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Biconjugate Residual Algorithm for Solving General Sylvester-transpose Matrix Equations

Masoud Hajarian[a]

[a]*Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, General Campus, Evin, Tehran 19839, Iran*

Abstract. The present paper is concerned with the solution of the coupled generalized Sylvester-transpose matrix equations

$$\begin{cases} A_1XB_1 + C_1XD_1 + E_1X^TF_1 = M_1, \\ A_2XB_2 + C_2XD_2 + E_2X^TF_2 = M_2, \end{cases}$$

including the well-known Lyapunov and Sylvester matrix equations. Based on a variant of biconjugate residual (BCR) algorithm, we construct and analyze an efficient algorithm to find the (least Frobenius norm) solution of the general Sylvester-transpose matrix equations within a finite number of iterations in the absence of round-off errors. Two numerical examples are given to examine the performance of the constructed algorithm.

1. Introduction

Linear matrix equations have variety of applications in applied mathematics and engineering [3–5, 25, 26, 29]. For instance, the Sylvester-transpose matrix equation

$$AX + X^TB = C, \tag{1.1}$$

has close relations with many problems in system theory, such as eigenstructure assignment, observer design, control of system with input constraint, and fault detection [13, 18–20, 32]. The generalized Sylvester matrix equation

$$AXB + CXD = E, \tag{1.2}$$

appears in the model reduction and stability, reachability, observability and controllability analysis of discrete-time and continuous-time linear systems [36, 38–40]. During the last two decades, the linear matrix equations have drawn much attention due to their wide applications [3, 14, 27, 28, 35, 42]. Li and Wang introduced the weighted steepest descent algorithms to solve the general linear matrix equation including the Lyapunov and Sylvester matrix equations [31]. In [37], the explicit expression of least squares Hermitian solution with the least norm of (1.2) was presented. By applying Galerkin projection with an extended Krylov subspace method, Shank et al. proposed an iterative method for the low-rank

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Email address: m_hajarian@sbu.ac.ir, mhajarian@aut.ac.ir, masoudhajarian@gmail.com (Masoud Hajarian)

approximate solution of the generalized Lyapunov equations [1]. In [2], Dopico et al. introduced several projection algorithms based on different Krylov spaces to find the low-rank approximations to the solution of the Sylvester-transpose matrix equation. The development of iterative methods proposed for $Ax = b$ has gained interest to compute numerical solutions of linear matrix equations in recent years [12, 21, 33, 45–47]. By developing the conjugate gradient normal equation residual (CGNR), the conjugate gradient normal equation error (CGNE) and the least-squares QR-factorization (LSQR) algorithms, various efficient methods were constructed for solving diverse linear matrix equations [12, 23, 24, 30]. In [15–17], by applying the hierarchical identification principle, the gradient based iterative algorithms were proposed for solving several linear matrix equations.

Our main result in this paper is to generalize a variant of BCR algorithm to compute the solution $X \in \mathrm{R}^{m\times n}$ of the coupled generalized Sylvester-transpose matrix equations

$$\begin{cases} A_1XB_1 + C_1XD_1 + E_1X^TF_1 = M_1, \\ A_2XB_2 + C_2XD_2 + E_2X^TF_2 = M_2, \end{cases} \tag{1.3}$$

where $A_i, C_i \in \mathrm{R}^{p_i\times m}$, $B_i, D_i \in \mathrm{R}^{n\times q_i}$, $E_i \in \mathrm{R}^{p_i\times n}$, $F_i \in \mathrm{R}^{m\times q_i}$ and $M_i \in \mathrm{R}^{p_i\times q_i}$ for $i = 1, 2$.

The coupled generalized Sylvester-transpose matrix equations (1.3) contain several matrix equations as special cases such as Lyapunov, Sylvester and Sylvester-transpose matrix equations.

The rest of this article is organized as follows. In Section 2, after a brief presentation of BCR algorithm, we obtain the generalization of the algorithm to solve the coupled generalized Sylvester-transpose matrix equations (1.3). In addition, we discuss the convergence properties of the generalized BCR algorithm. We provide two numerical evaluations of the obtained algorithm in Section 3. In Section 4, conclusions are given.

Some notations used in the sequel are presented as follows:

The symbols $\mathrm{tr}(A)$, A^T, $\mathcal{R}(A)$ and $A^\dagger$ are used to denote the trace, the transpose, the column space and pseudo inverse of A, respectively. The inner product of $A \in \mathrm{R}^{m\times n}$ and $B \in \mathrm{R}^{m\times n}$ is defined by $\langle A, B\rangle = \mathrm{tr}(B^TA)$. The associated norm is the well-known Frobenius norm denoted by $\|.\|$. For a matrix $A \in \mathrm{R}^{m\times n}$, the so-called stretching function $\mathrm{vec}(A)$ is defined by $\mathrm{vec}(A) = (a_1^T, a_2^T, ..., a_n^T)^T$, where a_k is the k-th column of A. The notation $A \otimes B$ stands for the Kronecker product of matrices A and B.

