Filomat 32:14 (2018), 5115–5122
https://doi.org/10.2298/FIL1814115L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

I_2-Convergence in T_0 Spaces

Jing Lu[a], Bin Zhao[a]

[a] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P.R. China

Abstract.
In this paper, we introduce the notion of I_2-convergence in T_0 spaces, and study the fundamental properties of I_2-topology which is determined by I_2-convergence according to the standard topological approach. Then, we give a sufficient condition for I_2-convergence to be topological. Finally, we introduce a special class of T_0 spaces, called *IDC*-spaces, and then present a sufficient and necessary condition for I_2-convergence to be topological in *IDC*-spaces.

1. Introduction

Convergence plays an important role in the research of general topology and order theory. Just because of this, the convergence problems have been considered by many researchers (see [1, 5, 6, 9–13, 15, 17–19]). In particular, an important convergence is the lim-inf convergence in complete lattices (see [6]), which is introduced by Scott to characterize continuous lattices. A net $(x_i)_{i\in I}$ in a complete lattice L is said to lim-inf converge to an element x if $x = sup\{inf\{x_i \mid i \geq k\} \mid k \in I\}$. A basic question arises naturally: is the lim-inf convergence in a complete lattice L topological? That is, there exists a topology $\mathcal{T}$ on a complete lattice L such that a net $(x_i)_{i\in I}$ in L lim-inf converges to x if and only if it converges to x with respect to the topology $\mathcal{T}$. It has been shown by Scott that the lim-inf convergence in a complete lattice L is topological if and only if L is a continuous lattice (see [6]). Later on, a general result showed that the lim-inf convergence (also be called $\mathcal{S}$-limit in [5]) in a dcpo L is topological if and only if L is a domain (see [5]). As a generalization of lim-inf convergence in dcpos, Zhao and Zhao (see [15]) introduced the lim-inf convergence in a partially ordered set, and proved that for a poset P the lim-inf convergence is topological if and only if P is a continuous poset.

In a recent invited talk at the Sixth International Symposium on Domain Theory, Lawson emphasized the need to develop the core of domain theory directed in T_0 spaces instead of posets. Towards this new direction, motivated by the definition of the Scott topology, Zhao and Ho [16] introduced a method of deriving a new topology out of a given one. They called this topology the irreducibly-derived topology (or simply, *SI*-topology). Furthermore, they introduced *SI*-continuous spaces, which lead to a generalization of the concept of continuous posets. In [7], Heckmann and Keimel presented a topological variant of Rudin's Lemma where irreducible sets replace directed sets. Moreover, in [14], as a generalization of lim-inf

2010 *Mathematics Subject Classification*. Primary 54D10; Secondary 06B35, 06B30
Keywords. I_2-convergence, T_0 space, *IDC*-space
Received: 15 June 2018, Revised: 10 December 2018; Accepted: 12 December 2018
Communicated by Ljubiša D.R. Kočinac
Corresponding author: Bin Zhao
Research supported by the National Natural Science Foundation of China (Grant no. 11531009)
Email addresses: lujing0926@126.com (Jing Lu), zhaobinmath@xjtu.edu.cn (Bin Zhao)

convergence in posets, the authors introduced the concept of *Irr*-convergence in a wider context of T_0 topological spaces, and present a sufficient and necessary condition for *Irr*-convergence to be topological in T_0 spaces.

Erné (see [4]) introduced the concepts of S_2-convergence in posets through filter and S_2-continuous posets by making use of the cut operator instead of join. The above notions have the advantage that not even the existence of directed joins has to be required. Moreover, Erné proved that the S_2-convergence in a poset P is topological if and only if P is an S_2-continuous poset. In this paper, we continue to respond to Lawson's call to develop the core of domain theory directly in topological spaces by establishing a topology parallel of the aforementioned result. More precisely, as a common generalization of both S_2-convergence and *Irr*-convergence, we introduce a new convergence in T_0 spaces, called I_2-convergence, and hope to find a satisfactory sufficient and necessary condition for I_2-convergence to be topological. We introduce the notion of I_2-continuous spaces, and prove that the I_2-convergence is topological in I_2-continuous spaces. Furthermore, we introduce a special class of T_0 spaces, called *IDC*-spaces, and then obtain the main result of this paper, that is, we give a sufficient and necessary condition for I_2-convergence to be topological in *IDC*-spaces, generalizing the known result of S_2-convergence to be topological in posets.

2. Preliminaries

Throughout the paper, we refer to [5] for domain theory, and to [2] for general topology.

Let P be a poset. A non-empty subset D of P is directed if every finite subset of D has an upper bound in D. A subset A of P is upper if $A = \uparrow A = \{x \in P : x \geq y \text{ for some } y \in A\}$. The Alexandroff topology $\Upsilon(P)$ on P is the topology consisting of all its upper subsets. A subset U of P is called Scott open if (i) $U = \uparrow U$ and (ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets on P form the Scott topology $\sigma(P)$. Obviously, a subset U of P is a Scott open set if and only if U is a Alexandroff open set and for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists.

