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Brzdęk's fixed point method for the Generalised Hyperstability of bi-Jensen functional equation in $(2, \beta)$-Banach spaces

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Abstract. Using the fixed point theorem [12, Theorem 1] in $(2, \beta)$-Banach spaces, we prove the generalized hyperstability results of the bi-Jensen functional equation

$$4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = f(x, y) + f(x, w) + f(z, y) + f(y, w).$$

Our main results state that, under some weak natural assumptions, functions satisfying the equation approximately (in some sense) must be actually solutions to it. The method we use here can be applied to various similar equations in many variables.

1. Introduction and Preliminaries

In this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the sets of all positive integers, real numbers, non-negative real numbers and complex numbers, respectively; and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $\mathbb{K}$ denote the fields of real or complex numbers. The next definition describes the notion of hyperstability that we apply here (A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

Definition 1.1 ([13]). *Let A be a nonempty set, (Z, d) be a metric space, $\gamma : A^n \to \mathbb{R}_+$, $B \subset A^n$ be nonempty, and $\mathcal{F}_1, \mathcal{F}_2$ map a nonempty $\mathcal{D} \subset Z^A$ into Z^{A^n}. We say that the conditional equation*

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in B \tag{1}$$

is γ-hyperstable provided every $\varphi_0 \in \mathcal{D}$, satisfying

$$d\Big(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\Big) \le \gamma(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in B, \tag{2}$$

is a solution to (1).

That notion is one of the notions connected with the well-known issue of Ulam's stability for various (e.g., difference, differential, functional, integral, operator) equations. Let us recall that the study of such problems was motivated by the following question of Ulam (cf. [32, 51]) asked in 1940.

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Ulam's question. *Let $(G_1, \diamond)$, $(G_2, \circledast)$ be two groups and $\rho : G_2 \times G_2 \to [0, \infty)$ be a metric. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if a function $g : G_1 \to G_2$ satisfies the inequality*

$$\rho(g(x \diamond y), g(x) \circledast g(y)) \leq \delta$$

for all $x, y \in G_1$, then there is a homomorphism $a : G_1 \to G_2$ with

$$\rho(g(x), a(x)) \leq \epsilon \quad \text{for all} \quad x \in G_1\,?$$

In 1941, Hyers [32] published the first answer to it, in the case of Banach space. The following theorem is the most classical result concerning the Hyers-Ulam stability of the Cauchy equation

$$f(x + y) = f(x) + f(y). \tag{3}$$

Theorem 1.2. *Let E_1 and E_2 be two normed spaces and let $f : E_1 \to E_2$ satisfy the inequality*

$$\left\| f(x + y) - f(x) - f(y) \right\| \leq \theta(\|x\|^p + \|y\|^p) \tag{4}$$

for all $x, y \in E_1 \setminus \{0\}$, where θ and p are real constants with $\theta > 0$ and $p \neq 1$. Then the following two statements are valid.

(a) If $p \geq 0$ and E_2 is complete, then there exists a unique solution $T : E_1 \to E_2$ of (3) such that

$$\left\| f(x) - T(x) \right\| \leq \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p, \qquad x \in E_1 \setminus \{0\}. \tag{5}$$

(b) If $p < 0$, then f is additive, i.e., (3) holds.

Note that Theorem 1.2 reduces to the first result of stability due to Hyers [32] if $p = 0$, Aoki [1] for $0 < p < 1$ (see also [46]). Afterward, Gajda [29] obtained this result for $p > 1$ and gave an example to show that Theorem 1.2 fails to hold whenever $p = 1$ thus answering a question of Th.M. Rassias. In addition, Rassias [47] proved Theorem 1.2 for $p < 0$ (see [48, page 326] and [5]). Now, it is well known that the statement (b) is valid, i.e., f must be additive in that case, which has been proved for the first time in [41] and next in [8] on the restricted domain. Since then, a further generalization of the Hyers-Ulam theorem has been extensively investigated by a number of mathematicians [11, 15–18, 23, 31, 33, 34, 36, 45, 49]. The first result on the stability of the classical Jensen equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ was given by Z. Kominek [40]. The first author, who investigated the stability problem on a restricted domain was F. Skof [50]. The stability of the Jensen equation and its generalizations were studied by numerous researchers, cf. [35, 42, 44].

The hyperstability term was used for the first time probably in [43]; however, it seems that the first hyperstability result was published in [4] and concerned the ring homomorphisms. For further information concerning the notion of hyperstability we refer to the survey paper [13] (for recent related results see, e.g., [3, 7–10, 14, 20–22, 24, 30, 37]).

The theory of 2-normed spaces was first developed by Gähler [27] in the mid 1960s, while that of 2-Banach spaces was studied later by Gähler [28] and White [52]. For more details, the readers refer to the papers [19, 25, 26].

Now, we give some basic concepts concerning $(2, \beta)$-normed spaces and some preliminary results.

Definition 1.3. *Let E be a linear space over $\mathbb{K}$ with dim $E > 1$ and $0 < \beta \leq 1$. A function $\|\cdot,\cdot\|_\beta : E \times E \to \mathbb{R}_+$ is called a $(2, \beta)$-norm on E if and only if it satisfies:*

(D1) $\|x, y\|_\beta = 0$ *if and only if x and y are linearly dependent;*

(D2) $\|x, y\|_\beta = \|y, x\|_\beta$;

(D3) $\|\lambda x, y\|_\beta = |\lambda|^\beta \|x, y\|_\beta$;

(D4) $\|x, y + z\|_\beta \le \|x, y\|_\beta + \|x, z\|_\beta$

for all $x, y, z \in E$ *and* $\lambda \in \mathbb{K}$. *The pair* $(E, \|\cdot,\cdot\|_\beta)$ *is called a* $(2,\beta)$*-normed space.*

If $x \in E$ and $\|x, y\|_\beta = 0$ for all $y \in E$, then $x = 0$. Moreover; the functions $x \to \|x, y\|_\beta$ are continuous functions of E into $\mathbb{R}_+$ for each fixed $y \in E$.

