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Generalized Drazin Inverses in a Ring

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Abstract. An element a in a ring R has generalized Drazin inverse if and only if there exists $b \in comm^2(a)$ such that $b = b^2a, a - a^2b \in R^{qnil}$. We prove that $a \in R$ has generalized Drazin inverse if and only if there exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$. An element a in a ring R has pseudo Drazin inverse if and only if there exists $b \in comm^2(a)$ such that $b = b^2a, a^k - a^{k+1}b \in J(R)$ for some $k \in \mathbb{N}$. We also characterize pseudo inverses by means of tripotents in a ring. Moreover, we prove that $a \in R$ has pseudo Drazin inverse if and only if there exists $b \in comm^2(a)$ and $m, k \in \mathbb{N}$ such that $b^m = b^{m+1}a, a^k - a^{k+1}b \in J(R)$.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm^2(a)\}$. We use $U(R)$ to denote the set of all units in R. Set $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. We say $a \in R$ is quasinilpotent if $a \in R^{qnil}$. The generalized Drazin inverse of $a \in R$ is the unique element $b \in R$ which satisfies

$$b \in comm^2(a), b = b^2a, a - a^2b \in R^{qnil}.$$

The set of all generalized Drazin invertible elements of R will be denoted by R^{gD}. Generalized Drazin inverse is extensively studied in matrix theory and Banach algebra (see [2, 3, 6, 7, 9] and [10]).

An element a in a ring R is quasipolar if there exists $e^2 = e \in comm^2(a)$ such that $a+e \in U(R)$ and $ae \in R^{qnil}$. As is well known, an element $a \in R$ has generalized Drazin inverse if and only if it is quasipolar (see [7, Theorem 4.2]). In Section 2, we shall characterize generalized Drazin inverse by means of tripotents p, i.e., $p^3 = p$. We prove that $a \in R$ has generalized Drazin inverse if and only if there exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$.

Following [8], an element a in a ring R has pseudo Drazin inverse if and only if there exists $b \in R$ such that

$$b \in comm^2(a), b = b^2a, a^k - a^{k+1}b \in J(R)$$

for some $k \in \mathbb{N}$. We may replace the double commutator by the commutator in the preceding definition for a Banach algebra (see [8, Remark 5.1]). If $a \in R$ has pseudo Drazin inverse, then it has a generalized

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Drazin inverse, but the converse is not true (see [8, Example 3.5]). Recently, many properties of pseudo inverses of matrices over a ring are explored (see [4, 8] and [11]). We also characterize pseudo inverses by means of tripotents in a ring. In Section 3, we prove that $a \in R$ has pseudo Drazin inverse if and only if there exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^k p \in R^{qnil}$ for some $k \in \mathbb{N}$. Moreover, we prove that $a \in R$ has pseudo Drrazin inverse if and only if there exists $b \in comm^2(a)$ and $m, k \in \mathbb{N}$ such that $b^m - b^{m+1}a, a^k - a^{k+1}b \in J(R)$.

Throughout the paper, all rings are associative with an identity and all Banach algebras of bounded linear operators are complex. We use $J(R)$ and $N(R)$ to denote the Jacobson radical of R and the set of all nilpotent elements in R, respectively. R^{gD} and R^{pD} denote the sets of all elements having generalized Drazin inverses and pesudo Drazin inverses in R, respectively. $\mathbb{N}$ stands for the set of all natural numbers.

2. Polar-like Characterizations

As is well known, generalized Drazin inverses in a ring can be characterized by quasipolar property. The aim of this section is to characterize such generalized inverses in terms of tripotents in a ring. We begin with

Lemma 2.1. *Let R be a ring, and let $a \in R, p^3 = p \in comm^2(a)$. If $ap \in R^{qnil}$, then $ap^2 \in R^{qnil}$.*

Proof. Let $x \in comm(ap^2)$. Then $(pxp)a = px(ap^2)p = p(ap^2)xp = a(pxp)$, and so $pxp \in comm(a)$. As $p \in comm^2(a)$, we have $pxp^2 = p^2xp$. Since $(ap)(p^2xp) = a(pxp) = (pxp)a = (pxp^2)(ap)$, we see that $pxp^2 \in comm(ap)$. By hypothesis, $ap \in R^{qnil}$, and so $1 - (ap)(pxp^2) \in U(R)$. In light of Jacobson's Lemma, $1 - p^2(ap^2)x \in U(R)$. That is, $1 - (ap^2)x \in U(R)$. Therefore $ap^2 \in R^{qnil}$, as asserted. □

