Filomat 32:14 (2018), 5131–5136
https://doi.org/10.2298/FIL1814131C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Dieudonne Complete Spaces

Asylbek A. Chekeev[a,b], Tumar J. Kasymova[a]

[a]*Mathematics and Informatics Faculty, Kyrgyz National University, 720033 Bishkek;*
[b]*Faculty of Science, Kyrgyz-Turkish Manas University, 720038 Bishkek, Kyrgyz Republic*

Abstract. In this paper, it is established a characterization of τ-normal coverings by means of approximation of the Čech complete paracompacta, which are the perfect preimages of complete metric spaces of weight $\leq \tau$. In particular, this characterization generalizes to an arbitrary cardinal the result of A. Garsia-Maynez [15].

1. Introduction and Preliminaries

All spaces are assumed to be Tychonoff. $C(X)$ is the set of all real-valued continuous functions on X. The set $Z(f) = \{x \in X : f(x) = 0\}$ is called *zero-set* of a function $f \in C(X)$. A family $\mathcal{Z}(X) = \{Z(f) : f \in C(X)\}$ is the set of all zero-sets on X. A family $C\mathcal{Z}(X) = \{X \setminus Z(f) : f \in C(X)\}$ is the set of all *cozero-sets* on X. For any function $f \in C(X)$ we will assume $coz f = X \setminus Z(f)$. A family $\mathcal{Z}(X)$ ($C\mathcal{Z}(X)$) forms a base of closed (open) sets of a space X [12]. A family $\mathcal{Z}(X)$ is a *separating nest-generated intersection ring* (s.n.-g.i.r.) [21] or a *strong delta normal base* [1], hence it is a *normal base* [10] and the Wallman compactification $\omega(X, \mathcal{Z}(X))$ coincides with the Stone–Čech compactification βX [1, 12]. The Hewitt–Nachbin–Shirota completion υX [13, 17, 20] is the Wallman realcompactification $\upsilon(X, \mathcal{Z}(X))$ [21]. A space βX consists of all z-ultrafilters (≡maximal centered systems on $\mathcal{Z}(X)$) with the Wallman normal base $\{\bar{Z} : Z \in \mathcal{Z}(X)\}$, where $\bar{Z} = \{p \in \beta X : Z \in p\}$ [1, 12, 21]. The realcompactification υX is a subspace of βX and it consists of all countably centered (CC) z-ultrafilters (≡maximal countably centered systems on $\mathcal{Z}(X)$) with a base $\{\bar{Z} \cap \upsilon X : Z \in \mathcal{Z}(X)\}$, where $\bar{Z} \cap \upsilon X = \{p \in \upsilon X : Z \in p\}$ [1, 12, 21]. Hence $\bar{Z} = [Z]_{\beta X}$ and $\bar{Z} \cap \upsilon X = [Z]_{\upsilon X}$ for all $Z \in \mathcal{Z}(X)$.

It is known from [7] that points of βX corresponding to the Dieudonne completion δX (by Curzer-Hager) was described as *co-locally finitely additive (co-LFA)* z-ultrafilters.

Below the important properties of co-LFA z-ultrafilters are established by Propositions 2.1, 2.3 and Theorem 2.6. Further, it is established a characterization of τ-normal coverings (Theorem 2.10), which implies some results of G. Vidossich [22] and A.Di Concilio [8]. Theorems 2.12 and 2.14 generalize to an arbitrary cardinal the result of A.Garsia-Maynez [15]. By using Theorem 2.10, we prove Theorems 2.18, 2.19, 2.20, where the known characterizations of Hewitt-Nachbin-Shirota completions and realcompact spaces are clarified.

Denote by $\mathbb{N}$ the set of all natural numbers, and by $\mathbb{R}$ the real line with the ordinary topology. The union and the intersection of a family $\alpha = \{U_s\}_{s \in S}$ of sets are denoted by $\cup_{s \in S} U_s$ and $\cap_{s \in S} U_s$ respectively; in the case of a sequence $\{U_n\}_{n \in \mathbb{N}}$ of sets we use the symbols $\cup_{n \in \mathbb{N}} U_n$ and $\cap_{n \in \mathbb{N}} U_n$, and in the case of a non-indexed