2. Description of the BCR Algorithm and its Convergence

In this section, we first briefly describe the BCR algorithm to solve $Ax = b$. Then we develop the new modified BCR algorithm to calculate the solution of the coupled generalized Sylvester-transpose matrix equations (1.3). Various iterative methods have been introduced for the solution of the nonsymmetric linear system $Ax = b$; see, e.g., [6–10, 43, 44] and references therein. Vespucci and Broyden in [34] presented the different computational variations of BCR algorithm without any convergence analysis. One of those variations is summarized as follows:

BCR algorithm

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Initial values: $x(1)$ and $s(1)$ arbitrary, $r(1) = Ax(1) - b$,
$u(1) = s(1)$, $v(1) = r(1)$, $w(1) = Au(1)$, and $z(1) = A^Tv(1)$.
Recursions:
$\alpha(k) = \frac{w(k)^Tr(k)}{w(k)^Tw(k)}$, $x(k+1) = x(k) - \alpha(k)u(k)$,
$r(k+1) = r(k) - \alpha(k)w(k)$,
$\beta(k) = \frac{z(k)^Ts(k)}{z(k)^Tz(k)}$, $s(k+1) = s(k) - \beta(k)z(k)$,
$\gamma(k) = \frac{w(k)^TAs(k+1)}{w(k)^Tw(k)}$, $u(k+1) = s(k+1) - \gamma(k)u(k)$,
$\eta(k) = \frac{z(k)^TA^Tr(k+1)}{z(k)^Tz(k)}$, $v(k+1) = r(k+1) - \eta(k)v(k)$,
$w(k+1) = Au(k+1)$,
$z(k+1) = A^Tv(k+1)$.
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In order to develop the above algorithm for solving (1.3), first we transform (1.3) into a linear system. By utilizing the stretching function and the Kronecker product, the coupled generalized Sylvester-transpose matrix equations (1.3) can be equivalently transformed into the following linear system

$$\underbrace{\begin{pmatrix} B_1^T \otimes A_1 + D_1^T \otimes C_1 + (F_1^T \otimes E_1)P \\ B_2^T \otimes A_2 + D_2^T \otimes C_2 + (F_2^T \otimes E_2)P \end{pmatrix}}_{A} \underbrace{\text{vec}(X)}_{x} = \underbrace{\begin{pmatrix} \text{vec}(M_1) \\ \text{vec}(M_2) \end{pmatrix}}_{b}, \tag{2.1}$$

where $P \in \mathrm{R}^{mn\times mn}$ is an orthogonal matrix. For more details about the form of the matrix P see [41]. Substituting the above system in the BCR algorithm leads to

$$\begin{aligned} r(1) = Ax(1) - b &= \begin{pmatrix} B_1^T \otimes A_1 + D_1^T \otimes C_1 + (F_1^T \otimes E_1)P \\ B_2^T \otimes A_2 + D_2^T \otimes C_2 + (F_2^T \otimes E_2)P \end{pmatrix} x(1) - \begin{pmatrix} \text{vec}(M_1) \\ \text{vec}(M_2) \end{pmatrix}, \\ w(k+1) = Au(k+1) &= \begin{pmatrix} B_1^T \otimes A_1 + D_1^T \otimes C_1 + (F_1^T \otimes E_1)P \\ B_2^T \otimes A_2 + D_2^T \otimes C_2 + (F_2^T \otimes E_2)P \end{pmatrix} u(k+1), \\ z(k+1) = A^T v(k+1) &= \begin{pmatrix} B_1^T \otimes A_1 + D_1^T \otimes C_1 + (F_1^T \otimes E_1)P \\ B_2^T \otimes A_2 + D_2^T \otimes C_2 + (F_2^T \otimes E_2)P \end{pmatrix}^T v(k+1) \\ &= \begin{pmatrix} B_1 \otimes A_1^T + D_1 \otimes C_1^T + P^T(F_1 \otimes E_1^T) & B_2 \otimes A_2^T + D_2 \otimes C_2^T + P^T(F_2 \otimes E_2^T) \end{pmatrix} v(k+1). \end{aligned} \tag{2.2}$$

According to (2.2), let us define

$$x(k) = \text{vec}(X(k)), \quad s(k) = \text{vec}(S(k)), \quad z(k) = \text{vec}(Z(k)), \quad u(k) = \text{vec}(U(k)), \tag{2.3}$$

$$v(k) = \begin{pmatrix} \text{vec}(V_1(k)) \\ \text{vec}(V_2(k)) \end{pmatrix}, \quad r(k) = \begin{pmatrix} \text{vec}(R_1(k)) \\ \text{vec}(R_2(k)) \end{pmatrix}, \quad w(k) = \begin{pmatrix} \text{vec}(W_1(k)) \\ \text{vec}(W_2(k)) \end{pmatrix}, \tag{2.4}$$