Let P be a poset. An upper (resp., a lower) bound of a subset A of P is an element x such that $y \leq x$ (resp., $x \leq y$) for all $y \in A$. The set of all upper (resp., lower) bounds of A will be denoted by A^u (resp., A^l). Given any two elements x and y in P, we say that $x \ll_2 y$ if for any directed set $D \subseteq P$ with $y \in D^{ul}$, there exists $d \in D$ such that $x \leq d$. The set $\{y \in P \mid y \ll_2 x\}$ will be denoted by $\Downarrow_2 x$. P is called S_2-continuous (see [4]) if for any $x \in P$, $\Downarrow_2 x$ is directed and $x = \bigvee \Downarrow_2 x$. In fact, we have that $x = \bigvee \Downarrow_2 x$ iff $x \in (\Downarrow_2 x)^{ul}$.

Proposition 2.1. ([3]) *Let P be a poset. Then the following statements hold:*
(1) Let A, B be subsets of P. If $A \subseteq B$, then $B^u \subseteq A^u$ and $B^l \subseteq A^l$;
(2) For all $a \in P$, $(\downarrow a)^{ul} = \downarrow a$.

Definition 2.2. ([16]) Let P be a poset. A subset U of P is called σ_2-open if the following conditions are satisfied:
(1) $U = \uparrow U$;
(2) For any directed set $D \subseteq P$, $D^{ul} \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$.

The collection of all σ_2-open subsets of P forms a topology, it will be called σ_2-topology of P and will be denoted by $\sigma_2(P)$.

Given a topological space (X, τ), a non-empty subset F of X is called a τ-irreducible set (or simply, irreducible set) if whenever $F \subseteq A \cup B$ for closed sets A, $B \subseteq X$, one has either $F \subseteq A$ or $F \subseteq B$. The set of all τ-irreducible sets of X will be denoted by $Irr_\tau(X)$. X is called sober if for every irreducible closed set F, there is a unique point $x \in X$ such that $F = cl(\{x\})$. Notice that every sober space is necessarily T_0.

For any T_0 space (X, τ), the specialization order $\leq$ on X is defined by $x \leq y$ if and only if $x \in cl(\{y\})$. Unless otherwise stated, throughout the paper, whenever an order concept is mentioned in the context of a T_0 space X, it is to be interpreted with respect to the specialization order on X.

Proposition 2.3. ([5]) *Let (X, τ) be a T_0 space. Then the following statements hold:*
(1) For all $a \in X$, $\downarrow a = \{x \in X \mid x \leq a\} = cl_X(\{a\})$.
(2) If $U \subseteq X$ is an open subset. Then we have $\uparrow U = U$.
(3) If $D \subseteq X$ is a directed set with respect to the specialization order, then D is irreducible.

Definition 2.4. ([16]) Let (X, τ) be a T_0 space. A subset U of X is called *SI*-open if the following conditions are satisfied:
(1) $U \in \tau$;
(2) For any $F \in Irr_\tau(X)$, $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists.

The set of all *SI*-open sets of (X, τ) is denoted by τ_{SI}. We can see that τ_{SI} is a topology on X. We call τ_{SI} the irreducibly-derived topology of τ. The space (X, τ_{SI}) will also be simply written as $SI(X)$. Moreover, complements of *SI*-open sets are called *SI*-closed sets.

Proposition 2.5. ([16]) *Let (X, τ) be a T_0 space. Then the specialization orders of spaces X and $SI(X)$ coincide, and $Irr_\tau(X) \subseteq Irr_{\tau_{SI}}(X)$.*

3. I_2-Topology

Based on the S_2-convergence in posets, we introduce the notion of I_2-convergence in T_0 spaces by replacing the directed subsets with irreducible subsets. In this section, we study the properties of I_2-convergence and the I_2-topology, which is obtained by I_2-convergence.

Let X be a set and $\mathcal{P}(X)$ the family of all subsets of X. By a filter $\mathcal{F}$ in X we mean a non-empty subfamily $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following conditions:
(1) $\emptyset \notin \mathcal{F}$;
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
(3) If $A \in \mathcal{F}$ and $A \subseteq B \in \mathcal{P}(X)$, then $B \in \mathcal{F}$.

Given a topological space (X, τ), a filter $\mathcal{F}$ in X is said to converge to $x \in X$ with respect to the topology τ if for any $U \in \tau$ with $x \in U$, $U \in \mathcal{F}$.

Definition 3.1. ([4]) Let P be a poset. A filter $\mathcal{F}$ in P S_2-converges to a point $x \in P$ if there exists an ideal D such that $x \in D^{ul}$ and for each $d \in D$, $\uparrow d \in \mathcal{F}$.

Definition 3.2. Let (X, τ) be a T_0 space. A filter $\mathcal{F}$ in X I_2-converges to $x \in X$ if there exists an irreducible subset F of X such that
(1) $x \in F^{ul}$;
(2) For each $e \in F$, $\uparrow e \in \mathcal{F}$.
In this case, we write $\mathcal{F} \xrightarrow{I_2} x$.

