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# A Certain Class of Surfaces on Product Time Scales with Interpretations from Economics

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**Abstract.** In this study, we consider a graph surface associated to Cobb-Douglas production function in economics on product time scales. We classify this surface based on the flatness and minimality properties for several product time scales. Then, we interpret the obtained results from the perspective of production theory in economics. Therefore, we extend the known results in Euclidean geometry by considering time scale calculus.

## 1. Introduction

Time scale calculus was first considered by Stefan Hilger [26] in 1988 in his doctoral dissertation under supervision of Bernard Aulbach [8], to unify the two approaches of dynamic modelling: difference and differential equations. However, similar ideas have been used before and go back at least to the introduction of the Rieamann-Stieltjes integral which unifies sums and integrals. More specifically, $\mathbb{T}$ is an arbitrary, non-empty, closed subset of $\mathbb{R}$. Many results related to differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be totally different in nature. Because of these reasons, the theory of dynamic equations is an active area of research. The time scale calculus can be applied to any fields in which dynamic processes are described by discrete or continuous time models. So, it has various applications involving non-continuous domains like modeling of certain bug populations, chemical reactions, phytoremediation of metals, wound healing, maximization problems in economics and traffic problems. In recent years, several authors have obtained many important results about different topics on time scales (see [1], [12], [13], [21], [23], [24], [27], [28], [30], [41]).

Although there are many applications about time scale theory in many areas, the implementation of the time scale calculus in geometry is quite new. In 2004, Bohner and Guseinov [10] studied partial differentiation on time scales by explaining geometric sense of complete differentiability in case of single variable functions on time scales. They used the results of Guseinov and Özyilmaz [25] about tangent lines of generalized regular curves parameterized by time scales. These studies constitute the infrastructure of application of time scale in differential geometry. After these studies, time scale calculus has attracted by the researchers in differential geometry. In next years, some authors obtained noteworthy results. For instance, Ciésliński [18] gave geometric definition of pseudospherical surfaces on time scales in 2007. Kusak

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and Caliskan [31] made an application of vector field and derivative mapping on time scale in 2008. They obtained some important results about the delta nature connection on time scales in 2011 [32]. Aktan and his coworkers [2] considered directional $\nabla$-derivative and curves on $n$-dimensional time scale in 2009. Bohner and Guseinov [11] studied surface areas and surface integrals on time scales in 2010. Atmaca [6] examined normal and osculating planes of delta regular curves on time scales in 2010. And, she [7] presented a theoretical framework for surfaces parameterized by the product of two arbitrary time scales in 2013 with Akgüller. Then, Samanci and Caliskan [35] studied level curves and surfaces by considering delta gradient functions on time scales in 2015. In 2016, Samanci [34] considered the delta shape operator of a surface parameterized by the product of two arbitrary time scales. All of these studies are related to generalizing some fundamental definitions and theorems in differential geometry by using time scale theory. In this respect, our study is completely different. The first aim of our study is to obtain some results about curvature properties of a certain surface on some product time scales and second to provide a link between differential geometry and economics. In classical sense, this relation was established by Vîlcu in 2011 [36]. Vîlcu obtained a relation between some basic concepts in the theory of production function and the differential geometry of hypersurfaces in Euclidean space. He proved that a Cobb-Douglas production function has constant return to scale iff the corresponding hypersurface is flat, namely the Gauss-Kronocker curvature vanishes identically. Some authors generalized these results to arbitrary homogeneous production functions (See [3], [9], [15], [16], [17], [37], [38], [39], [40]). The first source related to economics on time scales is given by Atici and his coworkers [5]. As much as we know, there is not any study which explains the relation between differential geometry and economics on time scales. Therefore, the present study will be the first attempt in this field. Here, we construct a differential geometrical theory of production models in economics on product time scales.

This study is arranged as follows: In section 2, we remind some basic and important definitions about multivariable functions on time scales. Then, we studied the local structure of the surfaces on product time scales in section 3. We consider certain graph surfaces on time scale in section 4. Eventually, we appreciate the obtained results from the economics perspective in section 5.

## 2. Preliminaries on Product Time Scales

Since the production function that we consider is of two variables, we need to remind basic definitions and theorems of differential calculus for multivariable functions on product time scales.