2010 *Mathematics Subject Classification*. Primary 54D35; Secondary 54D70, 54D80, 54E15
Keywords. zero- (cozero-)sets, locally finite cozero covering, z-ultrafilter, normal base, uniformity, completion
Received: 13 June 2017; Revised: 22 December 2018; Accepted: 28 December 2018
Communicated by Ljubiiša D.R. Kočinac
Email addresses: asyl.ch.top@mail.ru (Asylbek A. Chekeev), tumar2000@mail.ru (Tumar J. Kasymova)

family $\alpha = \{U\}_{U\in\alpha}$ of sets we write $\cup\alpha$ and $\cap\alpha$. If $\bigcup\alpha = X$, then the family α is a *covering* of X. A covering β is a refinement of a covering α if for every $B \in \beta$ there exists $A \in \alpha$ such that $B \subset A$. For a covering α of X the *star* of a set $D \subset X$ with respect to α is the set $\mathrm{St}(D,\alpha) = \{A \in \alpha : A \cap D \neq \emptyset\}$ and $\alpha(D) = \cup\mathrm{St}(D,\alpha)$. A covering β is a *strongly star refinement* of a covering α if covering $\{\beta(B) : B \in \beta\}$ is a refinement of α. If $\alpha = \{U_s\}_{s\in S}$ and $\gamma = \{V_t\}_{t\in T}$ are two coverings of X, then α *screens* γ in case $S = T$ and $U_s \subset V_s$ for all $s \in S$.

Let Y be a subspace of a space X, then $f|_Y$ is a restriction of a mapping $f : X \to Z$ on Y, and the set $[Y]_X$ is the closure of Y in X. Let $[Y]_X = X$ and U be open in Y. Then $Ex_X U = X \setminus [Y \setminus U]_X$ is the largest open subset of X whose intersection with Y is equal U. If U, V are open in Y, then $Ex_X(U \cap V) = Ex_X U \cap Ex_X V$, $U \subset V$ if and only if $Ex_X U \subset Ex_X V$ [9]. If $\alpha = \{U_s\}_{s\in S}$ is a covering of Y, then $Ex_X\alpha = \{Ex_X U_s\}_{s\in S}$ and $\bigcup Ex_X\alpha = \bigcup_{s\in S} Ex_X U_s$. For a covering α of a space X *inner intersection* is the set $\alpha \wedge Y = \{A \cap Y : A \in \alpha\}$.

A filter $\mathcal{F}$ is said to be *countably centered* (CC) if $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$ for any sequence $\{F_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$. A filter $\mathcal{F}$ is a *Cauchy filter* in a uniform space uX if for any uniform covering $\alpha \in u$ there exist $U \in \alpha$ and $F \in \mathcal{F}$ such that $F \subset U$.

If $\{Z_s\}_{s\in S}$ is a zero-sets family of $\mathcal{Z}(X)$ such that $\{X \setminus Z_s\}_{s\in S}$ is locally finite, then $\cap_{s\in S} Z_s$ is a zero-set. It follows from important Pasynkov Lemma [18].

Lemma 1.1. ([18]) *Let $f_s : X \to \mathbb{R}_s$ be a system of continuous functions from a space X into real lines $\mathbb{R}_s = \mathbb{R}$ ($s \in S$) with marked zero $0_s = 0$ such that the system $\alpha = \{coz f_s = f_s^{-1}(\mathbb{R}_s \setminus \{0_s\})\}_{s\in S}$ is locally finite in X. Then the diagonal mapping $f = \triangle_{s\in S} f_s : X \to H^\tau$, (where $H^\tau = \mathcal{M}\prod_{s\in S}(\mathbb{R}_s, 0_s)$ is the Hilbert space of weight $\tau = |S|$ obtained as the metric product of $\mathbb{R}_s$ with marked points $0_s = 0$ [18]) is defined and continuous.*

Everywhere we will follow the denotation μX of [16] for the Dieudonne completion of a space X.

Standard references for topological spaces are in the books [9], and for uniform spaces are in the books [3, 14]. Information on the normal bases is in [1, 10, 21].

2. Main Results

Remind that z-ultrafilter p is *co-locally finitely additive* whenever the family $co(p) = \{X \setminus Z : Z \in p\}$ is *locally finitely additive*, i.e. $\cup\mathcal{F} \in co(p)$, whenever $\mathcal{F} \subset co(p)$ and $\mathcal{F}$ is locally finite [7].