Theorem 2.2. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R^{gD}$.*
(2) *There exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$.*

Proof. (1) ⇒ (2) Obviously, we have $a^d \in comm^2(a), a^d = a^d a a^d, a - a^2a^d \in R^{qnil}$. Set $p = 1 - aa^d$. As in the proof of [6, Lemma 2.4], one easily checks that $p = p^2 \in comm^2(a)$ and $a + p \in U(R)$ and $ap \in R^{qnil}$, and so $p = p^3$, as desired.

(2) ⇒ (1) By hypothesis, there exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$. Set $b = (1 - p^2)(a + p)^{-1}$. Then $b \in comm^2(a)$ and

$$\begin{aligned} b^2a - b &= -(1 - p^2)(a + p)^{-2}p \\ &= (p^3 - p)(a + p)^{-2} \\ &= 0; \end{aligned}$$

hence, $b^2a = b$. Further,

$$\begin{aligned} a - a^2b &= a - a^2(1 - p^2)(a + p)^{-1} \\ &= a(a + p)^{-1}((a + p) - a(1 - p^2)) \\ &= a(a + p)^{-1}(ap^2 + p) \\ &= a(a + p)^{-1}(a + p)p^2 \\ &= ap^2. \end{aligned}$$

In light of Lemma 2.1, $a - a^2b \in R^{qnil}$. Therefore $a \in R$ has generalized Drazin inverse. □

Corollary 2.3. *Let R be a ring, let $a \in R$ and let $x \in comm(a) \cap U(R)$. Then the following are equivalent:*

(1) *$a \in R^{gD}$.*
(2) *There exists $p = p^3 \in comm^2(a)$ such that $a + xp \in U(R)$ and $ap \in R^{qnil}$.*

Proof. (1) ⇒ (2) This is obvious by [7, Proposition 4.7].

(2) ⇒ (1) Since $ap \in R^{qnil}$, we see that $1 + ap \in U(R)$. It is easy to check that

$$\begin{aligned} a + p &= (a + p)p^2 + (a + xp)(1 - p^2) \\ &= (1 + ap)p + (a + xp)(1 - p^2). \end{aligned}$$

Hence,

$$(a + p)^{-1} = (1 + ap)^{-1}p + (a + xp)^{-1}(1 - p^2).$$

This completes the proof. □

As an immediate consequence of Corollary 2.3, we prove that $a \in R^{gD}$ if and only if there exists $p = p^3 \in comm^2(a)$ such that $a + p$ or $a - p$ is invertible and $ap \in R^{qnil}$.

We now turn to consider generalized Drazin inverses in a Banach algebra of bounded linear operators. The following lemma is crucial.

Lemma 2.4. *Let A be a Banach algebra, $a, b \in A$ and $ab = ba$.*

(1) *If $a, b \in A^{qnil}$, then $a + b \in A^{qnil}$.*
(2) *If a or $b \in A^{qnil}$, then $ab \in A^{qnil}$.*

Proof. In a Banach algebra A, the preceding definition of quasinilpotent coincides with the usual definition of $\lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0$, which is equivalent to $\lambda \cdot 1_A - a \in U(A)$ for all complex $\lambda \neq 0$ (see [6]). Then we complete the proof by [5, Theorem 7.4.3]. □

Theorem 2.5. *Let A be a Banach algebra, and let $a \in A$. Then the following are equivalent:*

(1) $a \in A^{gD}$.
(2) *There exists $e^3 = e \in comm^2(a)$ such that $a + e \in U(A)$ and $\lim\limits_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = 0$.*
(3) *There exist idempotents $e, f \in comm^2(a)$ such that*

$$a - e + f \in U(A), \lim_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(af)^n\|^{\frac{1}{n}} = 0.$$

Proof. (1) ⇔ (2) This is obvious by Theorem 2.2 and [5, Page 251].

(1) ⇒ (3) By hypothesis, there exists an idempotents $f \in comm^2(a)$ such that $a + f \in U(A)$ and $af \in A^{qnil}$. Choose $e = 0$, thus proving (2).