The basic definitions of a $(2,\beta)$-Banach space are given as follows:

(a) A sequence $\{x_n\}$ in a $(2,\beta)$-normed space E is called a *Cauchy sequence* if there are $y, z \in E$ such that y and z are linearly independent, $\lim_{n,m\to\infty} \|x_n - x_m, y\|_\beta = 0$ and $\lim_{n,m\to\infty} \|x_n - x_m, z\|_\beta = 0$.

(b) A sequence $\{x_n\}$ in a linear $(2,\beta)$-normed space E is called a *convergent sequence* if there is an $x \in E$ such that $\lim_{n\to\infty} \|x_n - x, y\|_\beta = 0$ for all $y \in E$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(c) A $(2,\beta)$-normed space in which every Cauchy sequence is a convergent sequence is called a $(2,\beta)$-*Banach space*.

We remark that the concept of a linear $(2,\beta)$-normed space is a generalization of a linear 2-normed space ($\beta = 1$). Now, we present an example about $(2,\beta)$-normed space.

Example 1.4. *For* $x = (x_1, x_2), y = (y_1, y_2) \in E = \mathbb{R}^2$, *the* $(2,\beta)$*-norm on* E *is defined by*

$$\|x, y\|_\beta = |x_1 y_2 - x_2 y_1|^\beta,$$

where β *is a fixed real number with* $0 < \beta \le 1$.

Let X be a β-normed spaces and Y a $(2,\beta)$-normed spaces. A mapping $f : X \times X \to Y$ is called a *bi-Jensen mapping* if f satisfies the equation

$$4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = f(x,y) + f(x,w) + f(z,y) + f(z,w), \quad x, y, z, w \in X. \tag{6}$$

When $X = Y = \mathbb{R}$, the function $f(x, y) := axy + bx + cy + d$ is a solution of the functional equation (6), where a, b, c and d are arbitrary constants. Bae and Park [2] obtained the general solution of a bi-Jensen functional equation and its stability. Moreover, the stability problem for the bi-Jensen functional equation was discussed by a number of authors (see [38, 39]).

Let U be a nonempty subset of X. We say that a function $f : U^2 \to Y$ fulfils equation (6) on U (or is a solution to (6) on U) provided

$$4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = f(x,y) + f(x,w) + f(z,y) + f(z,w), \quad x, y, z, w \in U, \ \frac{x+z}{2}, \frac{y+w}{2} \in U, \tag{7}$$

if $U = X$, then we simply say that f fulfils (or is a solution to) equation (6) on X.

We consider functions $f : U^2 \to Y$ fulfilling (7) approximately, i.e., satisfying the inequality

$$\left\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w), u\right\|_\beta \le \gamma(x,y,z,w,u), \tag{8}$$
$$u \in Y, \ x, y, z, w, \in U, \ \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

with γ is a given non negative mapping. In this paper, we show that, for some natural particular forms of γ (and under some additional assumptions on U), the conditional functional equation (7) is γ-hyperstable in the class of functions $f : U^2 \to Y$, i.e., each $f : U^2 \to Y$ satisfying inequality (8) with such γ must fulfil equation (7).

2. A fixed point theorem

In this section, we rewrite the fixed point theorem [12, Theorem 1] in $(2,\beta)$-Banach space. For it we need to introduce the following hypotheses.

(H1) W is a nonempty set, Y is a $(2,\beta)$-Banach space.

(H2) $f_1,\dots,f_k : W \to W$ and $L_1,\dots,L_k : W\times Y \to \mathbb{R}_+$ are given maps.

(H3) $\mathcal{T} : Y^W \to Y^W$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\right\|_\beta \le \sum_{i=1}^{k} L_i(x,y)\left\|\xi(f_i(x)) - \mu(f_i(x)), y\right\|_\beta$$

for all $\xi,\mu \in Y^W$ and all $(x,y)\in W\times Y$.

(H4) $\Lambda : \mathbb{R}_+^{W\times Y} \to \mathbb{R}_+^{W\times Y}$ is a linear operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{k} L_i(x,y)\delta(f_i(x),y)$$

for all $\delta \in \mathbb{R}_+^W$ and $(x,y)\in W\times Y$.

The basic tool in this paper is the following fixed point theorem.

Theorem 2.1. *Let hypotheses* **(H1)-(H4)** *be valid and functions* $\varepsilon : W\times Y \to \mathbb{R}_+$ *and* $\varphi : W \to Y$ *fulfil the following two conditions:*

$$\left\|\mathcal{T}\varphi(x) - \varphi(x), y\right\|_\beta \le \varepsilon(x,y), \quad x\in W, \quad y \in Y,$$

$$\varepsilon^*(x,y) := \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x,y) < \infty, \quad x\in W, \quad y\in Y.$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\left\|\varphi(x) - \psi(x), y\right\|_\beta \le \varepsilon^*(x,y), \quad x\in W, \quad y\in Y.$$

Moreover

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \quad x\in W.$$

Proof. We can prove Theorem 2.1 analogously as [14, Theorem 1]. □

3. Hyperstability Results for Eq. (7)

In the remaining part of the paper, X is a β-normed spaces, Y is a $(2,\beta)$-Banach space, $X_0 := X\setminus\{0\}$, and $\mathbb{N}_{m_0}$ denotes the set of all integers greater than or equal to a given $m_0\in\mathbb{N}$.

The following theorems are the main results in this paper and concern the γ-hyperstability of (7). Namely, for

$$\gamma(x,y,z,w,u) = h_1(x,u)h_2(y,u)h_3(z,u)h_4(w,u),$$

with $h_i : U\times Y \to Y$ is a function for $i\in\{1,2,3,4\}$, and

$$\gamma(x,y,z,w,u) = h(x,u) + h(y,u) + h(z,u) + h(w,u),$$

with $h : U \times Y \to Y$ is a function, under some additional assumptions on the functions h, h_1, h_2, h_3, h_4 and on nonempty $U \subset X$, we show that the conditional functional equation (7) is γ-hyperstable in the class of functions f mapping U^2 to a $(2, \beta)$-Banach space. The method based on a fixed point Theorem 2.1 and patterned on the ideas provided in [6].