where $X(k), S(k), Z(k), U(k) \in \mathrm{R}^{m\times n}$ and $R_i(k), V_i(k), W_i(k) \in \mathrm{R}^{p_i\times q_i}$ for $i = 1, 2$. Substituting (2.3) and (2.4) into (2.2) gives us

$$\begin{gathered} \text{vec}(R_i(1)) = \text{vec}(A_iX(1)B_i + C_iX(1)D_i + E_iX(1)^TF_i - M_i), \quad i = 1, 2, \\ \text{vec}(W_i(k+1)) = \text{vec}(A_iU(k+1)B_i + C_iU(k+1)D_i + E_iU(k+1)^TF_i), \quad i = 1, 2, \\ \text{vec}(Z(k+1)) = \text{vec}(A_1^TV_1(k+1)B_1^T + C_1^TV_1(k+1)D_1^T + F_1V_1(k+1)^TE_1 \\ +A_2^TV_2(k+1)B_2^T + C_2^TV_2(k+1)D_2^T + F_2V_2(k+1)^TE_2). \end{gathered}$$

Now by considering the above discussion, we establish the matrix form of BCR algorithm for solving (1.3) as follows.

Algorithm 1. (Matrix form of BCR algorithm to solve (1.3))

Step 1. *Given the initial matrix $X(1) \in R^{m\times n}$ and nonzero arbitrary matrix $S(1) \in R^{m\times n}$. Input the tolerance $\tau > 0$;*

Step 2. *Compute*

$$\begin{aligned} &R_i(1) = A_iX(1)B_i + C_iX(1)D_i + E_iX(1)^TF_i - M_i, \quad i = 1, 2, \\ &U(1) = S(1),\ V_1(1) = R_1(1),\ V_2(1) = R_2(1), \\ &W_i(1) = A_iU(1)B_i + C_iU(1)D_i + E_iU(1)^TF_i, \quad i = 1, 2, \\ &Z(1) = A_1^TV_1(1)B_1^T + C_1^TV_1(1)D_1^T + F_1V_1(1)^TE_1 + A_2^TV_2(1)B_2^T + C_2^TV_2(1)D_2^T + F_2V_2(1)^TE_2; \end{aligned}$$

Set $k = 1$;

Step 3. *Exit if the stopping criterion $\sqrt{\|R_1(k)\|^2 + \|R_2(k)\|^2} \le \tau$ has been met;*

Step 4. *Compute*

$$\alpha(k) = \frac{\mathrm{tr}(W_1(k)^T R_1(k)) + \mathrm{tr}(W_2(k)^T R_2(k))}{\mathrm{tr}(W_1(k)^T W_1(k)) + \mathrm{tr}(W_2(k)^T W_2(k))},$$
$$X(k+1) = X(k) - \alpha(k)U(k),$$
$$R_i(k+1) = R_i(k) - \alpha(k)W_i(k), \ \ i = 1,2,$$
$$\beta(k) = \frac{\mathrm{tr}(Z(k)^T S(k))}{\mathrm{tr}(Z(k)^T Z(k))},$$
$$S(k+1) = S(k) - \beta(k)Z(k),$$
$$\gamma(k) = \frac{\sum_{i=1}^{2} \mathrm{tr}(W_i(k)^T (A_i S(k+1) B_i + C_i S(k+1) D_i + E_i S(k+1)^T F_i))}{\mathrm{tr}(W_1(k)^T W_1(k)) + \mathrm{tr}(W_2(k)^T W_2(k))},$$
$$U(k+1) = S(k+1) - \gamma(k)U(k),$$
$$\eta(k) = \frac{1}{\mathrm{tr}(Z(k)^T Z(k))}[\mathrm{tr}(Z(k)^T (A_1^T R_1(k+1) B_1^T + C_1^T R_1(k+1) D_1^T + F_1 R_1(k+1)^T E_1$$
$$+ A_2^T R_2(k+1) B_2^T + C_2^T R_2(k+1) D_2^T + F_2 R_2(k+1)^T E_2)],$$
$$V_i(k+1) = R_i(k+1) - \eta(k)V_i(k), \ \ i = 1,2,$$
$$W_i(k+1) = A_i U(k+1) B_i + C_i U(k+1) D_i + E_i U(k+1)^T F_i$$
$$= A_i S(k+1) B_i + C_i S(k+1) D_i + E_i S(k+1)^T F_i - \gamma(k) W_i(k), \ \ i = 1,2,$$
$$Z(k+1) = A_1^T V_1(k+1) B_1^T + C_1^T V_1(k+1) D_1^T + F_1 V_1(k+1)^T E_1 + A_2^T V_2(k+1) B_2^T + C_2^T V_2(k+1) D_2^T + F_2 V_2(k+1)^T E_2$$
$$= A_1^T R_1(k+1) B_1^T + C_1^T R_1(k+1) D_1^T + F_1 R_1(k+1)^T E_1 + A_2^T R_2(k+1) B_2^T + C_2^T R_2(k+1) D_2^T + F_2 R_2(k+1)^T E_2 - \eta(k) Z(k);$$

Step 5. *Set $k = k+1$ and go to Step 3.*

In order to provide a convergence theorem of Algorithm 1, we first present the following lemma which gives some key properties of the algorithm.