(3) ⇒ (1) Let $a \in R$ and suppose that there exist idempotents $e, f \in comm^2(a)$ such that

$$a - e + f \in U(R), \lim_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(af)^n\|^{\frac{1}{n}} = 0.$$

Set $g = e - f$. Since $e \in comm^2(a)$, we see that $ea = ae$. As $f \in comm^2(a)$, we get $ef = fe$. It follows from $ae, af \in A^{qnil}$ that $a(e - f) = ae - af \in A^{qnil}$ by Lemma 2.4. One easily checks that $(e - f)^3 = e - f$. Therefore we complete by Theorem 2.2. □

Corollary 2.6. *Let A be a Banach algebra, and let $a \in A$. Then the following are equivalent:*

(1) $a \in A^{gD}$.
(2) *There exist two orthogonal idempotents $e \in comm(a), f \in comm^2(a)$ such that*

$$a - e + f \in U(A), \lim_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(af)^n\|^{\frac{1}{n}} = 0.$$

Proof. (1) ⇒ (2) Let $a \in A$. By Theorem 2.5 it follows that there exist idempotents $g, h \in comm^2(a)$ such that $a - g + h \in U(A), ag, ah \in A^{qnil}$. Clearly, $gh = hg$. Let $e = g(1-h)$ and $f = h(1-g)$. Then $e, f \in A$ are orthogonal idempotents. Obviously, $a - e + f = a - g + h \in U(A)$. By using Lemma 2.4, $ae = (ag)(1-h) \in A^{qnil}$. In view of [5, Page 251], we see that $\lim\limits_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = 0$. Likewise, $\lim\limits_{n\to\infty} \|(af)^n\|^{\frac{1}{n}} = 0$, as desired.

(2) ⇒ (1) Let $a \in A$. Then there exist orthogonal idempotents $e \in comm(a), f \in comm^2(a)$ such that

$$u := a - e + f \in U(A), \lim_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(af)^n\|^{\frac{1}{n}} = 0.$$

Let $g = f - e$. Then $g = g^3 \in comm(a)$. In view of Lemma 2.4, we see that $ag \in A^{qnil}$, and so $a^2g^2 \in A^{qnil}$. Moreover, $v := a^2 + g^2 = u^2 - 2ag = u^2(1 - 2u^{-2}ag) \in U(A)$. It follows from $ga = ag$ that $(1-g^2)a^2 = v(1-g^2)$. Hence $1 - g^2 = ba^2 = a^2b$ where $b = v^{-1}(1-g^2)$. Let $h = g^2$ and $x \in comm(a)$. Then

$$xh - hxh = (1-h)xh = (1-h)^n xh = b^n a^{2n} xh = b^n x(a^2h)^n$$

for all $n \in \mathbb{N}$. Hence

$$\|xh - hxh\|^{\frac{1}{n}} \leq \|b\|\|x\|^{\frac{1}{n}}\|(a^2h)^n\|^{\frac{1}{n}}.$$

Since $a^2h \in A^{qnil}$, we see that

$$\lim_{n\to\infty} \|(a^2h)^n\|^{\frac{1}{n}} = 0,$$

and so

$$\lim_{n\to\infty} \|xh - hxh\|^{\frac{1}{n}} = 0.$$

Therefore $\|xh - hxh\| = 0$ giving $xh = hxh$ and similarly $hxh = hx$. Hence $xh = hx$, and so $h \in comm^2(a)$. Then $f + e = (f-e)^2 = g^2 \in comm^2(a)$. If $ya = ay$, then $yf = fy$ and $(f+e)y = y(f+e)$. It follows that $ye = ey$; hence, $e \in comm^2(a)$. Therefore we complete the proof by Theorem 2.5. □

3. p-Drazin Inverse

The goal of this section is to characterize p-Drazin inverse in a ring by means of tripotents and we thereby obtain new characterizations of such generalized inverse. We now derive

Theorem 3.1. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{pD}$.
(2) *There exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^kp \in J(R)$ for some $k \in \mathbb{N}$.*

Proof. (1) ⇒ (2) follows by [8, Theorem 3.2].

