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Successive Derivatives of Fibonacci Type Polynomials of Higher Order in Two Variables

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Abstract. The purpose of this work is to compute the successive derivatives of Fibonacci type polynomials in two variables; polynomials these introduced by G. Ozdemir,Y. Simsek in [3] and generalized by G. Ozdemir, Y. Simsek and G. Milovanović in [2] to a higher order. In addition we construct their recursive formula different of that given in Theorem 2.2 [3] p.6. Finally we define a novel generalized class of those polynomials similar to that given in [1] and found its recursive formula.

1. Introduction

Let k, h, m, n are positive integers and x, y are two variables, The Fibonacci type polynomials of higher order in two variables are introduced by the following definition.

Definition 1.1. *([2] p.969 Definition 1.1 and [3] p.6 Definition 2.1) The Fibonacci type polynomials of higher order in two variables $\mathcal{G}_v^{(h)}(x, y, k, m, n)$ are defined by*

$$\sum_{v\geq 0} \mathcal{G}_v^{(h)}(x, y, k, m, n)\, t^v = \frac{1}{\left(1 - x^k t - y^m t^{m+n}\right)^h}. \tag{1}$$

For $h = 1$, we write only $\mathcal{G}_v^{(1)}(x, y, k, m, n) = \mathcal{G}_v(x, y, k, m, n)$.

In this work we need the following two classical identities. The first is the **Leibnitz formula**:

$$(fg)^{(j)} = \sum_{i=0}^{j} \binom{j}{i} f^{(i)} g^{(j-i)} \tag{2}$$

where f, g are two functions of the variable t and $f^{(i)} = \frac{d^i f}{dt^i}$ the derivative at order i of f. If f, g are functions of more than one variable, we derive the following formula

$$\frac{\partial^j (fg)}{\partial t^j} = \sum_{i=0}^{j} \binom{j}{i} \frac{\partial^i f}{\partial t^i} \frac{\partial^{j-i} g}{\partial t^{j-i}}. \tag{3}$$

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The second is the **Cauchy product** of two series:

$$\left(\sum_{n\geq 0} a_n\right)\left(\sum_{n\geq 0} b_n\right) = \sum_{n\geq 0}\sum_{i=0}^{n} a_i b_{n-i}. \tag{4}$$

which can be extended to generating functions of polynomials of two variables by the following expression

$$\left(\sum_{n\geq 0} P_n(x,y)t^n\right)\left(\sum_{m\geq 0} Q_m(x,y)t^m\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} P_i(x,y)Q_{n-i}(x,y)\right)t^n. \tag{5}$$

This idea helps us to compute the product of the partial sum $\sum_{n=0}^{N} P_n(x,y)t^n$ with the series $\sum_{n\geq 0} Q_n(x,y)t^n$;

$$\left(\sum_{n=0}^{N} P_n(x,y)t^n\right)\left(\sum_{m\geq 0} Q_m(x,y)t^m\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{\min\{n,N\}} P_i(x,y)Q_{n-i}(x,y)\right)t^n. \tag{6}$$

In the proof of this equation, we can write

$$\sum_{n=0}^{N} P_n(x,y)t^n = \sum_{n\geq 0} P_n(x,y)t^n$$

with $P_n(x,y) = 0$ for $n \geq N+1$ and by using equation (4) of the Cauchy product we get the result (6).

2. Partial derivatives of higher order

For integers a and b, let us define the set

$$I_a(b) = \{i + bl /\ 0 \leq l \leq a - i \leq a\}$$

and the characteristic function $\chi_{a,b}$ of $I_a(b)$:

$$\chi_{a,b}(v) := \begin{cases} 1 & \text{if } v \in I_a(b) \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Denoting $A^{(h)}(v,x,y)$ the polynomial

$$A^{(h)}(v,x,y) = \sum_{\frac{v-h}{m+n-1}\leq l\leq \frac{v}{m+n}} \binom{h}{v-(m+n-1)l}\binom{v-(m+n-1)l}{l}(-1)^{v-(m+n-1)l}\chi_{h,m+n}(v)y^{ml}x^{k(v-(m+n)l)}. \tag{8}$$

In those conditions the partial derivative of $\mathcal{G}_v^{(h)}(x,y,k,m,n)$ of order j is stated in the following theorem.