Proposition 2.1. *For a z-ultrafilter p the following are equivalent:*

(1) *The family $co(p)$ is locally finitely additive;*
(2) *$\cap_{s\in S} Z_s \neq \emptyset$ for any locally finite subfamily $\{X \setminus Z_s\}_{s\in S}$ of $co(p)$.*

Proof. (1) $\Rightarrow$ (2). Let $\{X \setminus Z_s\}_{s\in S}$ be a locally finite subfamily of $co(p)$. Then $\cup_{s\in S} X \setminus Z_s = X \setminus \cap_{s\in S} Z_s \in co(p)$. Hence, $\cap_{s\in S} Z_s \neq \emptyset$ (by Lemma 1.1, $\cap_{s\in S} Z_s \in \mathcal{Z}(X)$).

(2) $\Rightarrow$ (1). Let $\cap_{s\in S} Z_s \neq \emptyset$, where $\{X \setminus Z_s\}_{s\in S}$ is a locally finite subfamily of $co(p)$. By Lemma 1.1, $Z = \cap_{s\in S} Z_s$ is a zero-set. Suppose, that $Z \notin p$. Then there exists $Z' \in p$ such that $Z \cap Z' = \emptyset$. But the family $\{X \setminus Z_s\}_{s\in S} \bigcup \{X \setminus Z\}$ is locally finite, hence $\bigcup_{s\in S} X \setminus Z_s \cup \{X \setminus Z\} = X \setminus (\cap_{s\in S} Z_s \cap Z') \in co(p)$, i.e. $Z \cap Z' \neq \emptyset$. Contradiction. □

Further, a co-locally finitely additive z-ultrafilter we will denoted by *co-LFA z-ultrafilter*.

A family $\{Z_s\}_{s\in S}$ of subsets of space X is called *co-locally finite (co-LF)* if the family $\{X \setminus Z_s\}_{s\in S}$ is locally finite in X.

Corollary 2.2. *Every co-LFA z-ultrafilter p is closed with respect to the intersections of co-LF subfamilies, i.e. $\cap_{s\in S} Z_s \in p$ for every co-LF subfamily $\{Z_s\}_{s\in S}$ of p.*

Proof. Let $\{Z_s\}_{s\in S}$ be a co-LF subfamily of the z-ultrafilter p, and $Z \in p$ be an arbitrary element. Then $\cap_{s\in S} Z_s \neq \emptyset$ and the family $\{Z_s\}_{s\in S} \cup \{Z\}$ is co-LF. Hence $\cap_{s\in S} Z_s \cap Z \neq \emptyset$. Let q be a z-ultrafilter such that $p \cup \{\cap_{s\in S} Z_s\} \subset q$. Then $p = q$ and $\cap_{s\in S} Z_s \in p$. □

Proposition 2.3. *Every co-LFA z-ultrafilter p is countably centered, i.e. $\cap_{n\in\mathbb{N}} Z_n \neq \emptyset$ for any sequence $\{Z_n\}_{n\in\mathbb{N}}$ of p.*

Proof. Suppose that there is a sequence $\{Z_n\}_{n\in\mathbb{N}}$ in p such that $\cap_{n\in\mathbb{N}}Z_n = \emptyset$. Then the family $\{X\backslash Z_n\}_{n\in\mathbb{N}}$ is a countable cozero covering of the space X. There exists a countable locally finite cozero covering $\{X\backslash Z(f_n)\}_{n\in\mathbb{N}}$ that screens $\{X\backslash Z_n\}_{n\in\mathbb{N}}$, i.e. $X\backslash Z(f_n) \subset X\backslash Z_n$ for any $n \in \mathbb{N}$ [1, Theorem 11.1]. Then $Z_n \subset Z(f_n)$ and the family $\{X\backslash Z(f_n)\}_{n\in\mathbb{N}}$ is a co-LF subfamily of p. But $\cap_{n\in\mathbb{N}}Z(f_n) = \emptyset$. Contradiction. □

Corollary 2.4. *Every co-LFA z-ultrafilter p is closed with respect to the intersections of countable subfamilies, i.e.* $\cap_{n\in\mathbb{N}}Z_n \in p$ *for every sequence* $\{Z_n\}_{n\in\mathbb{N}}$ *in p.*

Proof. It immediately follows from Proposition 2.3, and the proof is similar to the proof of Corollary 2.2. □

Corollary 2.5. *Let* μX *be a set of all co-LFA z-ultrafilters on X. Then* $\mu X \subset \upsilon X \subset \beta X$.