Theorem 3.1. *Assume that $U \subset X_0$ is nonempty and there is $n_0 \in \mathbb{N}$, $n_0 > 3$, with*

$$-x, nx \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geq n_0 - 1. \tag{9}$$

Let $h_1, h_2, h_3, h_4 : U \times Y \to \mathbb{R}_+$ be four functions such that

$$\mathcal{M}_0 := \Big\{ n \in \mathbb{N}_{n_0} \mid \alpha_n := 4^{-\beta}[s_1(n)s_3(n) + s_1(2-n)s_3(2-n)] \times [s_2(\ell)s_4(\ell) + s_2(2-\ell)s_4(2-\ell)] < 1 \Big\} \neq \emptyset, \tag{10}$$

where $\ell \in \mathbb{N}_{n_0}$ is fixed, and $s_i(\pm n) := \inf\{t \in \mathbb{R}_+ : h_i(\pm nx, u) \leq t h_i(x, u)$ for all $(x, u) \in U \times Y\}$ for $n \in \mathbb{N}_{n_0}$ and $i = 1, 2, 3, 4$, such that

$$\lim_{n\to\infty} s_1(\pm n)s_3(\pm n) = \lim_{n\to\infty} s_1(n)s_3(2-n) = 0. \tag{11}$$

Suppose that $f : U^2 \to Y$ satisfies the inequality

$$\begin{aligned} &\Big\| 4f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x,y) - f(x,w) - f(z,y) - f(z,w), u \Big\|_\beta \\ &\qquad \leq h_1(x,u)h_2(y,u)h_3(z,u)h_4(w,u), \quad u \in Y,\ \ x, y, z, w \in U,\ \ \frac{x+z}{2}, \frac{y+w}{2} \in U, \end{aligned} \tag{12}$$

then (7) *holds.*

Proof. Assume that $l \in \mathbb{N}_{n_0}$ is fixed. Replacing (x, z, y, w) by $(mx, (2-m)x, ly, (2-l)y)$ in (12), we get

$$\begin{aligned} &\Big\| 4^{-1}f(mx, ly) + 4^{-1}f(mx, (2-l)y) + 4^{-1}f((2-m)x, ly) + 4^{-1}f((2-m)x, (2-l)y) - f(x,y), u \Big\|_\beta \\ &\qquad \leq 4^{-\beta} h_1(mx, u)h_2(ly, u)h_3((2-m)x, u)h_4((2-l)y, u) \end{aligned} \tag{13}$$

for all $m \in \mathbb{N}_{n_0}$, $x, y \in U$, $u \in Y$. Fix $m \in \mathbb{N}_{n_0}$ and we define

$$\mathcal{T}_m\xi(x,y) := 4^{-1}\xi(mx, ly) + 4^{-1}\xi(mx, (2-l)y) + 4^{-1}\xi((2-m)x, ly) + 4^{-1}\xi((2-m)x, (2-l)y)$$

$$\Lambda_m\delta(x,y,u) := 4^{-\beta}\delta(mx, ly, u) + 4^{-\beta}\delta(mx, (2-l)y, u) + 4^{-\beta}\delta((2-m)x, ly, u) + 4^{-\beta}\delta((2-m)x, (2-l)y, u)$$

for every $(x, y, u) \in U \times U \times Y$, $\xi \in Y^{U\times U}$ and $\delta \in \mathbb{R}_+^{U\times U\times Y}$. Further, observe that

$$\begin{aligned} \varepsilon_m(x,y,u) &: = 4^{-\beta} h_1(mx, u)h_2(ly, u)h_3((2-m)x, u)h_4((2-l)y, u) \\ &\leq 4^{-\beta} s_1(m)s_2(l)s_3(2-m)s_4(2-l)h_1(x,u)h_2(y,u)h_3(x,u)h_4(y,u) \end{aligned} \tag{14}$$

for all $x, y \in U$ and $u \in Y$. Then inequality (13) takes the form

$$\Big\| \mathcal{T}_m f(x,y) - f(x,y), u \Big\|_\beta \leq \varepsilon_m(x,y,u), \quad x, y \in U,\ u \in Y,$$

and the operator Λ_m has the form described in (**H4**) with $k = 4$,

$$f_1(x,y) \equiv (mx, ly), \qquad f_2(x,y) \equiv (mx, (2-l)y), \qquad f_3(x,y) \equiv ((2-m)x, ly)$$

$$f_4(x,y) \equiv ((2-m)x, (2-l)y), \qquad L_1(x,y,u) \equiv L_2(x,y,u) \equiv L_3(x,y,u) \equiv L_4(x,y,u) \equiv 4^{-\beta}$$

for all $x, y \in U$ and $u \in Y$. Moreover, for every $\xi, \mu \in Y^{U\times U}$ and $x, y \in U$, $u \in Y$, we obtain

$$\begin{aligned}\left\|\mathcal{T}_m\xi(x,y) - \mathcal{T}_m\mu(x,y), u\right\|_\beta &\le 4^{-\beta}\left\|(\xi-\mu)(f_1(x,y)), u\right\|_\beta + 4^{-\beta}\left\|(\xi-\mu)(f_2(x,y)), u\right\|_\beta \\ &\quad + 4^{-\beta}\left\|(\xi-\mu)(f_3(x,y)), u\right\|_\beta + 4^{-\beta}\left\|(\xi-\mu)(f_4(x,y)), u\right\|_\beta \\ &= \sum_{i=1}^{4} L_i(x,y,u)\left\|(\xi-\mu)(f_i(x,y)), u\right\|_\beta,\end{aligned}$$

where $(\xi - \mu)(x, y) \equiv \xi(x, y) - \mu(x, y)$. So, **(H3)** is valid for $\mathcal{T}_m$.