Theorem 2.1. *If $z \in \{x,y\}$ then*

$$\frac{\partial^j \mathcal{G}_v^{(h)}(x,y,k,m,n)}{\partial z^j} = -\sum_{s=0}^{\min\{v,(n+m)h\}}\sum_{i=0}^{j-1}\sum_{\mu=0}^{s}\binom{j}{j-i}\frac{\partial^i \mathcal{G}_\mu^{(h)}}{\partial z^i}\frac{\partial^{j-i}A^{(h)}(v-s,x,y)}{\partial z^{j-i}}\mathcal{G}_{s-\mu}^{(h)}. \tag{9}$$

For any positive integer i let us define the function δ_i by the following expression

$$\delta_i(j) := \begin{cases} 1 & \text{if } 0 \leq j \leq i \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

In means of this function we obtain the following theorem

Theorem 2.2.

$$\frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial x^j} = \sum_{i=0}^{j-1}\sum_{\mu=0}^{v-1} \frac{j!}{i!}\binom{k}{j-i}\delta_k(j-i)x^{k-j+i}\mathcal{G}_{v-1-\mu}(x,y,k,m,n)\frac{\partial^i \mathcal{G}_\mu(x,y,k,m,n)}{\partial x^i}. \tag{11}$$

And for $v \geq m+n$

$$\frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial y^j} = \sum_{i=0}^{j-1}\sum_{\mu=0}^{v-m-n} \frac{j!}{i!}\binom{m}{j-i}\delta_m(j-i)y^{m-j+i}\mathcal{G}_{v-m-n-\mu}(x,y,k,m,n)\frac{\partial^i \mathcal{G}_\mu(x,y,k,m,n)}{\partial y^i}. \tag{12}$$

In the case $j = 1$, Theorem 3.1 and Theorem 3.2 replace this expression as "are expressed as follows which are given in [3] p.973."

Corollary 2.3.

$$\frac{\partial}{\partial x}\mathcal{G}_v(x,y,k,m,n) = kx^{k-1}\sum_{\mu=0}^{v-1}\mathcal{G}_\mu(x,y,k,m,n)\mathcal{G}_{v-1-\mu}(x,y,k,m,n) \tag{13}$$

$$\frac{\partial}{\partial y}\mathcal{G}_v(x,y,k,m,n) = my^{m-1}\sum_{\mu=0}^{v-m-n}\mathcal{G}_\mu(x,y,k,m,n)\mathcal{G}_{v-m-n-\mu}(x,y,k,m,n),\ v \geq m+n. \tag{14}$$

3. Proof of Theorem 2.1 and Theorem 2.2

First let us denoting

$$H := H(x,y,k,m,n,t) = \frac{1}{1-x^k t - y^m t^{m+n}}$$

and

$$f := f(x,y,k,m,n,t) = 1 - x^k t - y^m t^{m+n}$$

then we get

$$H^h f^h = 1. \tag{15}$$

Lemma 3.1. *For $j \geq 1$ and replace this expression as " $z \in \{x,y\}$ "*

$$\frac{\partial^j H^h}{\partial z^j} = -H^h \sum_{i=0}^{j-1}\binom{j}{j-i}\frac{\partial^i H^h}{\partial z^i}\frac{\partial^{j-i} f^h}{\partial z^{j-i}}. \tag{16}$$

Proof. The proof consists to use the well-known Leibnitz formula (3). Since $H^h f^h = 1$, we obtain

$$\frac{\partial^j (H^h f^h)}{\partial z^j} = \sum_{i=0}^{j}\binom{j}{i}\frac{\partial^i H^h}{\partial z^i}\frac{\partial^{j-i} f^h}{\partial z^{j-i}} = 0.$$

Furthermore

$$\frac{\partial^j H^h}{\partial z^j} f^h + \sum_{i=0}^{j-1}\binom{j}{i}\frac{\partial^i H^h}{\partial z^i}\frac{\partial^{j-i} f^h}{\partial z^{j-i}} = 0$$

and the result follows. □

Lemma 3.2.

$$\left(1 - x^k t - y^m t^{m+n}\right)^h = \sum_{v=0}^{(n+m)h} A^{(h)}(v, x, y)\, t^v. \tag{17}$$

Proof. By using the binomial formula, we get

$$f^h = \left(1 - x^k t - y^m t^{m+n}\right)^h = \sum_{j=0}^{h} \binom{h}{j} (-1)^j \left(1 - y^m t^{m+n}\right)^{h-j} x^{kj} t^j$$

furthermore

$$f^h = \sum_{j=0}^{h} \sum_{l=0}^{h-j} \binom{h}{j} \binom{h-j}{l} (-1)^{j+l} x^{kj} y^{ml} t^{(m+n)l+j}.$$