Proof. It immediately follows from Corollary 2.4, and construction of the Hewitt–Nachbin completion υX and construction of the Stone–Čech compactification βX [12]. □

Theorem 2.6. *Let p be a z-ultrafilter on a space X. Then the following are equivalent:*

(1) *p is a co-LFA z-ultrafilter on X;*
(2) *p is a Cauchy filter with respect to the fine uniformity* u_f *of X.*

Proof. (1) ⇒ (2). Let p be an arbitrary co-LFA z-ultrafilter. A fine uniformity u_f has a base $\mathcal{B}$ consisting of all locally finite cozero coverings [9, 14]. Let $\alpha = \{X\backslash Z(f_s)\}_{s\in S}$ be locally finite. Then $\cap_{s\in S}Z(f_s) = \emptyset$. Since the family $\{Z(f_s)\}_{s\in S}$ is co-LF, it is not contained into z-ultrafilter p. Hence there exists index $s_0 \in S$ such that $Z(f_{s_0}) \notin p$. Therefore, there exists $Z_{n_0} \in p$ such that $Z(f_{s_0}) \cap Z_{n_0} = \emptyset$. Thus, $Z_{n_0} \subset X\backslash Z(f_{s_0}) \in \alpha$ and p is a Cauchy filter with respect to the fine uniformity u_f.

(2) ⇒ (1). Let a z-ultrafilter p be a Cauchy filter with respect to the fine uniformity u_f. Suppose that there is a subfamily $\{Z_s\}_{s\in S}$ of p such that it is co-LF and $\cap_{s\in S}Z_s = \emptyset$. Then the family $\alpha = \{X\backslash Z_s\}_{s\in S}$ is a locally finite cozero covering of X. Hence $\alpha \in \mathcal{B}$. Therefore, there exist an index $s_0 \in S$ and $Z_0 \in p$ such that $Z_0 \subset X\backslash Z_{s_0} \in \alpha$. Since $Z_0 \cap Z_{s_0} = \emptyset$, we have a contradiction. □

Corollary 2.7. *X is Dieudonne complete if and only if every co-LFA z-ultrafilter converges.*

Proof. It follows immediately from Theorem 2.6. □

Corollary 2.8. μX *with topology induced by the Stone–Čech compactification* βX *is the Dieudonne completion of X and points of* μX *are co-LFA z-ultrafilters.*

From [8, 22] all uniform coverings of cardinality $\leq \tau$ of the fine uniformity u_f form the compatible uniformity. The natural problem arises: describe open coverings which are refinements of cozero coverings of cardinality $\leq \tau$.

Definition 2.9. An open covering α of a space X is said to be *τ-normal* if it has a cozero refinement β of cardinality $|\beta| \leq \tau$.

Theorem 2.10. *Let* α *be an open covering of a space X and* $\tau \geq \aleph_0$ *be an arbitrary cardinal. The following are equivalent:*

(1) α *is τ-normal;*
(2) *There exists* Y_α *such that* $X \subset Y_\alpha \subset \bigcup Ex_{\beta X}\alpha \subset \beta X$ *and* Y_α *is a perfect preimage of some complete metric space of weight* $\leq \tau$.

Proof. (1) ⇒ (2). Let a locally finite cozero covering β be a refinement of α and $|\beta| \leq \tau$. Then $\beta = \{coz f_s\}_{s\in S}$, where $coz f_s = f_s^{-1}(\mathbb{R} \setminus \{0\})$, $f_s \in C(X)$ and $|S| \leq \tau$. By Lemma 1.1, the mapping $f = \triangle_{s\in S} f_s$ continuously maps X into the Hilbert space H^τ. For any $s \in S$ we have $f_s = \pi_s \circ f$, where $\pi_s = p_s|_{f(X)}$ is the restriction of the natural projection $p_s : H^\tau \to \mathbb{R}_s, s \in S$. We note that $coz f_s = f^{-1}(coz \pi_s)$. Let $M = [f(X)]_{H^\tau}$. Then M is a complete metric subspace of H^τ. Let $F:\beta X\to\beta M$ be the extension of f on the Stone-Čech compactifications βX and βM, $Y_\alpha = F^{-1}(M)$. Assume $\varphi = F|_{Y_\alpha}$. Then $\varphi : Y_\alpha \to M$ is a perfect mapping and $\varphi^{-1}(M) = Y_\alpha$. Assume $Ex_M coz\pi_s = M\setminus [f(X)\setminus coz\pi_s]_M$. Then $\bigcup_{s\in S} f^{-1}(Ex_M coz\pi_s) = Y_\alpha$ and $f^{-1}(Ex_M coz\pi_s)\cap X = coz f_s$ for all $s\in S$. Hence, $f^{-1}(Ex_M coz\pi_s)\subset Ex_{\beta X} coz f_s$. Then $X\subset Y_\alpha\subset \bigcup Ex_{\beta X}\beta \subset \bigcup Ex_{\beta X}\alpha\subset \beta X$.