By using mathematical induction, we will show that for each $x, y \in U$ and $u \in Y$ we have

$$\Lambda_m^n \varepsilon_m(x,y,u) \le 4^{-\beta} s_1(m)s_2(l)s_3(2-m)s_4(2-l)\alpha_m^n h_1(x,u)h_2(y,u)h_3(x,u)h_4(y,u) \tag{15}$$

for all $n \in \mathbb{N}_0$ and $m \in \mathcal{M}_0$. From (14), we obtain that the inequality (15) holds for $n = 0$. Next, we will assume that (15) holds for $n = r$, where $r \in \mathbb{N}_0$. Then we have

$$\begin{aligned}&\Lambda_m^{r+1}\varepsilon_m(x,y,u) = \Lambda_m(\Lambda_m^r\varepsilon_m(x,y,u)) \\ &= 4^{-\beta}\Lambda_m^r\varepsilon_m(mx,ly,u) + 4^{-\beta}\Lambda_m^r\varepsilon_m(mx,(2-l)y,u) + 4^{-\beta}\Lambda_m^r\varepsilon_m((2-m)x,ly,u) + 4^{-\beta}\Lambda_m^r\varepsilon_m((2-m)x,(2-l)y,u) \\ &\le 4^{-2\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)\alpha_m^r h_1(mx,u)h_2(ly,u)h_3(mx,u)h_4(ly,u) \\ &\quad + 4^{-2\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)\alpha_m^r h_1(mx,u)h_2((2-l)y,u)h_3(mx,u)h_4((2-l)y,u) \\ &\quad + 4^{-2\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)\alpha_m^r h_1((2-m)x,u)h_2(ly,u)h_3((2-m)x,u)h_4(ly,u) \\ &\quad + 4^{-2\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)\alpha_m^r h_1((2-m)x,u)h_2((2-l)y,u)h_3((2-m)x,u)h_4((2-l)y,u) \\ &\le 4^{-\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)\alpha_m^{r+1}h_1(x,u)h_2(y,u)h_3(x,u)h_4(y,u).\end{aligned}$$

This shows that (15) holds for $n = r + 1$. Now we can conclude that the inequality (15) holds for all $n \in \mathbb{N}_0$. Therefore, we obtain that

$$\varepsilon_m^*(x,y,u) := \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x,y,u) \le \frac{4^{-\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)h_1(x,u)h_2(y,u)h_3(x,u)h_4(y,u)}{1-\alpha_m}$$

for all $u \in Y$, $x, y \in U$ and $m \in \mathcal{M}_0$. Thus, according to Theorem 2.1, for each $m \in \mathcal{M}_0$ the function $J_m : U \times U \to Y$, given by $J_m(x, y) = \lim_{n\to\infty} \mathcal{T}_m^n f(x, y)$ for $x, y \in U$, is a unique fixed point of $\mathcal{T}_m$, i.e.,

$$J_m(x,y) = 4^{-1}J_m(mx,ly) + 4^{-1}J_m(mx,(2-l)y) + 4^{-1}J_m((2-m)x,ly) + 4^{-1}J_m((2-m)x,(2-l)y)$$

for all $x, y \in U$; moreover

$$\left\|f(x,y) - J_m(x,y), u\right\|_\beta \le \frac{4^{-\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)h_1(x,u)h_2(y,u)h_3(x,u)h_4(y,u)}{1-\alpha_m}$$

for all $u \in Y$, $x, y \in U$ and $m \in \mathcal{M}_0$. We show that

$$\begin{aligned}&\left\|4\mathcal{T}_m^n f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - \mathcal{T}_m^n f(x,y) - \mathcal{T}_m^n f(x,w) - \mathcal{T}_m^n f(z,y) - \mathcal{T}_m^n f(z,w), u\right\|_\beta \\ &\qquad \le \alpha_m^n h_1(x,u)h_2(y,u)h_3(z,u)h_4(w,u)\end{aligned} \tag{16}$$

for every $n \in \mathbb{N}_0$, $u \in Y$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$.

Clearly, if $n = 0$, then (16) is simply (12). So, fix $n \in \mathbb{N}_0$ and suppose that (16) holds for n and every $u \in Y$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$. Then, for every $u \in Y$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$,

$$\begin{aligned}
&\Big\|4\mathcal{T}_m^{n+1} f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - \mathcal{T}_m^{n+1} f(x,y) - \mathcal{T}_m^{n+1} f(x,w) - \mathcal{T}_m^{n+1} f(z,y) - \mathcal{T}_m^{n+1} f(z,w), u\Big\|_\beta \\
&= \Big\|\mathcal{T}_m^n f\left(m\frac{x+z}{2}, l\frac{y+w}{2}\right) + \mathcal{T}_m^n f\left(m\frac{x+z}{2}, (2-l)\frac{y+w}{2}\right) + \mathcal{T}_m^n f\left((2-m)\frac{x+z}{2}, l\frac{y+w}{2}\right) \\
&\quad + \mathcal{T}_m^n f\left((2-m)\frac{x+z}{2}, (2-l)\frac{y+w}{2}\right) - 4^{-1}\mathcal{T}_m^n f(mx, ly) - 4^{-1}\mathcal{T}_m^n f(mx, (2-l)y) - 4^{-1}\mathcal{T}_m^n f((2-m)x, ly) \\
&\quad - 4^{-1}\mathcal{T}_m^n f((2-m)x, (2-l)y) - 4^{-1}\mathcal{T}_m^n f(mx, lw) - 4^{-1}\mathcal{T}_m^n f(mx, (2-l)w) - 4^{-1}\mathcal{T}_m^n f((2-m)x, lw) \\
&\quad - 4^{-1}\mathcal{T}_m^n f((2-m)x, (2-l)w) - 4^{-1}\mathcal{T}_m^n f(mz, ly) - 4^{-1}\mathcal{T}_m^n f(mz, (2-l)y) - 4^{-1}\mathcal{T}_m^n f((2-m)z, ly) \\
&\quad - 4^{-1}\mathcal{T}_m^n f((2-m)z, (2-l)y) - 4^{-1}\mathcal{T}_m^n f(mz, lw) - 4^{-1}\mathcal{T}_m^n f(mz, (2-l)w) \\
&\quad - 4^{-1}\mathcal{T}_m^n f((2-m)z, lw) - 4^{-1}\mathcal{T}_m^n f((2-m)z, (2-l)w), u\Big\|_\beta \\
&\le 4^{-\beta}\Big\|4\mathcal{T}_m^n f\left(m\frac{x+z}{2}, l\frac{y+w}{2}\right) - \mathcal{T}_m^n f(mx, ly) - \mathcal{T}_m^n f(mx, lw) - \mathcal{T}_m^n f(mz, ly) - \mathcal{T}_m^n f(mz, lw), u\Big\|_\beta \\
&\quad + 4^{-\beta}\Big\|\mathcal{T}_m^n f\left(m\frac{x+z}{2}, (2-l)\frac{y+w}{2}\right) - \mathcal{T}_m^n f(mx, (2-l)y) - \mathcal{T}_m^n f(mx, (2-l)w) - \mathcal{T}_m^n f(mz, (2-l)y) \\
&\quad - \mathcal{T}_m^n f(mz, (2-l)w), u\Big\|_\beta + 4^{-\beta}\Big\|\mathcal{T}_m^n f\left((2-m)\frac{x+z}{2}, l\frac{y+w}{2}\right) - \mathcal{T}_m^n f((2-m)x, ly) - \mathcal{T}_m^n f((2-m)x, lw) \\
&\quad - \mathcal{T}_m^n f((2-m)z, ly) - \mathcal{T}_m^n f((2-m)z, lw), u\Big\|_\beta + 4^{-\beta}\Big\|\mathcal{T}_m^n f\left((2-m)\frac{x+z}{2}, (2-l)\frac{y+w}{2}\right) \\
&\quad - \mathcal{T}_m^n f((2-m)x, (2-l)y) - \mathcal{T}_m^n f((2-m)x, (2-l)w) - \mathcal{T}_m^n f((2-m)z, (2-l)y) - \mathcal{T}_m^n f((2-m)z, (2-l)w), u\Big\|_\beta \\
&\le 4^{-\beta}\alpha_m^n h_1(mx)h_2(ly)h_3(mz)h_4(lw) + 4^{-\beta}\alpha_m^n h_1(mx,u)h_2((2-l)y,u)h_3(mz,u)h_4((2-l)w,u) \\
&\quad + 4^{-\beta}\alpha_m^n h_1((2-m)x,u)h_2(ly,u)h_3((2-m)z,u)h_4(lw,u) \\
&\quad + 4^{-\beta}\alpha_m^n h_1((2-m)x,u)h_2((2-l)y,u)h_3((2-m)z,u)h_4((2-l)w,u) \\
&\le (\alpha_m)^{n+1} h_1(x,u)h_2(y,u)h_3(z,u)h_4(w,u).
\end{aligned}$$

