2}}{M-m}\right|,$$

$$R(u) := \max\Big\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\Big\} = \frac{1}{2} + \Big|\frac{u-\frac{M+m}{2}}{M-m}\Big|,$$

$$Q_q^f(m,M) := (\frac{M+m}{2})^q f(\frac{M+m}{2}) - \frac{m^q f(m) + M^q f(M)}{2},$$

where $0 < m < M$. The concavity of the function $\Lambda(t) = t^q f(t)$ on $\mathbb{H}_q$ exemplifies that

$$Q_q^f(m,M) \geq 0 \tag{7}$$

for $m, M \in [\alpha, \beta] \subseteq \mathbb{H}_q$ with $0 < m < M$.

The following two Lemmas have been proved for a convex function. We state them for a concave function as follows:

Lemma 2.5. *[5, Lemma 1] Let $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a concave function. Then,*

$$\begin{aligned} 0 &\leq g((1-c)x + cy) - ((1-c)g(x) + cg(y)) && (8)\\ &\leq c(1-c)(y-x)(g'_+(x) - g'_-(y)) && (9) \end{aligned}$$

for all $x, y \in (a,b)$ with $x < y$ and $c \in [0,1]$.

Lemma 2.6. *[4, Theorem 1] If $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a concave function, then*

$$\begin{aligned} 2r\Big[g(\frac{x+y}{2}) - \frac{g(x)+g(y)}{2}\Big] &\leq g((1-c)x + cy) - ((1-c)g(x) + cg(y))\\ &\leq 2R\Big[g(\frac{x+y}{2}) - \frac{g(x)+g(y)}{2}\Big] \end{aligned}$$

for any $x, y \in (a,b)$ and $c \in [0,1]$, where $r = \min\{c, 1-c\}$ and $R = \max\{c, 1-c\}$.

Lemma 2.7. *[1, Lemma 2.2] Let $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. If there exist two constants γ_1, γ_2 such that $\gamma_1 \leq g''(t) \leq \gamma_2$ foe every $t \in (a,b)$, then*

$$\begin{aligned} \frac{1}{2}c(1-c)\gamma_1(y-x)^2 &\leq (1-c)g(x) + cg(y) - g((1-c)x + cy)\\ &\leq \frac{1}{2}c(1-c)\gamma_2(y-x)^2, && (10) \end{aligned}$$

where $c \in [0,1]$, $x, y \in (a,b)$ with $x < y$.

We now commence the main results of this section.

Theorem 2.8. *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{H}_q$ with $0 < m < M$. Then,*

$$\begin{aligned} 0 &\leq S_q^f(A|B) - \frac{m^q f(m)}{M-m}(MA - B) - \frac{M^q f(M)}{M-m}(B - mA)\\ &\leq \frac{m^{q-1}(mf'(m) + qf(m)) - M^{q-1}(Mf'(M) + qf(M))}{M-m} P_\Psi(A,B)\\ &\leq \frac{1}{4}(M-m)\Big(m^{q-1}(mf'(m) + qf(m)) - M^{q-1}(Mf'(M) + qf(M))\Big)A, \end{aligned}$$

where $\Psi(t) = (t-m)(M-t)$.

Proof. We apply Lemma 2.5 for the function $\Lambda(t) = t^q f(t)$, $t \in \mathbb{H}_q$. Then,

$$\begin{aligned} 0 &\le \Lambda((1-c)x+cy) - (1-c)\Lambda(x) - c\Lambda(y) \\ &\le c(1-c)(y-x)(\Lambda'_+(x) - \Lambda'_-(y)), \end{aligned} \tag{11}$$

where $c \in [0,1]$ and $x, y \in [m, M]$ with $x < y$. Replacing $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ in (11), we see that

$$\begin{aligned} 0 &\le \Lambda(u) - \frac{m^q f(m)}{M-m}(M-u) - \frac{M^q f(M)}{M-m}(u-m) \\ &\le \frac{m^{q-1}(mf'(m)+qf(m)) - M^{q-1}(Mf'(M)+qf(M))}{M-m}\Psi(u). \end{aligned} \tag{12}$$