Lemma 1. *Suppose that there exists a positive integer number r such that $\alpha(k) \neq 0$, $\alpha(k) \neq \infty$ and $[\|R_1(k)\| + \|R_2(k)\|] \neq 0$ for all $k = 1,2,...,r$. Then*

$$\mathrm{tr}(R_1(v)^T W_1(u)) + \mathrm{tr}(R_2(v)^T W_2(u)) = 0, \quad for \ \ u,v = 1,2,...,r, \quad v > u, \tag{2.5}$$
$$\mathrm{tr}(S(v)^T Z(u)) = 0, \quad for \ \ u,v = 1,2,...,r, \quad v > u, \tag{2.6}$$
$$\mathrm{tr}(Z(v)^T Z(u)) = 0, \quad for \ \ u,v = 1,2,...,r, \quad u \neq v, \tag{2.7}$$
$$\mathrm{tr}(W_1(v)^T W_1(u)) + \mathrm{tr}(W_2(v)^T W_2(u)) = 0, \quad for \ \ u,v = 1,2,...,r, \quad u \neq v. \tag{2.8}$$

Proof. It is described in the Appendix. □

Using the above lemma, we will be able to obtain the convergence theorem of Algorithm 1 as follows.

Theorem 1. *Let the coupled generalized Sylvester-transpose matrix equations (1.3) be consistent. Algorithm 1 can compute the solution of (1.3) within a finite number of iterations in the absence of round-off errors.*

Proof. Let us first consider the space $\mathrm{R}^{p_1 \times q_1} \times \mathrm{R}^{p_2 \times q_2}$ with a inner product defined as follows

$$\langle (M_1, M_2), (N_1, N_2) \rangle = \mathrm{tr}(M_1^T N_1) + \mathrm{tr}(M_2^T N_2) \ \ M_1, N_1 \in \mathrm{R}^{p_1 \times q_1}, \quad M_2, N_2 \in \mathrm{R}^{p_2 \times q_2}.$$

If $W_1(k) \neq 0$ or $W_2(k) \neq 0$ for $k = 1,2,...,p_1q_1 + p_2q_2$ then from Lemma 1 we deduce that $(W_1(k), W_2(k))$, $k = 1,2,...,p_1q_1 + p_2q_2$ is an orthogonal basis of the inner product space $\mathrm{R}^{p_1 \times q_1} \times \mathrm{R}^{p_2 \times q_2}$. From this result and (2.5), we infer that Algorithm 1 can obtain the solution of (1.3) in a finite number of iterations in the absence of round-off errors. □

In what follows, we show that Algorithm 1 with the special initial matrix can obtain the least Frobenius norm solution of (1.3).

Theorem 2. *Let the coupled generalized Sylvester-transpose matrix equations (1.3) be consistent. If we choose the initial matrix*

$$X(1) = A_1^T M_1(1) B_1^T + C_1^T M_1(1) D_1^T + F_1 M_1(1)^T E_1 + A_2^T M_2(1) B_2^T + C_2^T M_2(1) D_2^T + F_2 M_2(1)^T E_2, \tag{2.9}$$

and matrix

$$S(1) = A_1^T N_1(1) B_1^T + C_1^T N_1(1) D_1^T + F_1 N_1(1)^T E_1 + A_2^T N_2(1) B_2^T + C_2^T N_2(1) D_2^T + F_2 N_2(1)^T E_2, \tag{2.10}$$

where $M_1(1), N_1(1) \in R^{p_1 \times q_1}$ and $M_2(1), N_2(1) \in R^{p_2 \times q_2}$ are arbitrary, then the solution X^* generated by Algorithm 1 is the least Frobenius norm solution of (1.3).

Proof. From (2.9) and (2.10) together with Algorithm 1 it follows that there exist the matrices $M_1(k), N_1(k) \in R^{p_1 \times q_1}$ and $M_2(k), N_2(k) \in R^{p_2 \times q_2}$ such that

$$X(k) = A_1^T M_1(k) B_1^T + C_1^T M_1(k) D_1^T + F_1 M_1(k)^T E_1 + A_2^T M_2(k) B_2^T + C_2^T M_2(k) D_2^T + F_2 M_2(k)^T E_2, \tag{2.11}$$

and

$$S(k) = A_1^T N_1(k) B_1^T + C_1^T N_1(k) D_1^T + F_1 N_1(k)^T E_1 + A_2^T N_2(k) B_2^T + C_2^T N_2(k) D_2^T + F_2 N_2(k)^T E_2. \tag{2.12}$$