Thus, by induction, we have shown that (16) holds for all $u \in Y$ and $x, y, z, w \in U$ such that $\frac{x+z}{2}, \frac{y+w}{2} \in U$ and for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (16), we obtain that

$$4J_m\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = J_m(x,y) + J_m(x,w) + J_m(z,y) + J_m f(z,w) \tag{17}$$

for every $m \in \mathcal{M}_0$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$.

In this way, for each $m \in \mathcal{M}_0$, we obtain a function J_m such that (17) holds for $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$ and

$$\|f(x,y) - J_m(x,y), u\|_\beta \le \frac{4^{-\beta}s_1(m)s_2(l)s_3(2-m)s_4(2-l)h_1(x,u)h_2(y,u)h_3(x,u)h_4(y,u)}{1-\alpha_m}$$

for all $u \in Y$, $x, y \in U$ and $m \in \mathcal{M}_0$. Since

$$\lim_{m\to\infty} s_1(\pm m)s_3(\pm m) = \lim_{m\to\infty} s_1(m)s_3(2-m) = 0,$$

it follows, with $m \to \infty$, that f fulfils (7). □

Remark 3.2. *The Theorem 3.1 also provide γ-hyperstability results in each of the following cases:*

- $\gamma(x, y, z, w, u) = h_1(x, u), \quad x, y, z, w \in U,\ u \in Y;$

- $\gamma(x,y,z,w,u) = h_1(x,u)h_2(y,u), \quad x,y,z,w \in U, \ \ u \in Y;$
- $\gamma(x,y,z,w,u) = h_1(x,u)h_2(y,u)h_3(z,u), \quad x,y,z,w \in U, \ \ u \in Y.$

In a similar way we can prove the following theorem.

Theorem 3.3. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition* (9) *with some $n_0 \in \mathbb{N}$. Let $h : U \times Y \to \mathbb{R}_+$ be a function such that*

$$\mathcal{M}_0 := \left\{ n \in \mathbb{N}_{n_0} \mid b_n := 2^{1-2\beta}[s(n) + s(2-n)] < 1 \right\} \neq \emptyset, \tag{18}$$

where $s(\pm n) := \inf\{t \in \mathbb{R}_+ : h(\pm nx, u) \leq th(x,u) \ \text{for all} \ (x,u) \in U \times Y\}$ for $n \in \mathbb{N}_{n_0}$ such that

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} s(-n) = 0. \tag{19}$$

Suppose that $f : U^2 \to Y$ satisfies the inequality

$$\begin{aligned} &\left\| 4f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x,y) - f(x,w) - f(z,y) - f(z,w), u \right\|_\beta \\ &\quad \leq h(x,u) + h(y,u) + h(z,u) + h(w,u), \quad u \in Y, \ x,y,z,w \in U, \ \ \frac{x+z}{2}, \frac{y+w}{2} \in U, \end{aligned} \tag{20}$$

then (7) *holds.*

Proof. Replacing (x,z,y,w) by $(mx,(2-m)x,my,(2-m)y)$ in (20), we get

$$\begin{aligned} &\left\| 4^{-1}f(mx,my) + 4^{-1}f(mx,(2-m)y) + 4^{-1}f((2-m)x,(2-m)y) + 4^{-1}f((2-m)x,my) - f(x,y), u \right\|_\beta \\ &\qquad \leq 4^{-\beta}[h(mx,u) + h(my,u) + h((2-m)x,u) + h((2-m)y,u)] \end{aligned} \tag{21}$$

for all $m \in \mathbb{N}_{n_0}$, $u \in Y$ and $x,y \in U$. Let

$$\begin{aligned} \varepsilon_m(x,y,u) : &= 4^{-\beta}[h(mx,u) + h(my,u) + h((2-m)x,u) + h((2-m)y,u)] \\ &\leq 4^{-\beta}\left(s(m) + s(2-m)\right)[h(x,u) + h(y,u)] \end{aligned} \tag{22}$$

$$\mathcal{T}_m\xi(x,y) := 4^{-1}\xi(mx,my) + 4^{-1}\xi(mx,(2-m)y) + 4^{-1}\xi((2-m)x,my) + 4^{-1}\xi((2-m)x,(2-m)y)$$

for $(x,y,u) \in U \times U \times Y$, $m \in \mathbb{N}_{n_0}$ and $\xi \in Y^{U\times U}$. Then inequality (21) takes the form

$$\left\| \mathcal{T}_m f(x,y) - f(x,y), u \right\|_\beta \leq \varepsilon_m(x,y,u), \quad u \in Y, \ \ x,y \in U, \ \ m \in \mathbb{N}_{n_0}.$$