A simple verification shows that the maximum value of the function $\Psi(u)$ is $\frac{1}{4}(M-m)^2$. So,

$$\begin{aligned} &\frac{m^{q-1}(mf'(m)+qf(m)) - M^{q-1}(Mf'(M)+qf(M))}{M-m}\Psi(u) \\ &\le \frac{1}{4}(M-m)\Big(m^{q-1}(mf'(m)+qf(m) - M^{q-1}(Mf'(M)+qf(M))\Big). \end{aligned} \tag{13}$$

Combining inequalities (12), (13) and regarding Lemma 2.2 and taking the perspective, we conclude the result. □

The following corollary gives the bounds of the perspective of a twice differentiable function on its concavity domain. As a consequence of this corollary we obtain the bounds of relative operator entropy proved in [3, Theorem 3].

Corollary 2.9. *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{H}_0$ with $0 < m < M$. Then,*

$$\begin{aligned} 0 &\le P_f(A,B) - \frac{f(m)}{M-m}(MA-B) - \frac{f(M)}{M-m}(B-mA) \\ &\le \frac{f'(m)-f'(M)}{M-m}P_\Psi(A,B) \\ &\le \frac{1}{4}(M-m)\Big(f'(m)-f'(M)\Big)A, \end{aligned}$$

where $\Psi(t) = (t-m)(M-t)$.

Proof. It follows from Theorem 2.8 by setting $q = 0$ and noting that $S_0^f(A,B) = P_f(A,B)$. □

The following corollary is a straight forward consequence of Corollary 2.9:

Corollary 2.10. *[3, Theorem 3] Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M > 0$ with $m < M$. Then,*

$$\begin{aligned} 0 &\le S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \\ &\le \frac{4}{(M-m)^2}\Big(K(\frac{M}{m})-1\Big)(B-mA)A^{-1}(MA-B) \\ &\le \Big(K(\frac{M}{m})-1\Big)A, \end{aligned}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $h > 0$ is the Kantorovich's constant.

Proof. By applying Corollary 2.9 and setting $f(t) = \ln t$ one can note that $\mathbb{H}_{0^+} = (0, \infty)$ and

$$\frac{(M-m)^2}{4mM} = K(\frac{M}{m}) - 1,$$
$$P_\Psi(A, B) = (B - mA)A^{-1}(MA - B).$$

□

Theorem 2.11. *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{H}_q$ with $0 < m < M$. Then,*

$$\begin{aligned} 2Q_q^f(m, M)P_r(A, B) &\leq S_q^f(A|B) - \frac{m^q f(m)}{M-m}(MA - B) - \frac{M^q f(M)}{M-m}(B - mA) \\ &\leq 2Q_q^f(m, M)P_R(A, B). \end{aligned}$$

Proof. Consider the concave function $\Lambda(t) = t^q f(t)$, $t \in \mathbb{H}_q$ in Lemma 2.6 to get

$$\begin{aligned} &2r\Big[(\frac{x+y}{2})^q f(\frac{x+y}{2}) - \frac{x^q f(x) + y^q f(y)}{2}\Big] \\ &\leq ((1-c)x + cy)^q f((1-c)x + cy) - ((1-c)x^q f(x) + cy^q f(y)) \\ &\leq 2R\Big[(\frac{x+y}{2})^q f(\frac{x+y}{2}) - \frac{x^q f(x) + y^q f(y)}{2}\Big] \end{aligned} \tag{14}$$

for any $x, y \in [\alpha, \beta]$ and $c \in [0, 1]$, where $r = \min\{c, 1 - c\}$ and $R = \max\{c, 1 - c\}$. Replacing $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ with $u \in [m, M]$ in (14), we deduce

$$\begin{aligned} 2Q_q^f(m, M)r(u) &\leq \Lambda(u) - m^q f(m)\frac{M-u}{M-m} - M^q f(M)\frac{u-m}{M-m} \\ &\leq 2Q_q(m, M)R(u). \end{aligned} \tag{15}$$

Making use of Lemma 2.2 and taking the perspective, we obtain the desired inequalities. □

In the following corollary, we obtain the lower and upper bound of the perspective of a twice differentiable function on its concavity domain.