Let $n \in \mathbb{N}$ and each $\mathbb{T}_i$ be a time scale, $i = \overline{1,n}$. Then, the set

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n = \{(t_1, t_2, ..., t_n) : t_i \in \mathbb{T}_i \text{ where } i = \overline{1,n}\},$$

is called $n-$dimensional time scale or product time scale. This set is a complete Euclidean metric space with the metric $d$ defined by [10]

$$d(t,s) = \left( \sum_{i=1}^{n} |t_i - s_i|^2 \right)^{\frac{1}{2}},$$

for $t, s \in \Lambda^n$. Here, we introduce and investigate some basic concepts and partial derivatives on product time scale for the function $f : \Lambda^n \to \mathbb{R}$.

The forward jump operator $\sigma_i : \mathbb{T}_i \to \mathbb{T}_i$ and the backward jump operator $\rho_i : \mathbb{T}_i \to \mathbb{T}_i$ are defined by

$$\sigma_i(u) = \inf\{v \in \mathbb{T}_i : v > u\},$$

and

$$\rho_i(u) = \sup\{v \in \mathbb{T}_i : v < u\},$$

for $u \in \mathbb{T}_i$, respectively. Here, one puts $\sigma_i(\max \mathbb{T}_i) = \max \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite maximum, and $\rho_i(\min \mathbb{T}_i) = \min \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite minimum. If $\sigma_i(u) > u$, then one can say that $u$ is right-scattered in $\mathbb{T}_i$, while any $u$ with $\rho_i(u) < u$ is called left-scattered in $\mathbb{T}_i$. Furthermore, if $u < \max \mathbb{T}_i$ and $\sigma_i(u) = u$, $u$ is called right dense in $\mathbb{T}_i$, and if $u > \min \mathbb{T}_i$ and $\rho_i(u) = u$, $u$ is called left dense in $\mathbb{T}_i$. If $\mathbb{T}_i$ has a left scattered maximum $M$, one can define $\mathbb{T}_i^\kappa = \mathbb{T}_i \backslash \{M\}$, otherwise $\mathbb{T}_i^\kappa = \mathbb{T}_i$. If $\mathbb{T}_i$ has a right-scattered minimum $m$, one can define $(\mathbb{T}_i)_\kappa = \mathbb{T}_i \backslash \{m\}$, otherwise $(\mathbb{T}_i)_\kappa = \mathbb{T}_i$ [10].

Let us consider the function $f : \Lambda^n \to \mathbb{R}$. The partial delta derivative of $f$ with respect to $t_i \in \mathbb{T}_i^\kappa$ is defined by the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, ..., t_{i-1}, \sigma_i(t_i), t_{i+1}, ..., t_n) - f(t_1, ..., t_{i-1}, s_i, t_{i+1}, ..., t_n)}{\sigma_i(t_i) - s_i},$$

providing that this limit exists as a finite number, and is indicated by any of the below symbols:

$$\frac{\partial f(t_1, t_2, ..., t_n)}{\Delta_i t_i}, \frac{\partial f(t)}{\Delta_i t_i}, f_{t_i}^{\Delta_i}(t).$$

If $f$ has partial delta derivatives $\dfrac{\partial f(t)}{\Delta_1 t_1}, \dfrac{\partial f(t)}{\Delta_2 t_2}, ..., \dfrac{\partial f(t)}{\Delta_n t_n}$, then one can also consider their partial delta derivatives which are called second order partial delta derivatives. These derivatives can be expressed by

$$\frac{\partial^2 f(t)}{\Delta_i t_i^2}, \frac{\partial^2 f(t)}{\Delta_j t_j \Delta_i t_i},$$

with respect to $t_i$ and $t_j$, respectively [10].