Taking $v = (m+n)\,l + j$ then $\frac{v-h}{m+n-1} \le l \le \frac{v}{m+n}$ and $0 \le v \le (m+n)\,h$. Hence

$$f^h = \sum_{v=0}^{(m+n)h} \sum_{\frac{v-h}{m+n-1} \le l \le \frac{v}{m+n}} \binom{h}{v-(m+n)l} \binom{v-(m+n)l}{l} (-1)^{v-(m+n-1)l} \chi_{h,m+n}(v) x^{k(v-(m+n)l)} y^{ml} t^v.$$

□

Lemma 3.3. *For $j \ge 1$,*

$$\frac{\partial^j f(x, y, k, m, n, t)}{\partial x^j} = -\frac{k!\delta_k(j)}{(k-j)!} x^{k-j} t. \tag{18}$$

$$\frac{\partial^j f(x, y, k, m, n, t)}{\partial y^j} = -\frac{m!\delta_m(j)}{(m-j)!} y^{m-j} t^{m+n}. \tag{19}$$

Proof. The proof is left as an exercise. □

Lemma 3.4.

$$\frac{\partial^j H(x, y, k, m, n, t)}{\partial x^j} = \sum_{i=0}^{j-1} \frac{j!}{i!} \binom{k}{j-i} \delta_k(j-i) x^{k-j+i} \frac{\partial^i H(x, y, k, m, n, t)}{\partial x^i} H(x, y, k, m, n, t)\, t. \tag{20}$$

$$\frac{\partial^j H(x, y, k, m, n, t)}{\partial y^j} = \sum_{i=0}^{j-1} \frac{j!}{i!} \binom{m}{j-i} \delta_m(j-i) y^{m-j+i} \frac{\partial^i H(x, y, k, m, n, t)}{\partial y^i} H(x, y, k, m, n, t)\, t^{m+n}. \tag{21}$$

Proof. By using the relation (18) Lemma 3.3 we conclude that

$$\sum_{i=0}^{j-1} \binom{j}{j-i} \frac{\partial^i H}{\partial x^i} \frac{\partial^{j-i} f}{\partial x^{j-i}} = -\sum_{i=0, j-i \le k}^{j-1} \frac{j!}{i!} \binom{k}{j-i} \delta_k(j-i) x^{k-j+i} \frac{\partial^i H}{\partial x^i} t$$

Furthermore from relation (16) of Lemma 3.1 for $z = x$ and $h = 1$ we get

$$\frac{\partial^j H}{\partial x^j} = \sum_{i=0, j-i \le k}^{j-1} \frac{j!}{i!} \binom{k}{j-i} \delta_k(j-i) x^{k-j+i} \frac{\partial^i H}{\partial x^i} H t.$$

The same idea is used to prove the second expression (21), just taking $z = y$. □

3.1. *Proof of Theorem 2.1*

Let $z \in \{x, y\}$ the series expansion of $\frac{\partial^i H^h}{\partial z^i}$ is

$$\frac{\partial^i H^h}{\partial z^i} = \sum_{v\geq 0} \frac{\partial^i \mathcal{G}_v^{(h)}(x,y,k,m,n)}{\partial z^i} t^v.$$

Furthermore

$$\frac{\partial^i H^h}{\partial z^i} H^h = \left(\sum_{v\geq 0} \frac{\partial^i \mathcal{G}_v^{(h)}(x,y,k,m,n)}{\partial z^i} t^v\right)\left(\sum_{v\geq 0} \mathcal{G}_v^{(h)}(x,y,k,m,n)\, t^v\right)$$

then

$$\frac{\partial^i H^h}{\partial z^i} H^h = \sum_{v\geq 0}\left(\sum_{\mu=0}^{v} \frac{\partial^i \mathcal{G}_\mu^{(h)}}{\partial z^i} \mathcal{G}_{v-\mu}^{(h)}\right) t^v$$

and

$$\frac{\partial^i H^h}{\partial z^i} H^h \frac{\partial^{j-i} f^h}{\partial z^{j-i}} = \left(\sum_{v\geq 0}\left(\sum_{\mu=0}^{v} \frac{\partial^i \mathcal{G}_\mu^{(h)}}{\partial z^i} \mathcal{G}_{v-\mu}^{(h)}\right) t^v\right)\left(\sum_{v=0}^{(n+m)h} \frac{\partial^{j-i} A^{(h)}(v,x,y)}{\partial z^{j-i}} t^v\right)$$