Write

$$\Lambda_m\delta(x,y,u) = 4^{-\beta}\delta(mx,my,u) + 4^{-\beta}\delta(mx,(2-m)y,u) + 4^{-\beta}\delta((2-m)x,my,u) + 4^{-\beta}\delta((2-m)x,(2-m)y,u)$$

for $(x,y,u) \in U \times U \times Y$, $m \in \mathbb{N}_{n_0}$ and $\delta \in \mathbb{R}_+^{U\times U\times Y}$. Then, the operator Λ_m has the form described in **(H4)** with $k = 4$ and

$$f_1(x,y) \equiv (mx,my), \qquad f_2(x,y) \equiv (mx,(2-m)y), \qquad f_3(x,y) \equiv ((2-m)x,my)$$

$$f_4(x,y) \equiv ((2-m)x,(2-m)y), \qquad L_1(x,y,u) \equiv L_2(x,y,u) \equiv L_3(x,y,u) \equiv L_4(x,y,u) \equiv 4^{-\beta}$$

for all $x,y \in U$ and $u \in Y$. Moreover, for every $\xi,\mu \in Y^{U\times U}$, $m \in \mathbb{N}_{n_0}$ and $u \in Y$, $x,y \in U$, we have

$$\begin{aligned} \left\| \mathcal{T}_m\xi(x,y) - \mathcal{T}_m\mu(x,y), u \right\|_\beta &\leq 4^{-\beta}\left\|(\xi-\mu)(f_1(x,y)), u\right\|_\beta + 4^{-\beta}\left\|(\xi-\mu)(f_2(x,y)), u\right\|_\beta \\ &\quad + 4^{-\beta}\left\|(\xi-\mu)(f_3(x,y)), u\right\|_\beta + 4^{-\beta}\left\|(\xi-\mu)(f_4(x,y)), u\right\|_\beta \\ &= \sum_{i=1}^{4} L_i(x,y,u)\left\|(\xi-\mu)(f_i(x,y)), u\right\|_\beta. \end{aligned}$$

So, **(H3)** is valid for $\mathcal{T}_m$.

Next, it easily seen that, by induction on n, from (22) we obtain

$$\Lambda_m^n \varepsilon_m(x,y,u) \le 4^{-\beta} \left(s(m) + s(2-m)\right) b_m^n [h(x,u) + h(y,u)] \tag{23}$$

for all $n \in \mathbb{N}_0$ and $m \in \mathcal{M}_0$. Therefore, we obtain that

$$\varepsilon_m^*(x,y,u) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x,y,u) \le \frac{4^{-\beta} \left(s(m) + s(2-m)\right) [h(x,u) + h(y,u)]}{1 - b_m}, \quad u \in Y, \; x, y \in U, \; m \in \mathcal{M}_0.$$

Thus, according to Theorem 2.1, for each $m \in \mathcal{M}_0$ the function $J_m : U \times U \to Y$, given by $J_m(x,y) = \lim_{n\to\infty} \mathcal{T}_m^n f(x,y)$ for $x, y \in U$, is a unique fixed point of $\mathcal{T}_m$, i.e.,

$$J_m(x,y) = 4^{-1} J_m(mx, my) + 4^{-1} J_m(mx, (2-m)y) + 4^{-1} J_m((2-m)x, my) + 4^{-1} J_m((2-m)x, (2-m)y)$$

for all $x, y \in U$; moreover

$$\left\| f(x,y) - J_m(x,y), u \right\|_\beta \le \frac{4^{-\beta} \left(s(m) + s(2-m)\right) [h(x,u) + h(y,u)]}{1 - b_m}$$

for all $u \in Y$, $x, y \in U$ and $m \in \mathcal{M}_0$. Similarly as in the proof of Theorem 3.1, we show that

$$\begin{aligned} &\left\| 4\mathcal{T}_m^n f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - \mathcal{T}_m^n f(x,y) - \mathcal{T}_m^n f(x,w) - \mathcal{T}_m^n f(z,y) - \mathcal{T}_m^n f(z,w), u \right\|_\beta \\ &\qquad \le b_m^n [h(x,u) + h(y,u) + h(z,u) + h(w,u)] \end{aligned}$$

for every $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$, $u \in Y$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$. Also the remaining reasonings are analogous as in the proof of that theorem. □

Remark 3.4. *The Theorem 3.3 also provide γ-hyperstability results in each of the following cases:*

- $\gamma(x,y,z,w) = h(x,u), \quad x \in U, \; u \in Y;$
- $\gamma(x,y,z,w) = h(x,u) + h(y,u), \quad x, y \in U, \; u \in Y;$
- $\gamma(x,y,z,w) = h(x,u) + h(y,u) + h(z,u), \quad x, y, z \in U, \; u \in Y.$

By using Theorems 3.1, 3.3 and the same technique we get the following hyperstability results for the inhomogeneous bi-Jensen functional equation.