Corollary 2.12. *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{H}_0$ with $0 < m < M$. Then,*

$$\begin{aligned} 2Q_0^f(m, M)P_r(A, B) &\leq P_f(A, B) - \frac{f(m)}{M-m}(MA - B) - \frac{f(M)}{M-m}(B - mA) \\ &\leq 2Q_0^f(m, M)P_R(A, B). \end{aligned}$$

Proof. It follows from Theorem 2.11 by setting $q = 0$ and noting that $S_0^f(A, B) = P_f(A, B)$. □

Corollary 2.13. *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M > 0$ with $m < M$. Then,*

$$\begin{aligned} 2\ln\frac{m+M}{2\sqrt{mM}}P_r(A, B) &\leq S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA) \\ &\leq 2\ln\frac{m+M}{2\sqrt{mM}}P_R(A, B). \end{aligned}$$

Proof. It follows from Corollary 2.12 by putting $f(t) = \ln t$ and noting that $Q_0^{\ln}(m, M) = \ln\frac{m+M}{2\sqrt{mM}}$ and $\mathbb{H}_{0^+} = (0, \infty)$. □

Since $2\ln\frac{m+M}{2\sqrt{mM}} \le K(\frac{M}{m})$, combining Corollary 2.13 and [2, Theorem 2], we obtain a new and refined upper bound for the relative operator entropy. Indeed, the upper bound announced in [2, Theorem 2] for the relative operator entropy can be sharpen.

Corollary 2.14. *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M > 0$ with $m < M$. Then, we have*

$$\begin{aligned}K(\frac{M}{m})P_r(A,B) &\le S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA)\\ &\le 2\ln\frac{m+M}{2\sqrt{mM}}P_R(A,B).\end{aligned}$$

Theorem 2.15. *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha,\beta] \subseteq \mathbb{H}_q$ with $0 < m < M$. If there exist the constants γ_1, γ_2 such that $\gamma_1 \le \Lambda''(t) \le \gamma_2$ for every $t \in (\alpha,\beta)$, then*

$$\begin{aligned}\frac{1}{2}\gamma_1 P_\Psi(A,B) &\le \frac{m^q f(m)}{M-m}(MA-B) + \frac{M^q f(M)}{M-m}(B-mA) - S_q^f(A|B)\\ &\le \frac{1}{2}\gamma_2 P_\Psi(A,B) \le 0,\end{aligned}$$

where $\Psi(t) = (t-m)(M-t)$.

Proof. Applying Lemma 2.7 for the function $\Lambda(t) = t^q f(t)$, $t \in \mathbb{H}_q$, we get

$$\begin{aligned}\frac{1}{2}c(1-c)\gamma_1(y-x)^2 &\le (1-c)\Lambda(x) + c\Lambda(y) - \Lambda((1-c)x+cy)\\ &\le \frac{1}{2}c(1-c)\gamma_2(y-x)^2,\end{aligned} \tag{16}$$

where $c \in [0,1]$, $x, y \in [\alpha,\beta]$. Replace $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$, in (16), to get

$$\begin{aligned}\frac{1}{2}(u-m)(M-u)\gamma_1 &\le \frac{M-u}{M-m}m^q f(m) + \frac{u-m}{M-m}M^q f(M) - u^q f(u)\\ &\le \frac{1}{2}(u-m)(M-u)\gamma_2.\end{aligned} \tag{17}$$

Due to Lemma 2.2, we reach the desired inequalities. □

By setting $q = 0$ in Theorem 2.15, we find the following result:

Corollary 2.16. *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha,\beta] \subseteq \mathbb{H}_0$ with $0 < m < M$. If there exist the constants γ_1, γ_2 such that $\gamma_1 \le f''(t) \le \gamma_2$ for every $t \in (\alpha,\beta)$, then*

$$\begin{aligned}\frac{1}{2}\gamma_1 P_\Psi(A,B) &\le \frac{f(m)}{M-m}(MA-B) + \frac{f(M)}{M-m}(B-mA) - P_f(A,B)\\ &\le \frac{1}{2}\gamma_2 P_\Psi(A,B) \le 0,\end{aligned}$$

where $\Psi(t) = (t-m)(M-t)$.