hence

$$\frac{\partial^i H^h}{\partial z^i} H^h \frac{\partial^{j-i} f^h}{\partial z^{j-i}} = \sum_{v\geq 0}\left(\sum_{s=0}^{\min\{v,(n+m)h\}} \sum_{\mu=0}^{s} \frac{\partial^i \mathcal{G}_\mu^{(h)}}{\partial z^i} \frac{\partial^{j-i} A^{(h)}(v-s,x,y)}{\partial z^{j-i}} \mathcal{G}_{s-\mu}^{(h)}\right) t^v.$$

Finally

$$\frac{\partial^j \mathcal{G}_v^{(h)}(x,y,k,m,n)}{\partial z^j} = -\sum_{s=0}^{\min\{v,(n+m)h\}} \sum_{i=0}^{j-1} \sum_{\mu=0}^{s} \binom{j}{j-i} \frac{\partial^i \mathcal{G}_\mu^{(h)}}{\partial z^i} \frac{\partial^{j-i} A^{(h)}(v-s,x,y)}{\partial z^{j-i}} \mathcal{G}_{s-\mu}^{(h)}.$$

3.2. *Proof of Theorem 2.2*

Here we develop two methods for the proof of Theorem 2.2.

- Let $z \in \{x, y\}$, the first one consist to use the series expansion of $\frac{\partial^j H}{\partial z^j}$ which is

$$\frac{\partial^j H}{\partial z^j} = \sum_{v\geq 0} \frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial z^j} t^v.$$

Furthermore

$$\frac{\partial^i H}{\partial z^i} H = \left(\sum_{l\geq 0} \frac{\partial^j \mathcal{G}_l(x,y,k,m,n)}{\partial z^j} t^l\right)\left(\sum_{l\geq 0} \mathcal{G}_l(x,y,k,m,n)\, t^l\right).$$

Using the Cauchy product of two series (5) we deduce that

$$\frac{\partial^i H}{\partial z^i} H = \sum_{l\geq 0}\left(\sum_{\mu=0}^{l} \frac{\partial^i \mathcal{G}_\mu(x,y,k,m,n)}{\partial z^i} \mathcal{G}_{l-\mu}(x,y,k,m,n)\right) t^l.$$

From the relation (20) Lemma 3.4 we conclude that

$$\sum_{v\geq 0} \frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial x^j} t^v = \sum_{v\geq 0}\left(\sum_{i=0,}^{j-1} \frac{j!}{i!}\binom{k}{j-i}\delta_k(j-i)x^{k-j+i} \sum_{\mu=0}^{v} \frac{\partial^i \mathcal{G}_\mu(x,y,k,m,n)}{\partial x^i} \mathcal{G}_{v-\mu}(x,y,k,m,n)\right) t^{v+1}$$

and

$$\sum_{v\geq 1}\frac{\partial^j\mathcal{G}_v(x,y,k,m,n)}{\partial x^j}t^v=\sum_{v\geq 1}\left(\sum_{i=0,}^{j-1}\frac{j!}{i!}\binom{k}{j-i}\delta_k(j-i)x^{k-j+i}\sum_{\mu=0}^{v-1}\frac{\partial^i\mathcal{G}_\mu(x,y,k,m,n)}{\partial x^i}\mathcal{G}_{v-1-\mu}(x,y,k,m,n)\right)t^v.$$

After identification we obtain the desired result (11) Theorem 2.2.

Now from the relation (21) Lemma 3.4 we obtain

$$\sum_{v\geq 0}\frac{\partial^j\mathcal{G}_v(x,y,k,m,n)}{\partial y^j}t^v=\sum_{v\geq 0}\sum_{i=0}^{j-1}\sum_{\mu=0}^{v}\frac{j!}{i!}\binom{m}{j-i}\delta_m(j-i)y^{m-j+i}\frac{\partial^i\mathcal{G}_\mu(x,y,k,m,n)}{\partial y^i}\mathcal{G}_{v-\mu}(x,y,k,m,n)\,t^{v+m+n}$$

and

$$\sum_{v\geq 0}\frac{\partial^j\mathcal{G}_v(x,y,k,m,n)}{\partial y^j}t^v=\sum_{v\geq m+n}\sum_{i=0}^{j-1}\sum_{\mu=0}^{v-m-n}\frac{j!}{i!}\binom{m}{j-i}\delta_m(j-i)y^{m-j+i}\frac{\partial^i\mathcal{G}_\mu(x,y,k,m,n)}{\partial y^i}\mathcal{G}_{v-m-n-\mu}(x,y,k,m,n)\,t^{v}$$