Remark 3.3. (1) Let P be a poset. Then I_2-convergence in the Alexandroff topological space $(P, \Upsilon(P))$ coincides with S_2-convergence in P. In particular, when P is a dcpo, I_2-convergence in a T_0 space $(P, \Upsilon(P))$ coincides with S_3-convergence in P (see [4]).

(2) Let $\mathcal{F}_{\{x\}} = \uparrow\{\{x\}\}$ denote the filter generated by the filter-base $\{\{x\}\}$, then $\mathcal{F}_{\{x\}} \xrightarrow{I_2} x$.

(3) Let X be a T_0 space and D be a directed subset of X. Then $\uparrow\{\uparrow d \mid d \in D\}$ is a filter in X and $\uparrow\{\uparrow d \mid d \in D\} \xrightarrow{I_2} x$ for any $x \in D^{ul}$.

(4) Let $\mathcal{F} \xrightarrow{I_2} x$, and $y \leq x$. Then $\mathcal{F} \xrightarrow{I_2} y$.

Definition 3.4. Let (X, τ) be a T_0 space. Then

$$\tau_{I_2} = \{U \subseteq X \mid \text{ whenever } \mathcal{F} \xrightarrow{I_2} x \text{ and } x \in U, U \in \mathcal{F}\}$$

is a topology, called the I_2-topology on X. $U \in \tau_{I_2}$ is called I_2-open. Complements of I_2-open sets are called I_2-closed sets.

Remark 3.5. (1) Let (X, τ) be a T_0 space. If $\mathcal{F} \xrightarrow{I_2} x$, then $\mathcal{F}$ converges to x with respect to the topology τ_{I_2}.
(2) Let P be a poset. Then $\sigma_2(P) = \tau_{I_2}$, where T_0 space is the Alexandroff space.
(3) Let (X, τ) be a T_0 space. Then τ and τ_{I_2} are independent. Please see Example 3.7.

Proposition 3.6. *Let (X,τ) be a T_0 space and $U \subseteq X$. If for any irreducible set F, $F^{ul} \cap U \neq \emptyset$ implies $\uparrow e \subseteq U$ for some $e \in F$, then U is an I_2-open set.*

Proof. We first show that U is an upper set. Given $x \leq y$ with $x \in U$, consider $F = \{x\}$, which gives $\uparrow x \subseteq U$ and so $y \in U$. Let $\mathcal{F} \xrightarrow{I_2} x$ and $x \in U$. Then there exists an irreducible set F such that $x \in F^{ul}$, and for each $e \in F$, $\uparrow e \in \mathcal{F}$. Thus $F^{ul} \cap U \neq \emptyset$, and so $\uparrow e \subseteq U$ for some $e \in F$. Therefore $U \in \mathcal{F}$. □

Example 3.7. (1) Let ε be the usual topology on the set of real number $\mathbb{R}$. Then $(\mathbb{R},\varepsilon)$ is a T_2 space. Let $U = \{x\}$ for some $x \in \mathbb{R}$. Suppose that F is an irreducible set, and $F^{ul} \cap U \neq \emptyset$. Then F is a single point set. Without loss of generality, suppose that $F = \{a\}$. Then $F^{ul} = \{a\}$, and thus $a = x$. So we conclude that $\uparrow x \subseteq U$. It follows from Proposition 3.6 that $U \in \tau_{I_2}$. Obviously, $U \notin \varepsilon$.

(2) Let $X = \mathbb{N} \cup \{\omega_1, \omega_2\}$. Define the order on X as follows:

$$0 \leq 1 \leq \cdots \leq \cdots \leq \omega_1 \text{ and } 0 \leq 1 \leq \cdots \leq \cdots \leq \omega_2.$$

Then $(X, \Upsilon(X))$ is a T_0 space. Obviously, $\{\omega_1\} \in \Upsilon(X)$. By Remark 3.3(3), we have that $\uparrow\{\uparrow n \mid n \in \mathbb{N}\} \xrightarrow{I_2} \omega_1$, but $\{\omega_1\} \notin \uparrow\{\uparrow n \mid n \in \mathbb{N}\}$. Therefore, $\{\omega\}$ is not an I_2-open set.

Proposition 3.8. *Let (X,τ) be a T_0 space. Then (X, τ_{I_2}) is a T_0 space.*

Proof. Let $x \in X$. Suppose that F is an irreducible set with $F^{ul} \cap X \setminus \downarrow x \neq \emptyset$. Then there exists $y \in F^{ul} \cap X \setminus \downarrow x$. Assume that $F \cap (X \setminus \downarrow x) = \emptyset$. Then $F \subseteq \downarrow x$, and thus $x \in F^u$. It follows from $y \in F^{ul}$ that $y \leq x$, which is a contradiction. Therefore $F \cap (X \setminus \downarrow x) \neq \emptyset$, that is, there exists $e \in F \cap (X \setminus \downarrow x)$. Therefore $\uparrow e \subseteq X \setminus \downarrow x$. By Proposition 3.6, we have that $X \setminus \downarrow x$ is an I_2-open set. Then (X, τ_{I_2}) is a T_0 space. □

4. I_2-Continuous Spaces

Definition 4.1. Let X be a T_0 space. For $x, y \in X$, define $x \ll_{I_2} y$ if for every filter $\mathcal{F}$ in X which I_2-converges to y, $\uparrow x \in \mathcal{F}$.