As a simple consequence of Corollary 2.16, one can get [3, Theorem 4]. Indeed, if we let $f(t) = \ln t$ in Corollary 2.16, then we deduce the following result.

Corollary 2.17. *[3, Theorem 4] Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M > 0$ with $m < M$. Then,*

$$\begin{aligned}
0 &\leq \frac{1}{2M^2} P_\Psi(A, B) \\
&\leq S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA) \\
&\leq \frac{1}{2m^2} P_\Psi(A, B),
\end{aligned}$$

where $\Psi(t) = (t - m)(M - t)$.

Our methods able us to identify the bounds of the parametric extension of the Shannon entropy and of the generalized Tsallis relative operator entropy. To these facts consider $f(t) = \ln t$ or $f(t) = \frac{t^\lambda - 1}{\lambda}$, $0 < \lambda < 1$ in Theorems 2.8, 2.11, 2.15 and conclude the following results.

Corollary 2.18. *Let A and B be two strictly positive operators and $0 \leq q < 1$.*

(i) If $mA \leq B \leq MA$ for some $m, M \in [e^{\frac{2q-1}{q(1-q)}}, \infty)$ with $m < M$, then

$$\begin{aligned}
0 &\leq S_q(A|B) - \frac{m^q \ln m}{M-m}(MA - B) - \frac{M^q \ln M}{M-m}(B - mA) \\
&\leq \frac{m^{q-1}(1 + q\ln m) - M^{q-1}(1 + q \ln M)}{M-m} P_\Psi(A, B) \\
&\leq \frac{1}{4}(M - m)\Big(m^{q-1}(1 + q\ln m) - M^{q-1}(1 + q\ln M)\Big)A,
\end{aligned}$$

(ii) If $mA \leq B \leq MA$ for some $m, M \in [e^{\frac{2q-1}{q(1-q)}}, \infty)$ with $m < M$, then

$$\begin{aligned}
2Q_q^{\ln}(m, M)P_r(A, B) &\leq S_q(A|B) - \frac{m^q \ln m}{M-m}(MA - B) - \frac{M^q \ln M}{M-m}(B - mA) \\
&\leq 2Q_q^{\ln}(m, M)P_R(A, B),
\end{aligned}$$

(iii) If $mA \leq B \leq MA$ for some $m, M \in [e^{\frac{2q-1}{q(1-q)}}, \infty)$ with $m < M$, then

$$\begin{aligned}
&\frac{m^{q-2}}{2}(2q - 1 + q(q-1)\ln M)P_\Psi(A, B) \\
&\quad \leq \frac{m^q f(m)}{M-m}(MA - B) + \frac{M^q f(M)}{M-m}(B - mA) - S_q(A|B) \\
&\quad \leq \frac{M^{q-2}}{2}(2q - 1 + q(q-1)\ln m)P_\Psi(A, B) \leq 0.
\end{aligned}$$

Interested readers can check it whenever A and B are two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [0, e^{\frac{2q-1}{q(1-q)}}]$ with $m < M$ and $q > 1$ or $q < 0$ the inequalities in parts (i)-(iii) of Corollary 2.18 are fulfilled.

We may determine the bounds of the generalized Tsallis relative operator entropy as follows:

Corollary 2.19. *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\sqrt[\lambda]{\frac{(\mu+(k-1)\lambda)(\mu+(k-1)\lambda-1)}{(\mu+k\lambda)(\mu+k\lambda-1)}}, \infty)$ with $m < M$.*

(i) *If* $0 < \lambda < 1$, $k \in \mathbb{Z}$, $0 < \mu + k\lambda < 1$, *then*

$$\begin{aligned}
0 &\leq \tilde{T}_{\mu,k,\lambda}(A,B) - \frac{\beta(m)}{\lambda(M-m)}(MA-B) - \frac{\beta(M)}{\lambda(M-m)}(B-mA) \\
&\leq \frac{\alpha(m)-\alpha(M)}{\lambda(M-m)} P_\Psi(A,B) \\
&\leq \frac{M-m}{4\lambda}\Big(\alpha(m)-\alpha(M)\Big)A,
\end{aligned}$$