which means that

$$\frac{\partial^j\mathcal{G}_v(x,y,k,m,n)}{\partial y^j}=0,\ v<m+n$$

and

$$\frac{\partial^j\mathcal{G}_v(x,y,k,m,n)}{\partial y^j}=\sum_{i=0}^{j-1}\sum_{\mu=0}^{v-m-n}\frac{j!}{i!}\binom{m}{j-i}\delta_m(j-i)y^{m-j+i}\frac{\partial^i\mathcal{G}_\mu(x,y,k,m,n)}{\partial y^i}\mathcal{G}_{v-m-n-\mu}(x,y,k,m,n),\ v\geq m+n$$

- The second method consist to compute the expression (9) of Theorem 2.1 for $h=1$

$$\frac{\partial^j\mathcal{G}_v(x,y,k,m,n)}{\partial z^j}=-\sum_{s=0}^{\min\{v,n+m\}}\sum_{i=0}^{j-1}\sum_{\mu=0}^{s}\binom{j}{j-i}\frac{\partial^i\mathcal{G}_\mu}{\partial z^i}\frac{\partial^{j-i}A^{(1)}(v-s,x,y)}{\partial z^{j-i}}\mathcal{G}_{s-\mu}$$

$$A^{(1)}(v,x,y)\ =\ \sum_{\frac{v-1}{m+n-1}\leq l\leq\frac{v}{m+n}}\binom{1}{v-(m+n-1)l}\binom{v-(m+n-1)l}{l}(-1)^{v-(m+n-1)l}\chi_{1,m+n}(v)y^{ml}x^{k(v-(m+n)l)}$$

We have

$$I_1(m+n)=\{0,1,m+n\}$$

furthermore

$$A^{(1)}(0,x,y)=1,A^{(1)}(1,x,y)=-x^k\text{ and }A^{(1)}(m+n,x,y)=-y^m.$$

But for $0\leq i\leq j-1$,

$$\frac{\partial^{j-i}A^{(1)}(0,x,y)}{\partial x^{j-i}}=\frac{\partial^{j-i}A^{(1)}(m+n,x,y)}{\partial x^{j-i}}=0$$

and by the formula (18) Lemma 3.3

$$\frac{\partial^{j-i}A^{(1)}(1,x,y)}{\partial x^{j-i}}=-\frac{k!\delta_k(j-i)}{(k-j+i)!}x^{k-j+i}.$$

In those conditions the sum $\sum_{s=0}^{\min\{v,(n+m)h\}}$ will be reduced to $s = v-1$ and then

$$\frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial x^j} = -\sum_{i=0}^{j-1}\sum_{\mu=0}^{v-1}\binom{j}{j-i}\frac{\partial^i \mathcal{G}_\mu}{\partial x^i}\frac{\partial^{j-i}A^{(1)}(1,x,y)}{\partial x^{j-i}}\mathcal{G}_{v-1-\mu}.$$

Since

$$\binom{j}{j-i}\frac{\partial^{j-i}A^{(1)}(1,x,y)}{\partial x^{j-i}} = -\frac{j!}{i!}\binom{k}{j-i}\delta_k(j-i)x^{k-j+i}$$

then

$$\frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial x^j} = \sum_{i=0}^{j-1}\sum_{\mu=0}^{v-1}\frac{j!}{i!}\binom{k}{j-i}\delta_k(j-i)x^{k-j+i}\mathcal{G}_{v-1-\mu}(x,y,k,m,n)\frac{\partial^i \mathcal{G}_\mu(x,y,k,m,n)}{\partial x^i}.$$

We do the same for $\frac{\partial^j \mathcal{G}_v(x,y,k,m,n)}{\partial y^j}$.

4. Recursive expression of $\mathcal{G}_v^{(h)}(x,y,k,m,n)$

In this section, we state an interesting general recursive formula for the polynomials $\mathcal{G}_v^{(h)}(x,y,k,m,n)$ different of that given in Theorem 2.2 [2] p.6:

$$\mathcal{G}_v^{(h_1+h_2)}(x,y,k,m,n) = \sum_{i=0}^{v}\mathcal{G}_i^{(h_1)}(x,y,k,m,n)\,\mathcal{G}_{v-i}^{(h_2)}(x,y,k,m,n). \tag{22}$$

More precisely we get the following theorem.