(2) ⇒ (1). Suppose that an open covering α satisfies conditions from (2). Then the inner intersection $Ex_{\beta X}\alpha \wedge Y_\alpha$ is an open covering of paracompactum Y_α. Since Y_α is perfectly mapped onto a complete metric space of weight $\le\tau$, then there exists a locally finite cozero covering γ of cardinality $|\gamma|\le\tau$ that is a refinement of $Ex_{\beta X}\alpha\wedge Y_\alpha$ [2, Chapter VI, 42]. □

Applying the fact that every open covering of a paracompactum has a cozero strongly star refinement, we obtain

Corollary 2.11. ([8, 22]) *The collection $(u_f)_\tau$ of all τ-normal coverings of a space X is a compatible uniformity on X.*

The next theorem characterizes a completion $\mu_\tau X$ of a space X with respect to the uniformity $(u_f)_\tau$.

Theorem 2.12. *The completion $\mu_\tau X$ of X with respect to the uniformity $(u_f)_\tau$ may be viewed as a subspace of βX containing X, namely, as the intersection of all paracompact G_δ-subspaces of βX containing X, which are perfect preimages of complete metric spaces of weight $\le\tau$.*

Proof. The Stone-Čech precompact uniformity u_β [9, 8.5.8] is containing in $(u_f)_\tau$, hence the Samuel compactification of X with respect to the uniformity $(u_f)_\tau$ is the Stone-Čech compactification of βX. Then for any τ-normal covering α we have $X\subset Y_\alpha\subset\bigcup Ex_{\beta X}\alpha\subset\beta X$, where Y_α is a paracompact G_δ-subspace of βX as a perfect preimage of some complete metric space of weight $\le\tau$ (by Theorem 2.10). Hence, $\mu_\tau X = \bigcap\{Y_\alpha : \alpha\in(u_f)_\tau\}$ as $\mu_\tau X = \bigcap\{\bigcup Ex_{\beta X}\alpha : \alpha\in(u_f)_\tau\}$ [19]. □

Remark 2.13. Theorem 2.12 is an extension of a result [15] to an arbitrary cardinal

The next result clarifies some result of [11].

Theorem 2.14. *Let $\mu X\subset Y\subset\beta X$. Then the following are equivalent:*

(1) *$\mu X = Y$;*
(2) *For any point $x\in\beta X\setminus Y$ there exist a cardinal $\tau\ge\aleph_0$, a cozero covering α of X of cardinality $|\alpha|\le\tau$ and Y_x such that $X\subset Y\subset Y_x\subset\bigcup Ex_{\beta X}\alpha\subset\beta X\setminus\{x\}$ and Y_x is the perfect preimage of some complete metric space of weight $\le\tau$.*