By using (2.11) and (2.12), it is clear that

$$\begin{aligned}
\mathrm{vec}(X(k)) =& \mathrm{vec}(A_1^T M_1(k) B_1^T + C_1^T M_1(k) D_1^T + F_1 M_1(k)^T E_1 + A_2^T M_2(k) B_2^T + C_2^T M_2(k) D_2^T + F_2 M_2(k)^T E_2) \\
=& \begin{pmatrix} B_1 \otimes A_1^T + D_1 \otimes C_1^T + P^T(F_1 \otimes E_1^T) & B_2 \otimes A_2^T + D_2 \otimes C_2^T + P^T(F_2 \otimes E_2^T) \end{pmatrix} \begin{pmatrix} \mathrm{vec}(M_1(k)) \\ \mathrm{vec}(M_2(k)) \end{pmatrix} \\
=& \begin{pmatrix} B_1^T \otimes A_1 + D_1^T \otimes C_1 + (F_1^T \otimes E_1)P \\ B_2^T \otimes A_2 + D_2^T \otimes C_2 + (F_2^T \otimes E_2)P \end{pmatrix}^T \begin{pmatrix} \mathrm{vec}(M_1(k)) \\ \mathrm{vec}(M_2(k)) \end{pmatrix} \\
\in& \mathcal{R}\left(\begin{pmatrix} B_1^T \otimes A_1 + D_1^T \otimes C_1 + (F_1^T \otimes E_1)P \\ B_2^T \otimes A_2 + D_2^T \otimes C_2 + (F_2^T \otimes E_2)P \end{pmatrix}^T \right).
\end{aligned}$$

Therefore, it comes that the solution X^* generated by Algorithm 1 is the least Frobenius norm solution of (1.3). □

Remark 1. *It holds*

$$\begin{aligned}
\|R_1(k+1)\|^2 + \|R_2(k+1)\|^2 =& \mathrm{tr}((R_1(k) - \alpha(k)W_1(k))^T(R_1(k) - \alpha(k)W_1(k))) \\
&+ \mathrm{tr}((R_2(k) - \alpha(k)W_2(k))^T(R_2(k) - \alpha(k)W_2(k))) \\
=& \|R_1(k)\|^2 + \|R_2(k)\|^2 + \alpha(k)^2(\|W_1(k)\|^2 + \|W_2(k)\|^2) \\
&- 2\alpha(k)[\mathrm{tr}(W_1(k)^T R_1(k)) + \mathrm{tr}(W_2(k)^T R_2(k))] \\
=& \|R_1(k)\|^2 + \|R_2(k)\|^2 - \alpha(k)[\mathrm{tr}(W_1(k)^T R_1(k)) + \mathrm{tr}(W_2(k)^T R_2(k))] \\
=& \|R_1(k)\|^2 + \|R_2(k)\|^2 - \frac{[\mathrm{tr}(W_1(k)^T R_1(k)) + \mathrm{tr}(W_2(k)^T R_2(k))]^2}{\|W_1(k)\|^2 + \|W_2(k)\|^2} \\
\leq& \|R_1(k)\|^2 + \|R_2(k)\|^2.
\end{aligned}$$

This shows that Algorithm 1 guarantees the descent of the residual norms.

3. Numerical Experiments

This section presents two numerical examples to further illustrate the numerical effectiveness of BCR algorithm compared with the CGNR and CGNE methods. All computations are carried out in MATLAB with double machine precision.

Example 1. *As the first example, we consider the Sylvester-transpose matrix equation* $AXB + CX^TD = M$ *with the following parameters*

$$A = triu(rand(n,n),1) + diag(2 + diag(rand(n))), \quad B = triu(rand(n,n),1) + diag(2 + diag(rand(n))),$$

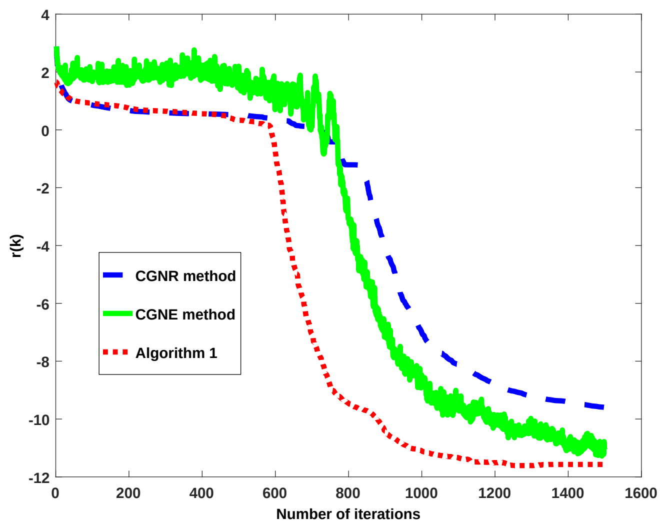
$$C = tril(rand(n,n),1) + diag(1.5 + diag(rand(n))), \quad D = triu(rand(n,n),1) + diag(1.5 + diag(rand(n))),$$

$$M = 10 * rand(n).$$

For $n = 15$, *we apply CGNR and CGNE methods and Algorithm 1 with the initial matrix* $X(0)$ *to solve this matrix equation. The obtained results are presented in Figure 1 where*

$$r(k) = \log_{10} \|M - AX(k)B - CX(k)^TD\|.$$

Figure 1: Numerical comparison of the testing methods for Example 1.