Theorem 4.1.

$$\mathcal{G}_v^{(h)}(x,y,k,m,n) = -\sum_{i=1}^{\min\{v,(m+n)h\}}A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n) \text{ for } v \geq 1. \tag{23}$$

Proof. From the formula (16) Lemma 3.1 and the expression (15) we deduce that

$$\left(\sum_{v=0}^{(n+m)h}A^{(h)}(v,x,y)\,t^v\right)\left(\sum_{v\geq 0}\mathcal{G}_v^{(h)}(x,y,k,m,n)\,t^v\right) = 1.$$

Using the Cauchy product (5) we get

$$\sum_{v\geq 0}\left(\sum_{i=0}^{\min\{v,(m+n)h\}}A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n)\right)t^v = 1.$$

Which means that $A^{(h)}(0,x,y)\,\mathcal{G}_0^{(h)}(x,y,k,m,n) = 1$ and for $v \geq 1$:

$$\sum_{i=0}^{\min\{v,(m+n)h\}}A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n) = 0.$$

But $A^{(h)}(0,x,y) = 1$ then $\mathcal{G}_0^{(h)}(x,y,k,m,n) = 1$ and

$$\mathcal{G}_v^{(h)}(x,y,k,m,n) = -\sum_{i=1}^{\min\{v,(m+n)h\}}A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n).$$

□

In the case $h = 1$, the recursive formula becomes.

Proposition 4.2.

$$\mathcal{G}_v(x, y, k, m, n) = x^{vk}, \ v < m + n. \tag{24}$$

$$\mathcal{G}_v(x, y, k, m, n) = x^k \mathcal{G}_{v-1}(x, y, k, m, n) + y^m \mathcal{G}_{v-m-n}(x, y, k, m, n), \ v \geq m + n. \tag{25}$$

Proof. The application of the relation (23) Theorem 4.1 to $h = 1$ helps us to deduce that

$$\mathcal{G}_v(x, y, k, m, n) = -\sum_{i=1}^{\min\{v, m+n\}} A^{(1)}(i, x, y)\, \mathcal{G}_{v-i}(x, y, k, m, n).$$

But

$$A^{(1)}(1, x, y) = -x^k, \ A^{(1)}(m + n, x, y) = -y^m$$

and the others are zero. Thus if $v < m + n$ we obtain

$$\mathcal{G}_v(x, y, k, m, n) = x^k \mathcal{G}_{v-1}(x, y, k, m, n)$$

furthermore

$$\mathcal{G}_v(x, y, k, m, n) = x^{vk}$$

and if $v \geq m + n$

$$\mathcal{G}_v(x, y, k, m, n) = x^k \mathcal{G}_{v-1}(x, y, k, m, n) + y^m \mathcal{G}_{v-m-n}(x, y, k, m, n).$$

□

The Theorem 2.2 in [3] is immediate from this proposition. More precisely we obtain the following corollary and it's proof is different of such given in the work [3] of G. Ozdemir and Y. Simsek.

Corollary 4.3. *([3] p.971. Theorem 2.2)*

$$\mathcal{G}_v(x, y, k, m, n) = \sum_{c=0}^{\lfloor \frac{v}{m+n} \rfloor} \binom{v - c(m + n - 1)}{c} y^{mc} x^{vk-mck-nck} \tag{26}$$

Proof. for $v < m + n$ the result is trivial. Now consider $v \geq m + n$, and the proof is given by the following recurrence processus. Suppose that

$$\mathcal{G}_{v-m-n}(x, y, k, m, n) = \sum_{c=0}^{\lfloor \frac{v-m-n}{m+n} \rfloor} \binom{v - m - n - c(m + n - 1)}{c} y^{mc} x^{(v-m-n)k-mck-nck}$$

and

$$\mathcal{G}_{v-1}(x, y, k, m, n) = \sum_{c=0}^{\lfloor \frac{v-1}{m+n} \rfloor} \binom{v - 1 - c(m + n - 1)}{c} y^{mc} x^{(v-1)k-mck-nck}.$$