Corollary 3.5. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition* (9) *with some $n_0 \in \mathbb{N}$. Let $F : U^4 \to Y$ be a given mapping and $h_1, h_2, h_3, h_4 : U \times Y \to \mathbb{R}_+$ be four functions such that* (10) *is an infinite set, where $s_i(\pm n) := \inf\{t \in \mathbb{R}_+ : h_i(\pm nx, u) \le t h_i(x,u) \text{ for all } (x,u) \in U \times Y\}$ for $n \in \mathbb{N}_{n_0}$ and $i = 1, 2, 3, 4$, such that*

$$\lim_{n\to\infty} s_1(\pm n) s_3(\pm n) = \lim_{n\to\infty} s_1(n) s_3(2-n) = 0.$$

Suppose that $f : U^2 \to Y$ satisfies the condition

$$\begin{aligned} &\left\| 4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w), u \right\|_\beta \\ &\qquad \le h_1(x,u) h_2(y,u) h_3(z,u) h_4(w,u), \quad u \in Y, \; x, y, z, w \in U, \; \frac{x+z}{2}, \frac{y+w}{2} \in U, \end{aligned} \tag{24}$$

and the functional equation

$$\begin{aligned} &4g\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = g(x,y) + g(x,w) + g(z,y) + g(z,w) + F(x,y,z,w), \\ &\qquad x, y, z, w \in U, \; \frac{x+z}{2}, \frac{y+w}{2} \in U, \end{aligned} \tag{25}$$

has a solution $f_0 : U^2 \to Y$. Then f is a solution of (25).

Proof. Let $f_1(x,y) := f(x,y) - f_0(x,y)$ for $x, y \in U$. Then

$$\begin{aligned}
&\left\|4f_1\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f_1(x,y) - f_1(x,w) - f_1(z,y) - f_1(z,w), u\right\|_\beta \\
&= \Big\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w) \\
&\quad - 4f_0\left(\frac{x+z}{2}, \frac{y+w}{2}\right) + f_0(x,y) + f_0(x,w) + f_0(z,y) + f_0(z,w) + F(x,y,z,w), u\Big\|_\beta \\
&\leq \left\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w), u\right\|_\beta \\
&\quad + \left\|4f_0\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f_0(x,y) - f_0(x,w) - f_0(z,y) - f_0(z,w) - F(x,y,z,w), u\right\|_\beta \\
&= \left\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w), u\right\|_\beta \\
&\leq h_1(x,u)h_2(y,u)h_3(z,u)h_4(w,u), \quad u \in Y, \;\; x,y,z,w \in U, \;\; \frac{x+z}{2}, \frac{y+w}{2} \in U.
\end{aligned}$$

It follows from Theorem 3.1 with f replaced by f_1 that f_1 satisfies the bi-Jensen functional equation (7). Therefore,

$$\begin{aligned}
&4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w) \\
&=4f_1\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f_1(x,y) - f_1(x,w) - f_1(z,y) - f_1(z,w) \\
&\quad + 4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w) = 0
\end{aligned}$$

for all $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$. □

Analogously we prove the following.

Corollary 3.6. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition (9) with some $n_0 \in \mathbb{N}$. Let $F : U^4 \to Y$ be a given mapping and $h : U \times Y \to \mathbb{R}_+$ be a function such that (18) is an infinite set, where $s(\pm n) := \inf\{t \in \mathbb{R}_+ : h(\pm nx, u) \leq th(x,u) \text{ for all } (x,u) \in U \times Y\}$ for $n \in \mathbb{N}_{n_0}$, such that*

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} s(-n) = 0.$$

Suppose that $f : U^2 \to Y$ satisfies the condition

$$\begin{aligned}
&\left\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w), u\right\|_\beta \\
&\qquad \leq h(x,u) + h(y,u) + h(z,u) + h(w,u), \quad u \in Y, \;\; x,y,z,w \in U, \;\; \frac{x+z}{2}, \frac{y+w}{2} \in U,
\end{aligned} \tag{26}$$

and the functional equation

$$\begin{aligned}
&4g\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = g(x,y) + g(x,w) + g(z,y) + g(z,w) - F(x,y,z,w), \\
&\qquad x,y,z,w \in U, \;\; \frac{x+z}{2}, \frac{y+w}{2} \in U,
\end{aligned} \tag{27}$$

has a solution $f_0 : U^2 \to Y$. Then f is a solution of (27).

4. Some particular cases and examples

According to Theorems 3.1, 3.3 and Corollaries 3.5, 3.6 with $h(x,u) := c\|x\|_\beta^p\|u,a\|_\beta$ and $h_i(x,u) := c_i\|x\|_\beta^{p_i}\|u,a\|_\beta$ for all $(x,u) \in U \times Y$ and for some arbitrary element $a \in Y$, where $c, p, c_i, p_i \in \mathbb{R}$ and $i = 1,2,3,4$, we derive some corollaries of our main results.

Corollary 4.1. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition* (9) *with some $n_0 \in \mathbb{N}$. If $f : U^2 \to Y$ satisfies the functional inequality*

$$\left\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w), u\right\|_\beta \le c\|x\|_\beta^p\|y\|_\beta^q\|z\|_\beta^r\|w\|_\beta^l\|u,a\|_\beta,$$

$$u \in Y,\ \ x,y,z,w \in U,\ \ \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

with some fixed number $a \in Y$, $c \ge 0$, and $p,q,r,l \in \mathbb{R}$ such that $p + r < 0$ or $q + l < 0$, then (7) *holds.*

Proof. Define $h_1, h_2, h_3, h_4 : U \times Y \to \mathbb{R}_+$ by $h_1(x,u) = c_1\|x\|_\beta^p\|u,a\|_\beta$, $h_2(x,u) = c_2\|x\|_\beta^q\|u,a\|_\beta$, $h_3(x,u) = c_3\|x\|_\beta^r\|u,a\|_\beta$ and $h_4(x,u) = c_4\|x\|_\beta^l\|u,a\|_\beta$ for an arbitrary element $a \in Y$, where $c_1,c_2,c_3,c_4 \in \mathbb{R}_+$ and $p,q,r,l \in \mathbb{R}$ such that $p + r < 0$ (the case $q + l < 0$ is analogous), $c = c_1c_2c_3c_4$ and fix $\ell \in \mathbb{N}_{n_0}$.