We will also need the following version of Chain rule for multivariable functions on time scales and its proof can be found in [13]. To get an extension to two-variable functions on time scales we start with a time scale $\mathbb{T}$. Denote its forward jump operator by $\sigma$ and its delta differentiation operator by $\Delta$. Let, further, two functions

$$\varphi, \psi : \mathbb{T} \longrightarrow \mathbb{R},$$

be given. Let us set

$$\varphi(\mathbb{T}) = \mathbb{T}_1 \text{ and } \psi(\mathbb{T}) = \mathbb{T}_2.$$

Denote by $\sigma_1, \Delta_1$ and $\sigma_2, \Delta_2$ the forward jump operators and delta operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Take a point $\xi_0 \in \mathbb{T}^\kappa$ and put

$$t_0 = \varphi(\xi_0) \text{ and } s_0 = \psi(\xi_0).$$

Assume that

$$\varphi(\sigma(\xi_0)) = \sigma_1(\varphi(\xi_0)) \text{ and } \psi(\sigma(\xi_0)) = \sigma_2(\varphi(\xi_0)).$$

**Theorem 2.1.** [10] *Let $f : \Lambda^2 \longrightarrow \mathbb{R}$ be $\sigma_1$−completely delta differentiable function at $(t_0, s_0)$. If the functions $\varphi$ and $\psi$ have delta derivatives at $\xi_0$, then the composite function*

$$F(\xi) = f(\varphi(\xi), \psi(\xi)), \xi \in \mathbb{T} \tag{2.1}$$

*has a delta derivative at that point which is expressed by*

$$F^\Delta(\xi_0) = \frac{\partial f(t_0, s_0)}{\Delta_1 t}\varphi^\Delta(\xi_0) + \frac{\partial f(\sigma_1(t_0), s_0)}{\Delta_2 s}\psi^\Delta(\xi_0).$$

**Theorem 2.2.** [10] *Let $f : \Lambda^2 \longrightarrow \mathbb{R}$ be $\sigma_2-$completely delta differentiable function at $(t_0, s_0)$. If the functions $\varphi$ and $\psi$ have delta derivatives at $\xi_0$, then the composite function $F$ defined by (2.1) has the delta derivative $F^{\Delta}(\xi_0)$ which is expressed by*

$$F^{\Delta}(\xi_0) = \frac{\partial f(t_0, \sigma_2(s_0))}{\Delta_1 t}\varphi^{\Delta}(\xi_0) + \frac{\partial f(t_0, s_0)}{\Delta_2 s}\psi^{\Delta}(\xi_0).$$

Concepts of $\sigma_1$ and $\sigma_2$ completely delta differentiable functions can be found in [10]. The next theorem gives us a sufficient condition for the independence of mixed delta derivatives of the order of differentiation.

**Theorem 2.3.** [10] *Let $f : \Lambda^2 \longrightarrow \mathbb{R}$ have the mixed partial delta derivatives $\dfrac{\partial^2 f(t,s)}{\Delta_1 t \Delta_2 s}$ and $\dfrac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}$ in some neighbourhood of the point $(t_0, s_0) \in \mathbb{T}_1^{\kappa} \times \mathbb{T}_2^{\kappa}$. If these derivatives are continuous at $(t_0, s_0)$, then*

$$\frac{\partial^2 f(t_0, s_0)}{\Delta_1 t \Delta_2 s} = \frac{\partial^2 f(t_0, s_0)}{\Delta_2 s \Delta_1 t}.$$

The above theorems have considerable importance to the advanced parts of the study.

### 3. Local Structure of Surfaces on Product Time Scales

Before giving the basic results, we need to remind some fundamental concepts related to surfaces on time scales.

**Definition 3.1.** [7] Let $S$ be a closed subset of $\mathbb{R}^3$. $S$ is a surface on time scale if for each point $P$ in $S$, there is a neighbourhood $A$ of $P$ and a function $\varphi : U \to S$, where $U$ is a closed set in $\mathbb{R}^2$ and an open set in time scale topology which satisfies following conditions:
i) $\varphi$ is a $\Delta-$differentiable function for each $(t, s) \in U$

$$\frac{\partial \varphi(t,s)}{\Delta_1 t} \times \frac{\partial \varphi(t,s)}{\Delta_2 s} \neq 0,$$

that is $\varphi$ is $\Delta-$regular.
ii) $\varphi(U) = S \cap A$ and $\varphi : U \to \varphi(U)$ is a homeomorphism.

Along with that Atmaca introduced metric properties of surfaces on time scales. Then, Samanci and Caliskan [35] considered the level curves and surfaces on time scales. Furthermore, Samanci [34] defined matrix presentation of delta shape operator on time scales.