We denote the set $\{x \in X \mid x \ll_{I_2} a\}$ by $\Downarrow_{I_2} a$, and the set $\{x \in X \mid a \ll_{I_2} x\}$ by $\Uparrow_{I_2} a$.

Remark 4.2. (1) Let P be a poset. Then $x \ll_2 y$ if and only if $x \ll_{I_2} y$, where the topology on P is the Alexandroff topology.

(2) Let X be a T_0 space, $x, y \in X$. Then $x \ll_{I_2} y$ implies $x \ll_2 y$.

Proposition 4.3. *Let X be a T_0 space. Then the following statements hold:*

(1) $x \ll_{I_2} y$ implies $x \leq y$ for all $x, y \in X$.

(2) $a \leq b \ll_{I_2} c \leq d$ implies $a \ll_{I_2} d$ for all $a, b, c, d \in X$.

Proof. (1) By Remark 3.3(2), we have that the filter $\uparrow\{\{y\}\}$ I_2-converges to y. Since $x \ll_{I_2} y$, we have that $\uparrow x \in \uparrow\{\{y\}\}$. Then $\{y\} \subseteq \uparrow x$, and thus $x \leq y$.

(2) Let $\mathcal{F} \xrightarrow{I_2} d$. It follows from $c \leq d$ that $\mathcal{F} \xrightarrow{I_2} c$. Since $b \ll_{I_2} c$, we have that $\uparrow b \in \mathcal{F}$. Then $\uparrow a \in \mathcal{F}$, and so we conclude that $a \ll_{I_2} d$. □

Proposition 4.4. *Let X be a T_0 space, $x, y \in X$. If for any irreducible set F, $y \in F^{ul}$ implies $x \leq e$ for some $e \in F$, then $x \ll_{I_2} y$.*

Proof. Let $x, y \in X$. Suppose that $\mathcal{F} \xrightarrow{I_2} y$. Then there exists an irreducible set F such that $y \in F^{ul}$, and for each $e \in F$, $\uparrow e \in \mathcal{F}$. By hypothesis, there exists $e \in F$ such that $x \leq e$. Then $\uparrow x \in \mathcal{F}$, and thus $x \ll_{I_2} y$. □

Definition 4.5. A T_0 space X is called an I_2-continuous space, if for each $a \in X$, the following conditions are satisfied:

(1) $\Downarrow_{I_2} a$ is an irreducible set and $a \in (\Downarrow_{I_2} a)^{ul}$;

(2) $\Uparrow_{I_2} a$ is an I_2-open set.

In fact, it follows from Proposition 4.3(1) that $a \in (\Downarrow_{I_2} a)^{ul}$ if and only if $a = \bigvee \Downarrow_{I_2} a$.

Remark 4.6. (1) Let P be a poset. Then P is an S_2-continuous poset if and only if $(P, \Upsilon(P))$ is an I_2-continuous space.

(2) Every T_2 space is an I_2-continuous space.

Theorem 4.7. *Let X be an I_2-continuous space. Then $\mathcal{F} \xrightarrow{I_2} x$ if and only if the filter $\mathcal{F}$ converges to x with respect to the topology τ_{I_2}.*

Proof. By Remark 3.5(1), the necessity is clear. Conversely, suppose that the filter $\mathcal{F}$ converges to x with respect to the topology τ_{I_2}. Since X is an I_2-continuous space, $\Downarrow_{I_2} x$ is an irreducible set and $x \in (\Downarrow_{I_2} x)^{ul}$. For all $y \in \Downarrow_{I_2} x$, we have that $x \in \Uparrow_{I_2} y$. Since $\Uparrow_{I_2} y$ is I_2-open, we have that $\Uparrow_{I_2} y \in \mathcal{F}$. It follows from $\Uparrow_{I_2} y \subseteq \uparrow y$ that $\uparrow y \in \mathcal{F}$. Therefore $\mathcal{F} \xrightarrow{I_2} x$. □

5. I_2-Convergence in *IDC*-Spaces

In this section, we introduce a special class of T_0 spaces, called *IDC*-spaces. The relationships between *IDC*-spaces and other spaces are investigated. I_2-convergence in *IDC*-spaces is also studied.

Definition 5.1. A T_0 space X is called an *IDC*-space if for each irreducible set F, there exists a directed set $D \subseteq \downarrow F$ such that $D^{ul} = F^{ul}$.