For each $n \in \mathbb{N}_{n_0}$ and for an arbitrary element $a \in Y$

$$\begin{aligned} s_1(\pm n) &= \inf\{t \in \mathbb{R}_+ : h_1(\pm nx, u) \le th_1(x,u) \ \text{ for all } \ (x,u) \in U \times Y\} \\ &= \inf\{t \in \mathbb{R}_+ : c_1\|nx\|_\beta^p\|u,a\|_\beta \le tc_1\|x\|_\beta^p\|u,a\|_\beta \ \text{ for all } \ (x,u) \in U \times Y\} \\ &= n^{\beta p}. \end{aligned}$$

Also, for each $n \in \mathbb{N}_{n_0}$, we have $s_3(\pm n) = n^{\beta r}$. So

$$\begin{aligned} &\lim_{n\to\infty} 4^{-\beta}[s_1(n)s_3(n) + s_1(2-n)s_3(2-n)] \times [s_2(\ell)s_4(\ell) + s_2(2-\ell)s_4(2-\ell)] \\ &\quad = \lim_{n\to\infty} 4^{-\beta}[n^{\beta(p+r)} + (n-2)^{\beta(p+r)}] \times [\ell^{\beta(q+l)} + (\ell-2)^{\beta(q+l)}] \\ &\quad = 0. \end{aligned}$$

Clearly, there is $n_1 \in \mathbb{N}_{n_0}$ such that

$$4^{-\beta}[n^{\beta(p+r)} + (n-2)^{\beta(p+r)}] \times [\ell^{\beta(q+l)} + (\ell-2)^{\beta(q+l)}] < 1, \quad n \ge n_1.$$

Also, we have

$$\lim_{n\to\infty} 4^{-\beta}s_1(n)s_2(\ell)s_3(2-n)s_4(2-\ell) = \lim_{n\to\infty} 4^{-\beta}n^{\beta p}\ell^{\beta q}(n-2)^{\beta r}(\ell-2)^{\beta l} = 0.$$

Thus, all the conditions in Theorem 3.1 are fulfilled. So, we get the desired results. □

Corollary 4.2. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition* (9) *with some $n_0 \in \mathbb{N}$. Let $F : U^4 \to Y$ be a given mapping, $c \ge 0$ and $p,q,r,l \in \mathbb{R}$ be such that $p + r < 0$ or $q + l < 0$. Suppose that $f : U^2 \to Y$ satisfies the condition*

$$\left\|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w), u\right\|_\beta \le c\|x\|_\beta^p\|y\|_\beta^q\|z\|_\beta^r\|w\|_\beta^l\|u,a\|_\beta,$$

$$u \in Y,\ \ x,y,z,w \in U,\ \ \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

for some fixed element $a \in Y$, and the functional equation

$$4g\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = g(x,y) + g(x,w) + g(z,y) + g(z,w) + F(x,y,z,w), \tag{28}$$

$$x,y,z,w \in U,\ \ \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

has a solution $f_0 : U^2 \to Y$. Then f is a solution of (28).

Corollary 4.3. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition* (9) *with some $n_0 \in \mathbb{N}$. If $f : U^2 \to Y$ satisfies the functional inequality*

$$\left\|4f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x,y) - f(x,w) - f(z,y) - f(z,w), u\right\|_\beta \le c(\|x\|_\beta^p + \|y\|_\beta^p + \|z\|_\beta^p + \|w\|_\beta^p)\|u,a\|_\beta,$$
$$u \in Y, \quad x,y,z,w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

with some fixed element $a \in Y$, $c \ge 0$, and $p < 0$, then (7) *holds.*

Corollary 4.4. *Let U be a nonempty subset of $X \setminus \{0\}$ fulfilling condition* (9) *with some $n_0 \in \mathbb{N}$. Let $F : U^4 \to Y$ be a given mapping, $c \ge 0$ and $p \in \mathbb{R}$ be such that $p < 0$. Suppose that $f : U^2 \to Y$ satisfies the condition*

$$\left\|4f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x,y) - f(x,w) - f(z,y) - f(z,w) - F(x,y,z,w), u\right\|_\beta$$
$$\le c(\|x\|_\beta^p + \|y\|_\beta^p + \|z\|_\beta^p + \|w\|_\beta^p)\|u,a\|_\beta, \quad u \in Y, \quad x,y,z,w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

for some fixed element $a \in Y$, and the functional equation

$$4g\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = g(x,y) + g(x,w) + g(z,y) + g(z,w) + F(x,y,z,w), \tag{29}$$
$$x,y,z,w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

has a solution $f_0 : U^2 \to Y$. Then f is a solution of (29).

Proof. The proofs of Corollary 4.2, 4.3 and 4.4 are direct consequences of Corollary 3.5, Theorem 3.3 and Corollary 3.6 respectively. □

Now, we give some examples which show that in the above theorems the additional assumption on U are necessary.

Example 4.5. *Let $X = Y = \mathbb{R}$, $U = [-1,1] \setminus \{0\}$, $p, r < 0$, and $f : U^2 \to \mathbb{R}$ be defined by $f(x,y) = |x|$. Let $h_1, h_2, h_3, h_4 : U \to \mathbb{R}_+$ be functions such that $h_1(x) = |x|^p$, $h_3(x) = |x|^r$ and $h_2(x) = h_4(x) = 2$. Then f satisfies*

$$\left|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w)\right| \le 4|x|^p|z|^r, \quad x,y,z,w \in U$$

but f is not solution of Eq. (7) *on U. We see that $0 \notin U$ and U does not satisfy the assumption of Theorem 3.1.*

Example 4.6. *Let $X = Y = \mathbb{R}$, $U = (0,\infty)$, $p < 0$ and $f : U^2 \to \mathbb{R}$ be defined by $f(x,y) = x^p + y^p$. Let $h : U \to \mathbb{R}_+$ be a function such that $h(x) = 2^{2-p}|x|^p$. Then f satisfies*

$$\left|4f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) - f(x,y) - f(x,w) - f(z,y) - f(z,w)\right| \le 2^{2-p}(|x|^p + |y|^p + |z|^p + |w|^p), \quad x,y,z,w \in U$$

but f is not solution of Eq. (7) *on U, which shows that in Theorem 3.3 the assumption that $-x \in U$ for every $x \in U$ is necessary.*

We end the paper with an open problem.

Problem. For the cases h_i and h are constants functions for $i = 1,2,3,4$, the method used in the proofs of the above theorems can not be applied, thus this is still an open problem.

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