Example 2. *In this example, we study the generalized Sylvester-transpose matrix equation* $AXB+CXD+EX^TF = M$ *where*

$$A = \begin{pmatrix} 4 & 40 & 4 & 7 & 9 & 1 & 0 & 10 \\ 4 & 400 & -99 & -2 & -2 & 2 & 3 & 4 \\ -2 & -2 & 100 & 5 & 600 & -1 & -5 & 5 \\ 100 & 2 & -2 & -2 & 5 & 1 & 200 & 2 \\ -90 & -9 & 10 & 5 & 200 & 3 & 1 & 3 \\ 10 & -20 & -1 & 50 & 4 & 5 & 3 & 10 \\ 20 & 3 & 900 & 6 & 3 & 5 & 9 & 4 \\ 20 & 3 & 233 & 6 & 3 & 5 & 9 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & -22 & 3 & 7 & 110 & -1 & 6 & 10 \\ 40 & -5 & 1 & -12 & 5 & 5 & 6 & 4 \\ -2 & 5 & 1 & 10 & 6 & -2 & 12 & 2 \\ 10 & 5 & -5 & -2 & 5 & -3 & 25 & 12 \\ 1 & -800 & 2 & 2 & 3 & 5 & 7 & 44 \\ 20 & -10 & -100 & 5 & 3 & 2 & 11 & 77 \\ 30 & 6 & 200 & 4 & 2 & 200 & 8 & 77 \\ 300 & 6 & 2 & 4 & 2 & 200 & 8 & 7700 \end{pmatrix},$$

$$C = \begin{pmatrix} -32 & 168 & -4 & -14 & -422 & 6 & -24 & -20 \\ -152 & 820 & -202 & 44 & -24 & -16 & -18 & -8 \\ 4 & -24 & 196 & -30 & 1176 & 6 & -58 & 2 \\ 160 & -16 & 16 & 4 & -10 & 14 & 300 & -44 \\ -184 & 3182 & 12 & 2 & 388 & -14 & -26 & -170 \\ -60 & 0 & 398 & 80 & -4 & 2 & -38 & -288 \\ -80 & -18 & 1000 & -4 & -2 & -790 & -14 & -300 \\ -1160 & -18 & 458 & -4 & -2 & -790 & -14 & -30792 \end{pmatrix},$$

$$D = \begin{pmatrix} -40 & -156 & -26 & -49 & -265 & -3 & -12 & -70 \\ -100 & -1990 & 493 & 34 & 0 & -20 & -27 & -28 \\ 14 & 0 & -502 & -45 & -3012 & 9 & 1 & -29 \\ -520 & -20 & 20 & 14 & -35 & 1 & -1050 & -34 \\ 448 & 1645 & -54 & -29 & -1006 & -25 & -19 & -103 \\ -90 & 120 & 205 & -260 & -26 & -29 & -37 & -204 \\ -160 & -27 & -4900 & -38 & -19 & -425 & -61 & -174 \\ -700 & -27 & -1169 & -38 & -19 & -425 & -61 & -15420 \end{pmatrix},$$

$$E = \begin{pmatrix} -28 & 208 & 0 & -7 & -413 & 7 & -24 & -10 \\ -148 & 1220 & -301 & 42 & -26 & -14 & -15 & -4 \\ 2 & -26 & 296 & -25 & 1776 & 5 & -63 & 7 \\ 260 & -14 & 14 & 2 & -5 & 15 & 500 & -42 \\ -274 & 3173 & 22 & 7 & 588 & -11 & -25 & -167 \\ -50 & -20 & 397 & 130 & 0 & 7 & -35 & -278 \\ -60 & -15 & 1900 & 2 & 1 & -785 & -5 & -296 \\ -1140 & -15 & 691 & 2 & 1 & -785 & -5 & -30788 \end{pmatrix},$$

$$F = \begin{pmatrix} -30 & -178 & -23 & -42 & -155 & -4 & -6 & -60 \\ -60 & -1995 & 494 & 22 & 5 & -15 & -21 & -24 \\ 12 & 5 & -501 & -35 & -3006 & 7 & 13 & -27 \\ -510 & -15 & 15 & 12 & -30 & -2 & -1025 & -22 \\ 449 & 845 & -52 & -27 & -1003 & -20 & -12 & -59 \\ -70 & 110 & 105 & -255 & -23 & -27 & -26 & -127 \\ -130 & -21 & -4700 & -34 & -17 & -225 & -53 & -97 \\ -400 & -21 & -1167 & -34 & -17 & -225 & -53 & -7720 \end{pmatrix}, \quad M = rand(8).$$

This generalized Sylvester-transpose matrix equation can be transformed into the following linear system