Example 5.2. (1) Let P be a poset. Then $(P, \Upsilon(P))$ is an *IDC*-space.

(2) Let $X = (\mathbb{N} \times (\mathbb{N} \cup \{\infty\}))$ with the partial order defined by

$$(m_1, n_1) \leq (m_2, n_2) \text{ iff } m_1 = m_2, n_1 \leq n_2 \leq \infty \text{ or } n_2 = \infty, n_1 \leq m_2.$$

Now, we consider the Scott topology space $(X, \sigma(X))$. It is proved in [8] that X is an irreducible set. Obviously, $X^{ul} = X$. Assume that there exists a directed set $D \subseteq X$ such that $D^{ul} = X^{ul} = X$. Since X is a dcpo, we have that $\bigvee D$ exists. Then $D^{ul} = \downarrow(\bigvee D)$, and thus $X = \downarrow(\bigvee D)$. But this is a contradiction. Thus $(X, \sigma(X))$ is not an *IDC*-space.

(3) Let X be a *C*-space. Then X is an *IDC*-space. But the converse may not be true. For example, let $X = \{a_i \mid i \in \mathbb{N}\} \cup \{b_i \mid i \in \mathbb{N}\} \cup \{\top\}$, where $\mathbb{N}$ denotes the set of all positive integers. The order $\leq$ on X is defined as follows:

(i) $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq \top$;

(ii) $b_1 \leq b_2 \leq \cdots \leq b_n \leq \cdots \leq \top$.

Then $(X, \leq)$ is a poset. Next, we shall prove that $(X, \sigma(X))$ is an *IDC*-space. If F is finite, then $\bigcup_{x \in F} \downarrow x$ is a finite union of closed sets. Since F is irreducible and $F \subseteq \bigcup_{x \in F} \downarrow x$, there is x in F such that $F \subseteq \downarrow x$. This x is the greatest element of F, and $F^{ul} = \{x\}^{ul}$. If F is infinite, then at least one of $F \cap \{a_i \mid i \in \mathbb{N}\}$ and $F \cap \{b_i \mid i \in \mathbb{N}\}$ must be infinite. Whenever $F \cap \{a_i \mid i \in \mathbb{N}\}$ is an infinite set, $F \cap \{a_i \mid i \in \mathbb{N}\}$ is a directed set and $F^{ul} = (F \cap \{a_i \mid i \in \mathbb{N}\})^{ul} = \downarrow \top$. Similarly, we can explain the case that $F \cap \{b_i \mid i \in \mathbb{N}\}$ is an infinite set. Therefore, $(X, \sigma(X))$ is an *IDC*-space. Suppose that $(X, \sigma(X))$ is a *C*-space. Then X is a continuous poset. In fact, $\Downarrow a_i = \emptyset$, then X is not a continuous poset. But this is a contradiction, so we conclude that $(X, \sigma(X))$ is not a *C*-space.

(4) Every T_2 space is an *IDC*-space. In fact, if F is an irreducible set, then F is a single point set. Thus F is a directed set. Therefore, X is an *IDC*-space. But the converse may not be true. Please see the following example.

Example 5.3. Let $X = \{a_i \mid i \in \mathbb{N}\}$. The order $\leq$ on X is defined as follows:

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

Then $(X, \leq)$ is a poset. By Example 5.2(1), we have that $(X, \Upsilon(X))$ is an *IDC*-space. Obviously, $(X, \Upsilon(X))$ is not a T_1 space. Moreover, one can see that $\{a_i \mid i \in \mathbb{N}\}$ is a directed set. But there does not exist $x \in X$ such that $\{a_i \mid i \in \mathbb{N}\} = \downarrow x$. Thus $(X, \Upsilon(X))$ is not a sober space.

Remark 5.4. (1) A T_1 space may not be an *IDC*-space (see Example 5.5(1)). Moreover, An *IDC*-space may not be a T_1 space (see Example 5.3).

(2) An *IDC*-space may not be a sober space (see Example 5.3). Moreover, a sober space need not be an *IDC*-space (see Example 5.5(2)).

Example 5.5. (1) Let X be an infinite set, and

$$\tau = \{A \subseteq X \mid \text{ the complement of } A \text{ is finite }\} \cup \{\emptyset\}.$$

Then (X, τ) is a T_1 space. Let A and B be closed sets, and $X \subseteq A \cup B$. Assume that $X \nsubseteq A$ and $X \nsubseteq B$. Hence $A \neq X$ and $B \neq X$. Since A and B are closed sets, A and B are finite sets. But this contradicts with the fact that X is an infinite set. Therefore, X is an irreducible set. Obviously, $X^{ul} = X$. Assume that there exists a directed set D such that $D^{ul} = X^{ul}$. Since D is a single point set, there exists $x \in X$ such that $D = \{x\}$. Then $D^{ul} = \{x\}^{ul} = \{x\} \neq X$. Therefore, X is not an *IDC*-space.