(2) ⇒ (1) By hypothesis, there exists $p^3 = p \in comm^2(a)$ such that $a + p \in U(R)$ and $a^kp \in J(R)$ for some $k \in \mathbb{N}$. Set $b = (1-p^2)(a+p)^{-1}$. As in the proof of Theorem 2.2, we see that $b \in comm^2(a), b^2a = b$ and $a - a^2b = ap^2$. Moreover, we check that

$$\begin{aligned} a^k - a^{k+1}b &= a^k(1-ab) \\ &= a^k(1-ab)^k \\ &= (a-a^2b)^k \\ &= (a^kp)p \in J(R). \end{aligned}$$

Therefore $a \in R$ has pseudo Drazin inverse. □

A ring R is a pseudopolar ring if every element in R is pseudopolar (see [8]). We now record the following.

Corollary 3.2. *A ring R is a pseudopolar ring if and only if for any $a \in R$ there exists $p = p^3 \in comm^2(a)$ such that $a + p \in U(R)$ and $a^kp \in J(R)$ for some $k \in \mathbb{N}$.*

Proof. This is obvious by Theorem 3.1. □

Recall that a ring R is polar (or strongly π-regular) if every element in R has Drazin inverse, i.e., for any $a \in R$, there exists $b \in comm^2(a)$ such that $b = b^2a$ and $a - a^2b \in N(R)$. For instances, every finite ring and every algebraic algebra over a field are polar.

Corollary 3.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is polar.*
(2) *For any $a \in R$ there exists $e^3 = e \in comm^2(a)$ such that $a + e \in U(R)$ and $ae \in N(R)$.*

Proof. (1) ⇒ (2) This is clear, by [8, Theorem 2.1].

(2) ⇒ (1) In view of Theorem 3.1, R is pseudopolar. Let $a \in R^{qnil}$. Then there exists $e^3 = e \in comm^2(a)$ such that $u := a + e \in U(R)$ and $ae \in N(R)$. Hence, $e = u - a = u(1 - u^{-1}a) \in U(R)$. This implies that $e^2 = 1$. It follows from $ae \in N(R)$ that $ae^2 \in N(R)$, and so $a \in N(R)$. Thus $R^{qnil} \subseteq N(R) \subseteq R^{qnil}$, i.e., $R^{qnil} = N(R)$. Therefore R is polar, by [8, Theorem 2.1]. □

Theorem 3.4. *Let A be a Banach algebra, and let $a \in A$. Then the following are equivalent:*

(1) *$a \in A^{pD}$.*
(2) *There exist idempotents $e, f \in comm^2(a)$ such that*

$$a - e + f \in U(A), a^k e, a^k f \in J(A)$$

for some $k \in \mathbb{N}$.

Proof. (1) ⇒ (2) This is obvious, by [8, Theorem 3.2].

(2) ⇒ (1) Let $a \in A$. Then there exist idempotents $e, f \in comm^2(a)$ such that

$$a - e + f \in U(A), a^k e, a^k f \in J(A)$$

for some $k \in \mathbb{N}$. Set $g = f - e$. Since $e \in comm^2(a)$, we see that $ea = ae$. It follows from $f \in comm^2(a)$ that $ef = fe$. Moreover, $a^k(f - e) = a^k f - a^k e \in J(A)$. One easily checks that $(f - e)^3 = f - e$. Therefore we complete by Theorem 3.1. □

As in the proof of Corollary 2.6, by using Corollary 3.4, we derive

Corollary 3.5. *Let A be a Banach algebra, and let $a \in A$. Then the following are equivalent:*

(1) *$a \in A^{pD}$.*
(2) *There exist two orthogonal idempotents $e \in comm(a), f \in comm^2(a)$ such that*

$$a - e + f \in U(A), a^k e, a^k f \in J(A)$$

for some $k \in \mathbb{N}$.

We are now ready to prove:

Theorem 3.6. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R^{pD}$.*
(2) *There exists $b \in comm^2(a)$ and $m, k \in \mathbb{N}$ such that*

$$b^m = b^{m+1}a, a^k - a^{k+1}b \in J(R).$$

Proof. $\Longrightarrow$ This is obvious by choosing $m = 1$.