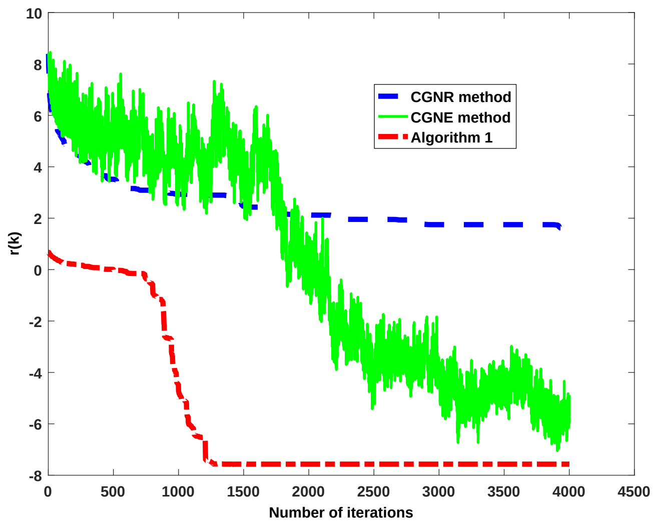
$$\underbrace{\left(B^T \otimes A + D^T \otimes C + (F^T \otimes E)P\right)}_{T} \underbrace{\mathrm{vec}(X)}_{x} = \underbrace{\mathrm{vec}(M)}_{d}.$$

Since the condition number $\kappa(T) = \|T\|\|T^{\dagger}\| = 1.8131 \times 10^6$, we see that the above system is ill-conditioned. By using the mentioned methods with $X(0)$, we obtain the sequence $\{X(k)\}$. In Figure 2, we depict the obtained results where

$$r(k) = \log_{10} \|M - AX(k)B - CX(k)D - EX(k)^T F\|.$$

The above both examples show that Algorithm 1 can solve the matrix equations efficiently.

Figure 2: Numerical comparison of the testing methods for Example 2.



4. Conclusions

In this paper, we have introduced the matrix form of BCR algorithm for solving the coupled generalized Sylvester-transpose matrix equations (1.3). We have shown that the introduced algorithm can find the least Frobenius norm solution of (1.3) within a finite number of iterations in the absence of round-off errors. It has been demonstrated by numerical examples that the introduced algorithm is better than some existing methods to solve linear matrix equation.

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Appendix

The proof of Lemma 1

We apply induction to prove (2.5)-(2.8). Since the inner product is commutative, we only need to demonstrate (2.5)-(2.8) for $1 \leq u < v \leq r$. When $v = 2$ and $u = 1$, we have

$$\mathrm{tr}(R_1(2)^T W_1(1)) + \mathrm{tr}(R_2(2)^T W_2(1))$$

$$= \mathrm{tr}(R_1(1)^T W_1(1)) + \mathrm{tr}(R_2(1)^T W_2(1)) - \mathrm{tr}(W_1(1)^T R_1(1)) - \mathrm{tr}(W_2(1)^T R_2(1)) = 0,$$

$$\mathrm{tr}(S(2)^T Z(1)) = \mathrm{tr}(S(1)^T Z(1)) - \mathrm{tr}(Z(1)^T S(1)) = 0,$$

$$\mathrm{tr}(Z(2)^T Z(1))$$

$$= \mathrm{tr}((A_1^T R_1(1) B_1^T + C_1^T R_1(1) D_1^T + F_1 R_1(1)^T E_1 + A_2^T R_2(1) B_2^T + C_2^T R_2(1) D_2^T + F_2 R_2(1)^T E_2 - \eta(1) Z(1))^T Z(1))$$

$$= \mathrm{tr}((A_1^T R_1(2) B_1^T + C_1^T R_1(2) D_1^T + F_1 R_1(2)^T E_1 + A_2^T R_2(2) B_2^T + C_2^T R_2(2) D_2^T + F_2 R_2(2)^T E_2)^T Z(1))$$

$$- \mathrm{tr}(Z(1)^T (A_1^T R_1(2) B_1^T + C_1^T R_1(2) D_1^T + F_1 R_1(2)^T E_1 + A_2^T R_2(2) B_2^T + C_2^T R_2(2) D_2^T + F_2 R_2(2)^T E_2)) = 0,$$

$$\mathrm{tr}(W_1(2)^T W_1(1)) + \mathrm{tr}(W_2(2)^T W_2(1))$$

$$= \mathrm{tr}((A_1S(2)B_1 + C_1S(2)D_1 + E_1S(2)^TF_1 - \gamma(1)W_1(1))^TW_1(1))$$
$$+\mathrm{tr}((A_2S(2)B_2 + C_2S(2)D_2 + E_2S(2)^TF_2 - \gamma(1)W_2(1))^TW_2(1))$$
$$= \mathrm{tr}((A_1S(2)B_1 + C_1S(2)D_1 + E_1S(2)^TF_1)^TW_1(1)) + \mathrm{tr}((A_2S(2)B_2 + C_2S(2)D_2 + E_2S(2)^TF_2)^TW_2(1))$$
$$-\sum_{i=1}^{2} \mathrm{tr}(W_i(1)^T(A_iS(2)B_i + C_iS(2)D_i + E_iS(2)^TF_i)) = 0.$$