Let $z \in C^{\infty}(\Lambda^2)$ be a real valued function where $C^{\infty}(\Lambda^2)$ denotes the space of all continuous functions which are completely delta differentiable. Then, its graph surface immersed in the Euclidean space $\mathbb{R}^3$ is locally defined by the mapping

$$r : \Lambda^2 \longrightarrow \mathbb{R}^3, \ (x, y) \longmapsto r(x, y) = (x, y, z(x, y)).$$

Such a surface is so called *Monge surface* (see [22]). Note that $r$ is $\Delta-$regular, i.e. $\dfrac{\partial r}{\Delta_1 x} \times \dfrac{\partial r}{\Delta_2 y} \neq 0$ everywhere on $\mathbb{R}^3$, where $\times$ denotes the cross product. Hence, the metric induced from $\mathbb{R}^3$ has the form [7]

$$ds^2 = I = \left[1 + \left(\frac{\partial z}{\Delta_1 x}\right)^2\right]dx^2 + 2\left(\frac{\partial z}{\Delta_1 x}\frac{\partial z}{\Delta_2 y}\right)dxdy + \left[1 + \left(\frac{\partial z}{\Delta_2 y}\right)^2\right]dy^2.$$

The normal vector field is defined by [7]

$$N = \frac{\frac{\partial r}{\Delta_1 x} \times \frac{\partial r}{\Delta_2 y}}{\left\| \frac{\partial r}{\Delta_1 x} \times \frac{\partial r}{\Delta_2 y} \right\|} = \frac{\left(-\frac{\partial z}{\Delta_1 x}, -\frac{\partial z}{\Delta_2 y}, 1\right)}{\sqrt{1 + \left(\frac{\partial z}{\Delta_1 x}\right)^2 + \left(\frac{\partial z}{\Delta_2 y}\right)^2}}.$$

Due to the Theorem 6.1 of [10], the following relation holds:

$$\frac{\partial^2 z}{\Delta_1 x \Delta_2 y} = \frac{\partial^2 z}{\Delta_2 y \Delta_1 x},$$

and therefore we have

$$\left\langle \frac{\partial^2 r}{\Delta_1 x \Delta_2 y}, N \right\rangle = \left\langle \frac{\partial^2 r}{\Delta_2 y \Delta_1 x}, N \right\rangle = -\left\langle \frac{\partial r}{\Delta_1 x}, \frac{\partial N}{\Delta_2 y} \right\rangle = -\left\langle \frac{\partial r}{\Delta_2 y}, \frac{\partial N}{\Delta_1 x} \right\rangle.$$

This provides us a symmetric second fundamental form, i.e. [7]

$$II = \left(\frac{\partial^2 z^{\sigma_1}}{\Delta_1 x^2}\right) dx^2 + 2\left(\frac{\partial^2 z}{\Delta_1 x \Delta_2 y}\right) dxdy + \left(\frac{\partial^2 z^{\sigma_2}}{\Delta_2 y^2}\right) dy^2.$$

As known, the Gaussian and the mean curvatures are defined by

$$K = \frac{\det II}{\det I}, \quad 2H = II \cdot I^{-1},$$

respectively where "$\cdot$" denotes the matrix multiplication and $I^{-1}$ is the inverse of $I$. Accordingly, we obtain the counterparts on the product time scale for equations of zero Gaussian and mean curvature types

$$\frac{\partial^2 z^{\sigma_1}}{\Delta_1 x^2} \frac{\partial^2 z^{\sigma_2}}{\Delta_2 y^2} - \left(\frac{\partial^2 z}{\Delta_1 x \Delta_2 y}\right)^2 = 0, \tag{3.1}$$

and

$$\left[1 + \left(\frac{\partial z}{\Delta_1 x}\right)^2\right] \frac{\partial^2 z^{\sigma_2}}{\Delta_2 y^2} - 2 \frac{\partial z}{\Delta_1 x} \frac{\partial z}{\Delta_2 y} \frac{\partial^2 z}{\Delta_1 x \Delta_2 y} + \left[1 + \left(\frac{\partial z}{\Delta_2 y}\right)^2\right] \frac{\partial^2 z^{\sigma_1}}{\Delta_1 x^2} = 0, \tag{3.2}$$

where $z$ is a $\sigma_1-$ and $\sigma_2-$completely delta differentiable function. It is well-known that the solutions of (3.1) and (3.2) represent *flat* and *minimal* graph surfaces, respectively. In classical sense, the equation (3.1) is also known as homogenous *Monge-Ampère* equation (see [20]).