$\Longleftarrow$ Let $a \in A$. Then there exists $b \in comm^2(a)$ and $m, k \in \mathbb{N}$ such that

$$b^m = b^{m+1}a, a^k - a^{k+1}b \in J(R).$$

Then

$$\begin{aligned}(ba - b^2a^2)^m &= (b - b^2a)^m a^m\\ &= (b - b^2a)^{m-1}(b - b^2a)a^m\\ &= (1 - ba)^{m-1}b^{m-1}(b - b^2a)a^m\\ &= (1 - ba)^{m-1}(b^m - b^{m+1}a)a^m\\ &= 0.\end{aligned}$$

Set $p = ba$. Then $p^m(1 - p)^m = 0$. It is easy to verify that

$$\begin{aligned}1 &= (p + (1 - p))^{2m}\\ &= \sum_{i=0}^{m}\binom{2m}{i}p^{2m-i}(1 - p)^i + \sum_{i=m+1}^{2m}\binom{2m}{i}p^{2m-i}(1 - p)^i.\end{aligned}$$

Take $e = \sum_{i=0}^{m}\binom{2m}{i}p^{2m-i}(1 - p)^i$ and $f = 1 - e$. Then $e + f = 1$ and $ef = fe = 0$. Thus $e \in comm^2(p)$ and $e - e^2 = ef = 0$. As $p \in comm^2(a)$, we have $e^2 = e \in comm^2(a)$, and so $f^2 = f \in comm^2(a)$. Since $a^k(1 - p) = a^k - a^{k+1}b \in J(R)$, we have

$$a^k f = \Big(\sum_{i=m+1}^{2m}\binom{2m}{i}p^{2m-i}(1 - p)^{i-1}\Big)(a^k(1 - p)) \in J(R).$$

Clearly, $(a+1-ab)(b+1-ab) = 1+(1-a)(b-ab^2)+(a-a^2b)$. We easily check that $(b-ab^2)^m = (1-ba)^{m-1}(b^{m-1}(b-b^2a)) = 0$; hence, $1 + (1 - a)(b - ab^2) \in U(R)$. Since $a^k - a^{k+1}b \in J(R)$, we have $a^{k-1}(a - a^2b) \in J(R)$, and then $(a - a^2b)^{2k+1} = a^{k-1}(a - a^2b)(1 - ab)^{k-1}(a - a^2b)^{k+1} \in J(R)$. Let $x = 1 + (1 - a)(b - ab^2)$. Then $\left(x^{-1}(a - a^2b)\right)^{2k+1} = x^{-(2k+1)}(a - a^2b)^{2k+1} \in J(R)$. It follows that $1 + \left(x^{-1}(a - a^2b)\right)^{2k+1} \in U(R)$, so $1 + x^{-1}(a - a^2b) \in U(R)$. Therefore

$$1 + (1 - a)(b - ab^2) + (a - a^2b) = x\Big(1 + x^{-1}(a - a^2b)\Big) \in U(R);$$

hence, $(a + 1 - ab)(b + 1 - ab) \in U(R)$. This implies that $a + 1 - p \in U(R)$. On the other hand,

$$\begin{aligned}p - e &= p - \sum_{i=0}^{m}\binom{2m}{i}p^{2m-i}(1 - p)^i\\ &= \sum_{i=0}^{2m-2}p^i(p - p^2) - \sum_{i=1}^{m}\binom{2m}{i}p^{2m-i}(1 - p)^i\\ &= z(p - p^2)\end{aligned}$$

for some $z \in comm^2(p)$. Since $(p-p^2)^m = 0$, we have $(p-e)^{2m+1} = 0$. Therefore $(a+1-p)^{2m+1}+(p-e)^{2m+1} \in U(R)$, and so $a + f = (a + 1 - p) + (p - e) \in U(R)$. Accordingly, a has pseudo Drazin inverse. □

Corollary 3.7. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{pD}$.
(2) *There exists $b \in comm^2(a)$ and $k \in \mathbb{N}$ such that*

$$b^k = b^{k+1}a, a^k - a^{k+1}b \in J(R).$$

Proof. This is obvious by Theorem 3.6. □

As an immediate consequence of Corollary 3.7, we have

Corollary 3.8. *A ring R is a pseudopolar ring if and only if for any $a \in R$ there exists $b \in comm^2(a)$ and $k \in \mathbb{N}$ such that*

$$b^k = b^{k+1}a, a^k - a^{k+1}b \in J(R).$$

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