Furthermore from the relation (25) Proposition 4.2 we deduce that

$$\begin{aligned}\mathcal{G}_v(x, y, k, m, n) &= \sum_{c=0}^{\lfloor \frac{v-1}{m+n} \rfloor} \binom{v - 1 - c(m + n - 1)}{c} y^{mc} x^{vk-mck-nck} \\ &+ \sum_{c=0}^{\lfloor \frac{v-m-n}{m+n} \rfloor} \binom{v - m - n - c(m + n - 1)}{c} y^{m(c+1)} x^{(v-m-n)k-mck-nck}\end{aligned}$$

The Euclidean division of v over $m+n$ conducts to $v = a + (m+n)\,b$ with $1 \le a < m+n$ then $\left\lfloor \frac{v}{m+n} \right\rfloor = b$. It follows that $\left\lfloor \frac{v-m-n}{m+n} \right\rfloor = b - 1 = \left\lfloor \frac{v}{m+n} \right\rfloor - 1$ and

$$\begin{aligned}\mathcal{G}_v(x,y,k,m,n) &= \sum_{c=0}^{\left\lfloor \frac{v}{m+n} \right\rfloor} \binom{v-1-c(m+n-1)}{c} y^{mc} x^{vk-mck-nck} \\ &+ \sum_{c=0}^{\left\lfloor \frac{v}{m+n} \right\rfloor - 1} \binom{v-m-n-c(m+n-1)}{c} y^{m(c+1)} x^{(v-m-n)k-mck-nck}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_v(x,y,k,m,n) &= \sum_{c=0}^{\left\lfloor \frac{v}{m+n} \right\rfloor} \binom{v-1-c(m+n-1)}{c} y^{mc} x^{vk-mck-nck} \\ &+ \sum_{c=1}^{\left\lfloor \frac{v}{m+n} \right\rfloor} \binom{v-1-c(m+n-1)}{c-1} y^{mc} x^{vk-mck-nck}\end{aligned}$$

From the identity

$$\binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}$$

we deduce that

$$\mathcal{G}_v(x,y,k,m,n) = x^{vk} + \sum_{c=1}^{\left\lfloor \frac{v}{m+n} \right\rfloor} \binom{v-c(m+n-1)}{c} y^{mc} x^{vk-mck-nck}.$$

Finally

$$\mathcal{G}_v(x,y,k,m,n) = \sum_{c=0}^{\left\lfloor \frac{v}{m+n} \right\rfloor} \binom{v-c(m+n-1)}{c} y^{mc} x^{vk-mck-nck}.$$

□

5. New generalization of Fibonacci type polynomials of higher order in two variables

Let $A(x,y)$ be a polynomial of two variables, here we give a generalization of Fibonacci type polynomials of higher order in two variables similar to such given in [1] for Catalan polynomials.

Definition 5.1. *Generalized two variable Fibonacci type polynomials of higher order $(x,y) \mapsto \mathcal{G}_v^{h,A}(x,y,k,m,n)$ are defined by the following generating function*

$$\sum_{v \ge 0} \mathcal{G}_v^{h,A}(x,y,k,m,n)\, t^v = \frac{1 + A(x,y)\,t}{\left(1 - x^k t - y^m t^{m+n}\right)^h}. \tag{27}$$

It's obvious that

$$\mathcal{G}_v^{h,A}(x,y,k,m,n) = \mathcal{G}_v^{(h)}(x,y,k,m,n) + A(x,y)\,\mathcal{G}_{v-1}^{(h)}(x,y,k,m,n). \tag{28}$$

Theorem 5.2.

$$\mathcal{G}_v^{h,A}(x,y,k,m,n) = -A^{(h)}(v,x,y) - \sum_{i=1}^{v-1} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{h,A}(x,y,k,m,n),\ v \le (m+n)h \tag{29}$$

$$\mathcal{G}_v^{h,A}(x,y,k,m,n) = -\sum_{i=1}^{(m+n)h} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{h,A}(x,y,k,m,n),\ v > (m+n)h \tag{30}$$

Proof. From the relation (28) and the expression (23) of $\mathcal{G}_v^{h,A}(x,y,k,m,n)$ in Theorem 4.1 we deduce that

$$\begin{aligned}\mathcal{G}_v^{h,A}(x,y,k,m,n) &= -\sum_{i=1}^{\min\{v,(m+n)h\}} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n)\\ &- A(x,y)\sum_{i=1}^{\min\{v-1,(m+n)h\}} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i-1}^{(h)}(x,y,k,m,n)\end{aligned}$$