## 4. Certain Graph Surfaces on Some Product Time Scales

We purpose to observe the graph surfaces of the following function

$$z : \Lambda^2 \longrightarrow \mathbb{R},\ z(x, y) = Ax^{\alpha} y^{\beta},\ A, \alpha, \beta \in \mathbb{R},\ A \neq 0. \tag{4.1}$$

Since the roles of $\alpha$ and $\beta$ are symmetric, we shall only discuss the situations based on $\alpha$ throughout this section. Thereby we look for the solutions of the equations (3.1) and (3.2) on product time scales $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z} \times \mathbb{R}$ and $h\mathbb{Z} \times h\mathbb{Z}$, respectively.

**1.** $\Lambda^2 = q^{\mathbb{N}} \times q^{\mathbb{N}}$, $q > 1$.

**1.1.** The equation (3.1) associated with (4.1) turns to

$$\left(q^{\alpha}-1\right)\left(q^{\beta}-1\right)\left[\left(q^{\alpha-1}-1\right)\left(q^{\beta-1}-1\right)-\left(q^{\alpha}-1\right)\left(q^{\beta}-1\right)\right]=0, \tag{4.2}$$

which immediately implies that $\alpha=0$. Otherwise, i.e. $\alpha\beta\neq 0$, (4.2) reduces to

$$\left(q^{\alpha-1}-1\right)\left(q^{\beta-1}-1\right)-\left(q^{\alpha}-1\right)\left(q^{\beta}-1\right)=0 \tag{4.3}$$

or

$$1+q^{-1}=\frac{q^{\alpha}+q^{\beta}}{q^{\alpha+\beta}}.$$

The solution of (4.3) with respect to $\alpha$ yields that $\alpha=0$, which is not our case. Therefore the graph surface of (4.1) has locally the parameterization

$$r(x,y)=\left(x,y,y^{\beta}\right)=x(1,0,0)+\left(0,y,y^{\beta}\right),$$

namely it is a generalized cylinder over the curve $\left(0,y,y^{\beta}\right)$ (see [22], pp. 439).

**1.2.** By a direct calculation, the equation (3.2) associated with (4.1) becomes

$$\begin{aligned}&\left(q^{\alpha-1}-1\right)\left(q^{\alpha}-1\right)(q-1)^{2}y^{2}+\left(q^{\beta-1}-1\right)\left(q^{\beta}-1\right)(q-1)^{2}x^{2}\\&-A^{2}\left(\tfrac{q-1}{q}\right)\left(q^{\alpha}-1\right)\left(q^{\beta}-1\right)\left[q^{\alpha}\left(q^{\beta}-1\right)+q^{\beta}\left(q^{\alpha}-1\right)\right]x^{2\alpha}y^{2\beta}=0.\end{aligned} \tag{4.4}$$

The partial delta derivative of (4.4) with respect to $x$ and $y$ gives

$$\left(q^{2\alpha}-1\right)\left(q^{2\beta}-1\right)\left(q^{\alpha}-1\right)\left(q^{\beta}-1\right)\left[q^{\alpha}\left(q^{\beta}-1\right)+q^{\beta}\left(q^{\alpha}-1\right)\right]=0. \tag{4.5}$$

The situation vanishing $\alpha$ is a solution for (4.5). Considering it into (4.4) gives $\beta=1$. Hence, the graph surface of (4.1) is a plane. Otherwise, i.e. $\alpha\neq 0$, we conclude $2=2^{-\alpha}+2^{-\beta}$, which leads to a contradiction.

**2.** $\Lambda^{2}=\mathbb{Z}\times\mathbb{R}$.

**2.1.** The equation (3.1) associated with (4.1) turns to

$$\beta(\beta-1)x^{\alpha}\left[(x+2)^{\alpha}-2(x+1)^{\alpha}+x^{\alpha}\right]-\beta^{2}\left[(x+1)^{\alpha}-x^{\alpha}\right]^{2}=0. \tag{4.6}$$

For $x=0$ in (4.6), we have $\beta=0$ and thus $\alpha\neq 0$ due to the symmetry. This implies that the graph of (4.1) to be a generalized cylinder.