Proof. (1) ⇒ (2). Let $\mu X = Y$ and $x\in\beta X\setminus Y$ be an arbitrary point. Then there exists a unique z-ultrafilter p_x on X such that $\{x\} = \bigcap\{[Z]_{\beta X} : Z\in p_x\}$ [12] and p_x is not co-LFA. Hence there exists a co-LF subfamily $\{Z_s\}_{s\in S}$ of p_x such that $|S|\le\tau$ and $\bigcap_{s\in S} Z_s = \emptyset$. By Lemma 1.1, the mapping $f = \triangle_{s\in S} f_s$ maps X into the Hilbert space H^τ, where $Z_s = Z(f_s)$. As in the proof of implication (1) ⇒ (2) from Theorem 2.10, we have $f_s = \pi_s\circ f$ and $coz f_s = f^{-1}(coz\pi_s)$. The closure M of $f(X)$ in H^τ is a complete metric space of weight $\le\tau$. Let $F:\beta X\to\beta M$ be an extension of f to the Stone-Čech compactifications βX and βM, $Y_\alpha = f^{-1}(M)$ and $\varphi = F|_{Y_\alpha}$. Then $\varphi : Y_\alpha\to M$ is a perfect mapping and $\varphi^{-1}(M) = Y_\alpha$. Suppose $Ex_M coz\pi_s = M\setminus[f(X)\setminus coz\pi_s]_M$. Then $\bigcup_{s\in S} f^{-1}(Ex_M coz\pi_s) = Y_\alpha$ and $f^{-1}(Ex_M coz\pi_s)\cap X = coz f_s$ for all $s\in S$. Hence $f^{-1}(Ex_M coz\pi_s)\subset Ex_{\beta X} coz f_s$. Then $\mu X = Y\subset Y_\alpha\subset\bigcup Ex\alpha$. It is clear, $x\notin Ex_{\beta X} coz f_s = \beta X\setminus[X\setminus coz f_s]_{\beta X}$ for all $s\in S$. So, $X\subset Y\subset Y_\alpha\subset\bigcup Ex_{\beta X}\alpha\subset\beta X\setminus\{x\}$.

(2) ⇒ (1). It is clear, $\mu X = \bigcap\{Y_x : x\in\beta X\setminus Y\} = Y$. □

The following statement clarifies the result of [11].

Corollary 2.15. *The following are equivalent:*

(1) *X is Dieudonne complete;*

(2) *For any point $x \in \beta X \setminus X$ there exists a cardinal $\tau \geq \aleph_0$, a cozero covering α of X of cardinality $|\alpha| \leq \tau$ and Y_x such that $X \subset Y_x \subset \bigcup Ex_{\beta X}\alpha \subset \beta X \setminus \{x\}$ and Y_x is the perfect preimage of some complete metric space of weight $\leq \tau$.*

Proof. It follows immediately from Theorem 2.14, as $X = \mu X$. □

All countable coverings of the fine uniformity u_f of a space X form the uniformity u_ω [14]. By Shirota [20] it was proved that the completion of the space X with respect to the uniformity u_ω is the realcompactification υX and X is realcompact if and only if $X = \upsilon X$. The family $\mathcal{B}_\omega$ of all countable cozero coverings forms a base of the uniformity u_ω [20].

Theorem 2.16. *Let p be a z-ultrafilter on a space X. The following are equivalent:*

(1) *p is a CC z-ultrafilter on X;*
(2) *p is a Cauchy filter with respect to the uniformity u_ω of X.*

Proof. (1) ⇒ (2). Let p be an arbitrary CC z-ultrafilter. The uniformity u_ω has a base $\mathcal{B}_\omega$ of all countable cozero coverings [20]. For any $\alpha = \{X \backslash Z(f_n)\}_{n\in\mathbb{N}}$ of $\mathcal{B}_\omega$ we have $\cap_{n\in\mathbb{N}} Z(f_n) = \emptyset$. Then the family $\{Z(f_n)\}_{n\in\mathbb{N}}$ is not contained in z-ultrafilter p. Hence there exists index $n_0 \in \mathbb{N}$ such that $Z(f_{n_0}) \notin p$. Therefore, there exists $Z_{n_0} \in p$ such that $Z_{n_0} \cap Z(f_{n_0}) = \emptyset$. Thus, $Z_{n_0} \subset X \backslash Z(f_{n_0}) \in \alpha$ and p is a Cauchy filter with respect to the uniformity u_ω.

(2) ⇒ (1). Let p be a Cauchy filter with respect to the uniformity u_ω. Suppose that there is a sequence $\{Z_n\}_{n\in\mathbb{N}}$ of p such that $\cap_{n\in\mathbb{N}} Z_n = \emptyset$. Then the family $\alpha = \{X \backslash Z_n\}_{n\in\mathbb{N}}$ is a countable cozero covering of the space X. Hence $\alpha \in u_\omega$. Therefore there exist index $n_0 \in \mathbb{N}$ and $Z_0 \in p$ such that $Z_0 \subset X \backslash Z_{n_0} \in \alpha$. Since $Z_0 \cap Z_{n_0} = \emptyset$, we have a contradiction. □

The next corollary is well known [9].