If $v \le (m+n)h$ then

$$\begin{aligned}\mathcal{G}_v^{h,A}(x,y,k,m,n) &= -\sum_{i=1}^{v} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n)\\ &- A(x,y)\sum_{i=1}^{v-1} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i-1}^{(h)}(x,y,k,m,n)\end{aligned}$$

and

$$\mathcal{G}_v^{h,A}(x,y,k,m,n) = -A^{(h)}(v,x,y) - \sum_{i=1}^{v-1} A^{(h)}(i,x,y)\left(\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n) + A(x,y)\,\mathcal{G}_{v-i-1}^{(h)}(x,y,k,m,n)\right)$$

Furthermore

$$\mathcal{G}_v^{h,A}(x,y,k,m,n) = -A^{(h)}(v,x,y) - \sum_{i=1}^{v-1} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{h,A}(x,y,k,m,n).$$

If $v > (m+n)h$ then

$$\begin{aligned}\mathcal{G}_v^{h,A}(x,y,k,m,n) &= -\sum_{i=1}^{(m+n)h} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i}^{(h)}(x,y,k,m,n)\\ &- A(x,y)\sum_{i=1}^{(m+n)h} A^{(h)}(i,x,y)\,\mathcal{G}_{v-i-1}^{(h)}(x,y,k,m,n)\end{aligned}$$

and the result (30) Theorem 5.2 follows. □

5.1. Application to generalized Catalan and Humbert polynomials

Now we extend this result to the generalized Catalan polynomials $\mathcal{P}_{v,m}^{h,A}(y)$ defined in [1] by the following generating function

$$f_{m,h,A}(y,t) = \frac{1 + A(y)\,t}{(1 - mt + yt^m)^h} = \sum_{v\ge 0} \mathcal{P}_{v,m}^{h,A}(y)\,t^v.$$

First let us denoting

$$H^{h,A}(x,y;k,m,n,t) = \frac{1 + A(x,y)\,t}{(1 - x^k t - y^m t^{m+n})^h}.$$

Without lost generalities we can only consider all the powers of x in the polynomial $A(x,y)$ are zero, and write $A(x,y) = A(y)$. Then

$$H^{h,A}(m,-y;1,1,m-1,t) = \frac{1 + A(y)\,t}{(1 - mt + yt^m)^h} = f_{m,h,A(m,y)}(y,t).$$

Furthermore

$$\mathcal{G}_v^{h,A}(m,-y,1,1,m-1)=\mathcal{P}_{v,m}^{h,A}(y)$$

We already proved the following Corollary concerning the recursive formula satisfied by the generalized Catalan polynomials.

Corollary 5.3.

$$\mathcal{P}_{v,m}^{h,A}(y)=-A^{(h)}(v,m,-y)-\sum_{i=1}^{v-1}A^{(h)}(i,m,-y)\mathcal{P}_{v-i,m}^{h,A}(y),\ v\le mh \tag{31}$$

$$\mathcal{P}_{v,m}^{h,A}(y)=-\sum_{i=1}^{v-1}A^{(h)}(i,m,-y)\mathcal{P}_{v-i,m}^{h,A}(y),\ v>mh \tag{32}$$

One can found the similar recursive formula for the generalized Humbert's polynomials $\Pi_{n,m}^{h,A}(x)$ defined in [1] in means of the following generating function

$$\frac{1+A(x)t}{(1-mxt+t^m)^h}=\sum_{v\ge 0}\Pi_{v,m}^{h,A}(x)t^v.$$

Comparing with the generating function $H^{h,A}(x,y;k,m,n;t)$ we get

$$\frac{1+A(mx)t}{(1-mxt+t^m)^h}=H^{h,A}(mx,-1;1,1,m-1;t)$$

it follows that

$$\Pi_{v,m}^{h,A(mx)}(x)=\mathcal{G}_v^{h,A}(mx,-1,1,1,m-1)$$

Finally we deduce that

$$\Pi_{v,m}^{h,A(mx)}(x)=-A^{(h)}(v,mx,-1)-\sum_{i=1}^{v-1}A^{(h)}(i,mx,-1)\Pi_{v-i,m}^{h,A}(x),\ v\le mh \tag{33}$$

$$\mathcal{G}_v^{h,A}(x,y,k,m,n)=-\sum_{i=1}^{v-1}A^{(h)}(i,mx,-1)\Pi_{v-i,m}^{h,A}(x),\ v>mh. \tag{34}$$

References

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[3] G., Ozdemir, Y. Simsek, Generating functions for two-variable polynomials related to a family of Fibonacci type polynomials and numbers, Filomat 30 (4) (2016) 969-975.