(2) Let $X = (\mathbb{N} \times (\mathbb{N} \cup \{\infty\})) \cup \{\top\}$. The order $\leq$ on X is defined as follows:

(i) for any $(m, n) \in \mathbb{N} \times (\mathbb{N} \cup \{\infty\}), (m, n) \leq \top$;

(ii) for any (m_1, n_1), $(m_2, n_2) \in \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, $(m_1, n_1) \leq (m_2, n_2)$ iff $m_1 = m_2$, $n_1 \leq n_2 \leq \infty$ or $n_2 = \infty$, $n_1 \leq m_2$.

Then $(X, \leq)$ is a poset, and thus $(X, \sigma(X))$ is a T_0 space. Let $F = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. One can conclude that F is an irreducible Scott closed set. Then F is an irreducible set in $SI(X)$. Since $\bigvee F = \top$, we have that $F^{ul} = \downarrow\top$. But there does not exist directed set $D \subseteq F$ such that $D^{ul} = \downarrow\top$. Thus, $SI(X)$ is not an *IDC*-space. One can conclude that every irreducible closed set in $SI(X)$ is exactly a principle ideal. Therefore, $SI(X)$ is a sober space.

Proposition 5.6. *Let X be an IDC-space. Then for any irreducible set F, there exists a filter $\mathcal{F}$ such that $\mathcal{F} \xrightarrow{I_2} x$ for all $x \in F^{ul}$.*

Proof. Let F be an irreducible set. Since X is an *IDC*-space, there exists a directed set $D \subseteq \downarrow F$ such that $D^{ul} = F^{ul}$. By Remark 3.3(3), we have that $\uparrow\{\uparrow d \mid d \in D\}$ is a filter and $\uparrow\{\uparrow d \mid d \in D\} \xrightarrow{I_2} y$ for all $y \in D^{ul}$. Therefore, $\uparrow\{\uparrow d \mid d \in D\} \xrightarrow{I_2} x$ for all $x \in F^{ul}$. □

The above proposition may fail for a T_0 space. Please see Example 5.8.

Lemma 5.7. *Let X be a T_0 space. If a filter $\mathcal{F}$ such that $\mathcal{F} \xrightarrow{I_2} x$, then there exists a net $(x_i)_{i\in I}$ satisfying the following conditions:*

(1) There exists an irreducible set F such that $x \in F^{ul}$;

(2) $\forall\, e \in F$, there exists $i_0 \in I$ such that $e \leq x_i$ for any $i \geq i_0$.

Proof. Let $I = \{(a, A) \mid a \in A \in \mathcal{F}\}$. Then $I \neq \emptyset$. The pre-order $\leq$ on I is defined as follows:

$$(a, A) \leq (b, B) \text{ if and only if } B \subseteq A.$$

Then I is a directed set. Let $x_i = a$ for $i = (a, A) \in I$. Then $(x_i)_{i\in I}$ is a net. Since $\mathcal{F} \xrightarrow{I_2} x$, there exists an irreducible set F such that $x \in F^{ul}$, and $\uparrow e \in \mathcal{F}$ for all $e \in F$. $\forall\, e \in F$, let $i_0 = (e, \uparrow e) \in I$. Then $b \in B \subseteq \uparrow e$ for any $i = (b, B) \geq i_0$, and thus $e \leq x_i$ for all $i \geq i_0$. □

Example 5.8. Let $(X, \mathcal{T})$ be a T_1 space defined in Example 5.5(1). By Example 5.5(1), we have that $(X, \mathcal{T})$ is not an *IDC*-space and X is an irreducible set. Assume that there exists a filter $\mathcal{F}$ such that $\mathcal{F} \xrightarrow{I_2} x$ for all $x \in X^{ul} = X$. By Lemma 5.7, there exists a net $(x_i)_{i\in I}$ and for any $x \in X^{ul}$, the net $(x_i)_{i\in I}$ satisfies the following conditions:

(1) there exists an irreducible set F such that $x \in F^{ul}$;

(2) $\forall\, e \in F$, there exists $i_0 \in I$ such that $e \leq x_i$ for any $i \geq i_0$.

Suppose that $|F| \geq 2$. Then F is an infinite set. Take $e_1 \in F$. By (2), there exists $i_1 \in I$ such that $e_1 \leq x_i$ for any $i \geq i_1$. Take $e_2 \in F$ such that $e_1 \neq e_2$. By (2), there exists $i_2 \in I$ such that $e_2 \leq x_i$ for any $i \geq i_2$. Since I is a directed set, there exists $i_3 \in I$ such that $i_1, i_2 \leq i_3$. Then $e_1, e_2 \leq x_i$ for any $i \geq i_3$. Since X is a T_1 space, we have that $e_1 = e_2$. But this is a contradiction. We conclude that $|F| = 1$. Since $x \in F^{ul}$, we have that $F = \{x\}$. Then there exists $i_0 \in I$ such that $x \leq x_i$ for any $i \geq i_0$, and thus $x = x_i$ for any $i \geq i_0$. Since X is a infinite set, there exists $y \in X$ such that $x \neq y$. Repeat the above process, we also have that there exists $i_4 \in I$ such that $x_i = y$ for any $i \geq i_4$. Then there exists $i_5 \in I$ such that $x_i = y = x$ for any $i \geq i_5$. But this is a contradiction. Therefore, there does not exist filter $\mathcal{F}$ such that $\mathcal{F} \xrightarrow{I_2} x$ for all $x \in X^{ul} = X$.