**2.2.** By a straightforward computation, the equation (3.2) associated with (4.1) becomes

$$\begin{aligned}&\left[1+A^{2}\beta^{2}x^{2\alpha}y^{2\beta-2}\right]\left[(x+2)^{\alpha}-2(x+1)^{\alpha}+x^{\alpha}\right]\\&-2A^{2}\beta^{2}x^{\alpha}y^{2\beta-2}\left[(x+1)^{\alpha}-x^{\alpha}\right]^{2}+\\&+\beta(\beta-1)\left[y^{-2}x^{\alpha}+A^{2}y^{2\beta-2}x^{\alpha}\left\{(x+1)^{\alpha}-x^{\alpha}\right\}^{2}\right]=0.\end{aligned} \tag{4.7}$$

Taking partial delta derivative of (4.7) on $y$ and dividing it by $y^{2\beta-3}$ yields

$$\begin{aligned}&2A^{2}(\beta-1)\beta^{2}x^{2\alpha}\left[(x+2)^{\alpha}-2(x+1)^{\alpha}+x^{\alpha}\right]\\&-4A^{2}(\beta-1)\beta^{2}x^{\alpha}\left[(x+1)^{\alpha}-x^{\alpha}\right]+\\&+2\beta(\beta-1)\left[-2y^{-2\beta}x^{\alpha}+2(\beta-1)A^{2}x^{\alpha}\left\{(x+1)^{\alpha}-x^{\alpha}\right\}^{2}\right]^{2}=0.\end{aligned} \tag{4.8}$$

After again taking partial delta derivative of (4.8), we deduce $\beta(\beta-1)=0$. If $\beta=0$, then (4.7) yields $\alpha=1$. If $\beta=1$, by symmetry we have $\alpha\neq 0$ and thus (4.7) reduces to

$$\begin{aligned}&\left[1+A^{2}x^{2\alpha}\right]\left[(x+2)^{\alpha}-2(x+1)^{\alpha}+x^{\alpha}\right]\\&-2A^{2}x^{\alpha}\left[(x+1)^{\alpha}-x^{\alpha}\right]^{2}=0.\end{aligned} \tag{4.9}$$

The twice partial delta derivative of (4.9) yields a contradiction due to $\alpha\neq 0$.

**3.** $\Lambda^2 = h\mathbb{Z} \times h\mathbb{Z}, h > 0.$

**3.1.** The equation (3.1) associated with (4.1) turns to

$$\begin{aligned} & x^{\alpha} y^{\beta}\left[(x+2h)^{\alpha}-2(x+h)^{\alpha}+x^{\alpha}\right]\left[(y+2h)^{\beta}-2(y+h)^{\beta}+y^{\beta}\right] \\ & -\left[(x+h)^{\alpha}-x^{\alpha}\right]^{2}\left[(y+h)^{\beta}-y^{\beta}\right]^{2}=0. \end{aligned} \tag{4.10}$$

For $(x, y) = (0, 0)$ in (4.10), we get the contradiction $h = 0$.

**3.2.** By a calculation, the equation (3.2) associated with (4.1) becomes

$$\begin{aligned} & x^{\alpha}\left\{1+\tfrac{A^{2} y^{2\beta}}{h^{2}}\left[(x+h)^{\alpha}-x^{\alpha}\right]^{2}\right\}\left[(y+2h)^{\beta}-2(y+h)^{\beta}+y^{\beta}\right] \\ & -2\tfrac{A^{2} x^{\alpha} y^{\beta}}{h^{2}}\left[(x+h)^{\alpha}-x^{\alpha}\right]^{2}\left[(y+h)^{\beta}-y^{\beta}\right]^{2} \\ & +y^{\beta}\left\{1+\tfrac{A^{2} x^{2\alpha}}{h^{2}}\left[(y+h)^{\beta}-y^{\beta}\right]^{2}\right\}\left[(x+2h)^{\alpha}-2(x+h)^{\alpha}+x^{\alpha}\right]=0. \end{aligned} \tag{4.11}$$

For $x = 0$, we have $\alpha = 1$. Substituting it into (4.11) yields

$$x\left\{1+A^{2} y^{2\beta}\right\}\left[(y+2h)^{\beta}-2(y+h)^{\beta}+y^{\beta}\right]-2A^{2} x y^{\beta}\left[(y+h)^{\beta}-y^{\beta}\right]=0. \tag{4.12}$$

Also, for $y = 0$ in (4.12), we obtain $\beta = 1$.