(ii) *If* $0 < \lambda < 1$, $k \in \mathbb{Z}$, $0 < \mu + k\lambda < 1$, *then*

$$\begin{aligned}
0 &\leq 2Q_{\mu+(k-1)\lambda}^{\frac{t^\lambda-1}{\lambda}}(m,M)P_r(A,B) \\
&\leq \tilde{T}_{\mu,k,\lambda}(A,B) - \frac{\beta(m)}{\lambda(M-m)}(MA-B) - \frac{\beta(M)}{\lambda(M-m)}(B-mA) \\
&\leq 2Q_{\mu+(k-1)\lambda}^{\frac{t^\lambda-1}{\lambda}}(m,M)P_R(A,B),
\end{aligned}$$

(iii) *If* $0 < \lambda < 1$, $k \in \mathbb{Z}$, *and* $0 < \mu + k\lambda < 1$, *then*

$$\begin{aligned}
&\frac{1}{2}(\mu+k\lambda)(\mu+k\lambda-1)M^\lambda P_\Psi(A,B) \\
&\quad \leq \frac{\beta(m)}{\lambda(M-m)}(MA-B) + \frac{\beta(M)}{\lambda(M-m)}(B-mA) - \tilde{T}_{\mu,k,\lambda}(A,B) \\
&\quad \leq \frac{1}{2}(\mu+k\lambda)(\mu+k\lambda-1)m^\lambda P_\Psi(A,B) \leq 0,
\end{aligned}$$

where $\alpha(t) = t^{\mu+(k-1)\lambda-1}((\mu+k\lambda)t^\lambda - \mu - (k-1)\lambda)$, $\beta(t) = t^{\mu+k\lambda} - t^{\mu+(k-1)\lambda}$, *and* $\Psi(t) = (t-m)(M-t)$.

Proof. Consider $f(t) = \frac{t^\lambda-1}{\lambda}$ and $q = \mu + (k-1)\lambda$ in Theorems 2.8, 2.11, 2.15 respectively and deduce the desired results. □

In particular, when we put $\mu = 0$, $k = 1$ in Corollary 2.19, we achieve the bounds of the Tsallis relative operator entropy.

Corollary 2.20. *Let* A *and* B *be two strictly positive operators such that* $mA \leq B \leq MA$ *for some* m, M *with* $0 < m < M$.

(i) *[16, Theorem 6] If* $0 < \lambda < 1$, *then*

$$\begin{aligned}
0 &\leq T_\lambda(A,B) - \frac{m^\lambda-1}{\lambda(M-m)}(MA-B) - \frac{M^\lambda-1}{\lambda(M-m)}(B-mA) \\
&\leq \frac{m^{\lambda-1}-M^{\lambda-1}}{M-m}P_\Psi(A,B) \leq \frac{M-m}{4}(m^{\lambda-1}-M^{\lambda-1})A,
\end{aligned}$$

(ii) *[16, Theorem 7] If* $0 < \lambda < 1$, *then*

$$\begin{aligned}
0 &\leq 2Q_0^{\frac{t^\lambda-1}{\lambda}}(m,M)P_r(A,B) \\
&\leq T_\lambda(A,B) - \frac{m^\lambda-1}{\lambda(M-m)}(MA-B) - \frac{M^\lambda-1}{\lambda(M-m)}(B-mA) \\
&\leq 2Q_0^{\frac{t^\lambda-1}{\lambda}}(m,M)P_R(A,B),
\end{aligned}$$

(iii) If $0 < \lambda < 1$, *then*

$$\begin{aligned}
&\frac{1}{2}\lambda(\lambda-1)M^{\lambda}P_{\Psi}(A,B)\\
&\leq \frac{m^{\lambda}-1}{\lambda(M-m)}(MA-B)+\frac{M^{\lambda}-1}{\lambda(M-m)}(B-mA)-T_{\lambda}(A,B)\\
&\leq \frac{1}{2}\lambda(\lambda-1)m^{\lambda}P_{\Psi}(A,B)\leq 0,
\end{aligned}$$

where $\Psi(t) = (t-m)(M-t)$.

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