Proposition 5.9. *Let (X, τ) be an IDC-space and $U \subseteq X$. Then $U \in \tau_{I_2}$ if and only if the following two conditions are satisfied:*

(1) U is an upper set;

(2) For any irreducible set F, $F^{ul} \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$.

Proof. By Proposition 3.6, sufficiency is clear. It suffices to prove the necessity. By Proposition 3.8, we have that (X, τ_{I_2}) is a T_0 space. By Proposition 2.3(2) again, we have that U is an upper set. Let F be an irreducible set. Then there exists a directed set $D \subseteq \downarrow F$ such that $D^{ul} = F^{ul}$ and $\uparrow\{\uparrow d \mid d \in D\} \xrightarrow{I_2} a$ for all $a \in D^{ul}$. Since $F^{ul} \cap U \neq \emptyset$, there exists $a \in F^{ul} \cap U$. Then $\uparrow\{\uparrow d \mid d \in D\} \xrightarrow{I_2} a \in U$, and thus $U \in \uparrow\{\uparrow d \mid d \in D\}$. Hence, $\uparrow d \subseteq U$ for some $d \in D$. Obviously, there exists $m \in F$ such that $d \leq m$. Therefore, $m \in F \cap U$, i.e., $F \cap U \neq \emptyset$. □

Proposition 5.10. *Let X be an IDC-space and $x, y \in X$. Then the following statements hold:*

(1) For any irreducible set F, $y \in F^{ul}$ implies $x \leq e$ for some $e \in F$;

(2) $x \ll_{I_2} y$.

Proof. By Proposition 4.4, (1)⟹(2) is clear.

(2)⟹(1) Let $x \ll_{I_2} y$, and let F be an irreducible set with $y \in F^{ul}$. Since X is an *IDC*-space, there exists a directed set $D \subseteq \downarrow F$ such that $D^{ul} = F^{ul}$. By Remark 3.3(3), we have that $\uparrow\{\uparrow d \mid d \in D\} \xrightarrow{I_2} y$. Since $x \ll_{I_2} y$, we have that $\uparrow x \in \uparrow\{\uparrow d \mid d \in D\}$. Then $\uparrow d \subseteq \uparrow x$ for some $d \in D$, and thus $x \leq d \leq m$ for some $m \in F$. □

Proposition 5.11. *Let X be an IDC-space. Then the filter $\mathcal{F}$ in X I_2-converges to $x \in X$ if and only if the filter $\mathcal{F}$ in X S_2-converges to x under the specialization order.*

Proof. Sufficiency is clear. Next, we shall prove the necessity. Let the filter $\mathcal{F}$ in X I_2-converges to $x \in X$. Then there exists a filter F such that $x \in F^{ul}$, and for any $e \in F$, $\uparrow e \in \mathcal{F}$. Since X is an *IDC*-space, there exists a directed set $D \subseteq \downarrow F$ such that $D^{ul} = F^{ul}$. Then for any $d \in D$, there exists $e \in F$ such that $d \leq e$, and thus $\uparrow e \subseteq \uparrow d$. So we conclude that $\uparrow d \in \mathcal{F}$. Therefore, $\mathcal{F}$ S_2-converges to x under the specialization order. □

Proposition 5.12. *Let X be an IDC-space. Then X is an I_2-continuous space if and only if X is an S_2-continuous poset under the specialization order.*

Proof. Necessity. Let $x \in X$. By Remark 4.2(2), we have that $\Downarrow_{I_2} x \subseteq \Downarrow_2 x$. Since X is an I_2-continuous space, we have that $\Downarrow_{I_2} x$ is an irreducible set and $x \in (\Downarrow_{I_2} x)^{ul}$. Then there exists a directed set $D \subseteq \Downarrow_{I_2} x$ such that $D^{ul} = (\Downarrow_{I_2} x)^{ul}$, and thus $\Downarrow_2 x$ is an irreducible set and $x \in (\Downarrow_2 x)^{ul}$. Therefore, X is an S_2-continuous poset under the specialization order.