Thus, in summary, we have proved the below theorems with the above quite confused and long computations:

**Theorem 4.1.** *The function given by (4.1) is a solution of the equation of zero Gaussian curvature type on the product time scales, $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z} \times \mathbb{R}$ and $h\mathbb{Z} \times h\mathbb{Z}$ if and only if its graph is a generalized cylinder.*

**Theorem 4.2.** *The function given by (4.1) is a solution of the equation of zero mean curvature type on the product time scales, $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z} \times \mathbb{R}$ and $h\mathbb{Z} \times h\mathbb{Z}$ if and only if its graph is a plane.*

**Example 4.3.** *Let us consider the function $z(x, y) = y^2$ where $A = 1, \alpha = 0$ and $\beta = 2$. Its graph is a parabolic cylinder. It is easy to say that (4.2), (4.6) and (4.10) are satisfied for this surface on the product time scales $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z} \times \mathbb{R}$ and $h\mathbb{Z} \times h\mathbb{Z}$, respectively. So, the given surface is flat.*

## 5. A Look at the Results in Terms of Economics

We shall interpret the obtained results in previous section by using the notion of production function which is a key concept in economics (For details see [3], [4], [14], [19], [33]). A *production function* is a mathematical model which indicates the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. It is usually assumed to be a twice differentiable mapping given by

$$\begin{cases} z : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+, \quad z = z(x_1, ..., x_n), \\ \mathbb{R}_+^n = \{(x_1, ..., x_n) : x_i > 0, i = 1, ..., n\}, \end{cases}$$

where $z$ is the quantity of output, $n$ is the number of inputs and $x_1, x_2, ..., x_n$ are the inputs (e.g. labor, capital, land, raw materials etc.). Recall that the production function both is strictly increasing and has decreasing efficiency with respect to any factor of production, namely $\frac{\partial f}{\partial x_i} > 0$ and $\frac{\partial^2 f}{\partial x_i^2} < 0$, for $i = 1, ..., n$, respectively. Another important property is that the production function is to be homogeneous. Explicitly, one is said to be *homogeneous of degree d* if

$$z(tx_1, tx_2, ..., tx_n) = t^d z(x_1, x_2, ..., x_n), \tag{5.1}$$

holds for each $t \in \mathbb{R}_+$. In particular case $d = 1$, it is said to be *linearly homogeneous.* The function exhibits increasing return to scale (resp. decreasing return to scale) if $d > 1$ (resp. $d < 1$). If $d = 1$, it exhibits constant return to scale (cf. [15]).

Among homogeneous production functions, most famous one is the *Cobb-Douglas production function* introduced by Cobb and Douglas in 1928 [19] in order to describe the distrubition of the national income of the United States via production functions. In general form, it is expressed as

$$f(x) = A \prod_{j=1}^{n} x_j^{\alpha_j}, x = (x_1, x_2, ..., x_n),$$

where $A, \alpha_1, ..., \alpha_n \in \mathbb{R}_+$. Note that the homogeneity degree is $\sum_{i=1}^{n} \alpha_i$.

A production function can be identified by the graph hypersurface of $\mathbb{R}^{n+1}$, namely

$$r(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, z(x_1, x_2, ..., x_n)),$$

which is called *production hypersurface.* Many geometric classifications regarding to homogeneous production hypersurfaces were presented in [9], [16], [17], [29]. In particular a Cobb-Douglas surface is the graph surface of the function $z(x, y) = Ax^{\alpha}y^{\beta}$, $A, \alpha, \beta \in \mathbb{R}_+$. In $\mathbb{R}^{n+1}$, the following results were provided:

**Theorem 5.1.** [40] *There does not exist a minimal generalized Cobb-Douglas production hypersurface in $\mathbb{R}^{n+1}$.*

**Theorem 5.2.** [36] *A Cobb-Douglas function has constant return to scale if and only if its graph hypersurface is flat.*