Corollary 2.17. *X is realcompact if and only if every CC z-ultrafilter converges.*

Proof. It follows from Theorem 2.16. □

From Theorem 2.10 in the case $\tau = \aleph_0$ we obtain the next

Theorem 2.18. *Let α be an open covering of a space X. The following are equivalent:*

(1) *α is a uniform covering with respect to u_ω;*
(2) *There exists Y_α such that $X \subset Y_\alpha \subset \bigcup Ex_{\beta X}\alpha \subset \beta X$ and Y_α is the perfect preimage of some complete metric space of countable weight.*

Proof. It is similar to the proof of Theorem 2.10, assuming $\tau = \aleph_0$. □

The next theorem characterizes the completion υX of a space X with respect to the uniformity u_ω.

Theorem 2.19. *The completion υX of X with respect to the uniformity u_ω may be viewed as a subspace of βX containing X, namely, as the intersection of all Lindelöf G_δ-subspaces of βX containing X, which are perfect preimages of complete metric spaces of countable weight.*

Proof. The Stone-Čech precompact uniformity u_β [9, 8.5.8] is contained in u_ω, hence the Samuel compactification of X with respect to the uniformity u_ω is the Stone-Čech compactification of βX. Then for any countable normal covering α we have $X \subset Y_\alpha \subset \bigcup Ex_{\beta X}\alpha \subset \beta X$, where Y_α is a Lindelöf G_δ-subspace of βX as the perfect preimage of some complete metric space of countable weight (Theorem 2.10). Hence, $\upsilon X = \bigcap\{Y_\alpha : \alpha \in u_\omega\}$ as $\upsilon X = \bigcap\{\bigcup Ex_{\beta X}\alpha : \alpha \in u_\omega\}$ [19]. □

Theorem 2.20. *Let $\upsilon X \subset Y \subset \beta X$. Then the following are equivalent:*

(1) *$\upsilon X = Y$;*

(2) *For any point* $x \in \beta X \setminus Y$ *there exist countable cozero covering* α *of* X *and* Y_x *such that* $X \subset Y \subset Y_x \subset \bigcup Ex_{\beta X}\alpha \subset \beta X \setminus \{x\}$ *and* Y_x *is the perfect preimage of some complete metric space of countable weight.*

Proof. It is similar to the proof of Theorem 2.14, assuming $\tau = \aleph_0$. □

Corollary 2.21. *The following are equivalent:*

(1) X *is realcompact;*
(2) *For any* $x \in \beta X \setminus X$ *there exist a countable cozero covering* α *and a Lindelöf* G_δ*-subspace* Y_x *of* βX *such that* $X \subset Y_x \subset \bigcup Ex_{\beta X}\alpha \subset \beta X \setminus X$.

For every space X with a uniformity u the set $\mathcal{Z}_u$ of all zero-sets of uniformly continuous real-valued functions of a uniform space uX form s.n.-g.i.r. [21] and the characterizations of the Wallman β-like compactification $\omega(X, \mathcal{Z}_u) = \beta_u X$ and realcompactification $\upsilon(X, \mathcal{Z}_u) = \upsilon_u X$ are given in [4–6]. A characterization of the Wallman-Dieudonne completion $\mu_u X$ is given in [6]. Thus, the next problem arises:

Problem 2.22. *Let* $\mathcal{Z} \subset \mathcal{Z}(X)$ *be an arbitrary s.n.-g.i.r. on a space* X *and* $\mu(X, \mathcal{Z})$ *be the set of all co-LFA-ultrafilters on* $\mathcal{Z}$*. It is clear that* $\mu(X, \mathcal{Z}) \subset \omega(X, \mathcal{Z})$*. Is the following true:*

(1) $\mu(X, \mathcal{Z}) \subset \upsilon(X, \mathcal{Z})$*, where* $\upsilon(X, \mathcal{Z})$ *is the Wallman realcompactification ?*
(2) $\mu(X, \mathcal{Z})$ *is Dieudonne complete in the induced topology from the compactification* $\omega(X, \mathcal{Z})$*?*

Acknowledgement

The authors wish to thank the referees for corrections of the original draft of the paper and for many helpful remarks.

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