Sufficiency. Let $x, y \in X$ satisfying $x \ll_2 y$. Let F be an irreducible set with $y \in F^{ul}$. Then there exists a directed set $D \subseteq \downarrow F$ such that $y \in D^{ul} = F^{ul}$, and thus $x \leq d$ for some $d \in D$. So we conclude that $x \leq m$ for some $m \in F$. By Proposition 5.10, we have that $x \ll_{I_2} y$. It follows from that $\Downarrow_{I_2} x = \Downarrow_2 x$ is an irreducible set and $x \in (\Downarrow_{I_2} x)^{ul}$. It suffices to prove that $\Uparrow_{I_2} x$ is an I_2-open set. Let G be an irreducible set satisfying $G^{ul} \cap \Uparrow_{I_2} x \neq \emptyset$. Then there exists $y \in G^{ul} \cap \Uparrow_{I_2} x$, that is, $x \ll_{I_2} y \in G^{ul}$. Since X is an *IDC*-space, there exists a directed set $D_1 \subseteq \downarrow G$ such that $D_1^{ul} = G^{ul}$, so we conclude that $x \ll_2 y \in D_1^{ul}$. Then $x \ll_2 d_1$ for some $d_1 \in D_1$, and thus $x \ll_{I_2} z$ for some $z \in G$. Hence $\Uparrow_{I_2} x \cap G \neq \emptyset$, and so we conclude that X is an I_2-continuous space. □

In the following, we consider I_2-convergence in *IDC*-spaces, and prove that when X is an *IDC*-space, the I_2-convergence is topological in X if and only if X is an I_2-continuous space.

Proposition 5.13. *Let X be an IDC-space. If the I_2-convergence in X is topological, then X is an I_2-continuous space.*

Proof. Suppose that I_2-convergence in X is topological. Then there exists a topology $\mathcal{T}$ on X such that $\mathcal{F} \xrightarrow{I_2} x$ if and only if $\mathcal{F}$ converges to x with respect to the topology $\mathcal{T}$. By Proposition 5.11, we have that the filter $\mathcal{F}$ S_2-converges to x under the specialization order if and only if $\mathcal{F}$ converges to x with respect to the topology $\mathcal{T}$. Then X is an S_2-continuous poset under the specialization order. By Proposition 5.12, we have that X is an I_2-continuous space. □

By Theorem 4.7 and Proposition 5.13, we have the following theorem immediately.

Theorem 5.14. *Let X be an IDC-space. Then the I_2-convergence is topological if and only if X is an I_2-continuous space.*

Example 5.15. (1) Let P be a poset. Then the S_2-convergence is topological if and only if P is an S_2-continuous poset.

(2) Let X be a T_2 space. Then the I_2-convergence in X is topological.

Acknowledgement

The authors would like to thank the anonymous referee for his/her valuable comments and suggestions.

References

[1] G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloq. Publ. 25, 1940.
[2] R. Engelking, General Topology, Polish Scientific Publishers, Warszawa, 1977.
[3] M. Erné, The Dedekind-MacNeille completion as a reflector, Order 8 (1991) 159–73.
[4] M. Erné, Scott convergence and Scott topology on partially ordered sets II, In: B. Banaschewski, R.E. Hoffman (Eds.), Continuous Lattices, Bremen 1979, In: Lecture Notes in Math., vol. 871, Springer-Verlag, Berlin, Heidelberg, New York, (1981) 61–96.
[5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, Continuous Lattices and Domain, Encylopedia of Mathematics and its Applications 93, Cambridge Univ. Press, 2003.
[6] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer-Verlag, Berlin, 1980.
[7] R. Heckmann, K. Keimel, Quasicontinuous domains and the Smyth Powerdomain, Electronic Notes Theoret. Comput. Sci. 298 (2013) 215–232.
[8] P.T. Johnstone, Scott is not always sober, in Continuous Lattices, Lecture Note in Mathematics 871 (1981) 282–283.
[9] J.C. Mathews, R.F. Anderson, A comparison of two modes of order convergence, Proc. Amer. Math. Soc. 18 (1967) 100–104.
[10] V. Olejček, Order convergence and order topology on a poset, Internat. J. Theoret. Physics 38 (1999) 557–561.
[11] T. Sun, Q. Li, L. Guo, Birkhoff's order-convergence in partially ordered sets, Topology Appl. 207 (2016) 156–166.
[12] K. Wang, B. Zhao, Some further results on order-convergence in posets, Topology Appl. 160 (2013) 82–86.
[13] E.S. Wolk, On order-convergence, Proc. Amer. Math. Soc. 12 (1961) 379–384.
[14] B. Zhao, J. Lu, K. Wang, Irreducible convergences in T_0 spaces, Rocky Mountain J. Math., 2018, accepted.
[15] B. Zhao, D. Zhao, Lim-inf convergence in partially ordered sets, J. Math. Anal. Appl. 309 (2005) 701–708.
[16] D. Zhao, W.K. Ho, On topologies defined by irreducible sets, J. Logical Algeb. Methods Program. 84 (2015) 185–195.
[17] B. Zhao, K. Wang, Order topology and bi-Scott topology on a poset, Acta Math. Sinica, Eng. Ser. 27 (2011) 2101–2106.
[18] B. Zhao, J. Li, O_2-convergence in posets, Topology Appl.s 153 (2006) 2971–2975.
[19] Y. Zhou, B. Zhao, Order-converence and lim-inf$_M$-convergence in posets, J. Math. Anal. Appl. 325 (2007) 655–664.