The Cobb-Douglas surfaces expressed via Theorem 4.1. and Theorem 4.2. can be reinterpreted on product time scales by the following result:

**Theorem 5.3.** *The graph surface of a Cobb-Douglas function whose inputs come from the product time scales, $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z}^+ \times \mathbb{R}^+$ and $h\mathbb{Z}^+ \times h\mathbb{Z}^+$ can be neither minimal nor flat.*

**Remark 5.4.** *Since the negative inputs are no sense in economics, we have considered $\mathbb{Z}^+$ and $\mathbb{R}^+$ instead of $\mathbb{Z}$ and $\mathbb{R}$ in Theorem 5.3.*

**Conclusion 5.5.** *Unlike Euclidean spaces, there is not a flat graph surface of a Cobb-Douglas production function on product time scales, $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z} \times \mathbb{R}$ and $h\mathbb{Z} \times h\mathbb{Z}$, respectively.*

**Example 5.6.** *Let us consider the function $z(x, y) = xy^2$ where $A = 1, \alpha = 1$ and $\beta = 2$. We can calculate the Gaussian and the mean curvatures for this surface on the product time scales $q^{\mathbb{N}} \times q^{\mathbb{N}}$, $\mathbb{Z} \times \mathbb{R}$ and $h\mathbb{Z} \times h\mathbb{Z}$, respectively.*

Let $\Lambda^2 = q^{\mathbb{N}} \times q^{\mathbb{N}}$. Since

$$\frac{\partial z}{\Delta_1 x} = y^2, \quad \frac{\partial z}{\Delta_2 y} = xy(q+1), \quad \frac{\partial^2 z^{\sigma_1}}{\Delta_1 x^2} = 0, \quad \frac{\partial^2 z^{\sigma_2}}{\Delta_1 y^2} = x(q+1), \quad \frac{\partial^2 z}{\Delta_1 x \Delta_1 y} = y(q+1),$$

we get

$$K_{q^{\mathbb{N}}\times q^{\mathbb{N}}} = \frac{-y^2(q+1)^2}{1 + y^4 + x^2y^2(q+1)^2}, \quad H_{q^{\mathbb{N}}\times q^{\mathbb{N}}} = \frac{x\left(1 + y^4\right)(q+1) - 2xy^4(q+1)^2}{2\left(1 + y^4 + x^2y^2(q+1)^2\right)} \; .$$

Similarly, if $\Lambda^2 = \mathbb{Z} \times \mathbb{R}$ and $\Lambda^2 = h\mathbb{Z} \times h\mathbb{Z}$, we get

$$K_{\mathbb{Z}\times\mathbb{R}} = \frac{-4y^2}{1 + 4x^2y^2 + y^4}, \quad H_{\mathbb{Z}\times\mathbb{R}} = \frac{x(1 - 3y^4)}{1 + 4x^2y^2 + y^4} \; .$$

and

$$K_{h\mathbb{Z}\times h\mathbb{Z}} = \frac{-(2y+h)^2}{1+x^2(2y+h)^2+y^4}, \quad H_{h\mathbb{Z}\times h\mathbb{Z}} = \frac{x(1+y^4)-xy^2(2y+h)^2}{1+x^2(2y+h)^2+y^4},$$

respectively. Since all curvatures depend on $x$ and $y$, it denotes that the given surface is not flat on $q^{\mathbb{N}}\times q^{\mathbb{N}}$, $\mathbb{Z}\times\mathbb{R}$ and $h\mathbb{Z}\times h\mathbb{Z}$, respectively. Here, $K_{q^{\mathbb{N}}\times q^{\mathbb{N}}}, K_{\mathbb{Z}\times\mathbb{R}}, K_{h\mathbb{Z}\times h\mathbb{Z}}, H_{q^{\mathbb{N}}\times q^{\mathbb{N}}}, H_{\mathbb{Z}\times\mathbb{R}}, H_{h\mathbb{Z}\times h\mathbb{Z}}$ denote the Gaussian and the mean curvatures for the given surface on $q^{\mathbb{N}}\times q^{\mathbb{N}}$, $\mathbb{Z}\times\mathbb{R}$ and $h\mathbb{Z}\times h\mathbb{Z}$.

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