These imply that (2.5)-(2.8) hold for $v = 2$ and $u = 1$. Now for $u < w < r$, we assume that

$$\mathrm{tr}(R_1(w)^TW_1(u)) + \mathrm{tr}(R_2(w)^TW_2(u)) = 0, \quad \mathrm{tr}(S(w)^TZ(u)) = 0, \tag{4.1}$$

$$\mathrm{tr}(Z(w)^TZ(u)) = 0, \ \ \mathrm{tr}(W_1(w)^TW_1(u)) + \mathrm{tr}(W_2(w)^TW_2(u)) = 0. \tag{4.2}$$

By making use of (4.1) and (4.2), we can obtain

$$\mathrm{tr}(R_1(w+1)^TW_1(u)) + \mathrm{tr}(R_2(w+1)^TW_2(u))$$
$$= \mathrm{tr}((R_1(w) - \alpha(w)W_1(w))^TW_1(u)) + \mathrm{tr}((R_2(w) - \alpha(w)W_2(w))^TW_2(u)) = 0,$$
$$\mathrm{tr}(S(w+1)^TZ(u)) = \mathrm{tr}((S(w) - \beta(w)Z(w))^TZ(u)) = 0,$$
$$\mathrm{tr}(Z(w+1)^TZ(u))$$
$$= \mathrm{tr}((A_1^TR_1(w+1)B_1^T + C_1^TR_1(w+1)D_1^T + F_1R_1(w+1)^TE_1 + A_2^TR_2(w+1)B_2^T$$
$$+C_2^TR_2(w+1)D_2^T + F_2R_2(w+1)^TE_2 - \eta(w)Z(w))^TZ(u))$$
$$= \frac{1}{\beta(u)}[\mathrm{tr}(R_1(w+1)^T(A_1(S(u) - S(u+1))B_1 + C_1(S(u) - S(u+1))D_1 + E_1(S(u) - S(u+1))^TF_1))$$
$$+\mathrm{tr}(R_2(w+1)^T(A_2(S(u) - S(u+1))B_2 + C_2(S(u) - S(u+1))D_2 + E_2(S(u) - S(u+1))^TF_2))]$$
$$= \frac{1}{\beta(u)}[\mathrm{tr}(R_1(w+1)^T(W_1(u) + \gamma(u-1)W_1(u-1))) - \mathrm{tr}(R_1(w+1)^T(W_1(u+1) + \gamma(u)W_1(u)))$$
$$+\mathrm{tr}(R_2(w+1)^T(W_2(u) + \gamma(u-1)W_2(u-1))) - \mathrm{tr}(R_2(w+1)^T(W_2(u+1) + \gamma(u)W_2(u)))]$$
$$= -\frac{1}{\beta(u)}[\mathrm{tr}(R_1(w+1)^TW_1(u+1)) + \mathrm{tr}(R_2(w+1)^TW_2(u+1))], \tag{4.3}$$

$$\mathrm{tr}(W_1(w+1)^TW_1(u)) + \mathrm{tr}(W_2(w+1)^TW_2(u))$$
$$= \mathrm{tr}((A_1S(w+1)B_1 + C_1S(w+1)D_1 + E_1S(w+1)^TF_1 - \gamma(w)W_1(w))^TW_1(u))$$
$$+\mathrm{tr}((A_2S(w+1)B_2 + C_2S(w+1)D_2 + E_2S(w+1)^TF_2 - \gamma(w)W_2(w))^TW_2(u))$$
$$= \mathrm{tr}(S(w+1)^T(A_1^TW_1(u)B_1^T + C_1^TW_1(u)D_1^T + F_1W_1(u)^TE_1 + A_2^TW_2(u)B_2^T + C_2^TW_2(u)D_2^T + F_2W_2(u)^TE_2))$$
$$= \frac{1}{\alpha(u)}[\mathrm{tr}(S(w+1)^T(A_1^T(R_1(u) - R_1(u+1))B_1^T + C_1^T(R_1(u) - R_1(u+1))D_1^T + F_1(R_1(u) - R_1(u+1))^TE_1$$
$$+A_2^T(R_2(u) - R_2(u+1))B_2^T + C_2^T(R_2(u) - R_2(u+1))D_2^T + F_2(R_2(u) - R_2(u+1))^TE_2))]$$
$$= \frac{1}{\alpha(u)}[\mathrm{tr}(S(w+1)^T(Z(u) + \eta(u-1)Z(u-1)) - Z(u+1) - \eta(u)Z(u)))]$$
$$= -\frac{1}{\alpha(u)}[\mathrm{tr}(S(w+1)^TZ(u+1))]. \tag{4.4}$$

Also for $u = w$, we can get

$$\mathrm{tr}(R_1(w+1)^TW_1(w)) + \mathrm{tr}(R_2(w+1)^TW_2(w)) = 0,$$

$$\begin{aligned}
&\mathrm{tr}(S(w+1)^T Z(w)) = 0,\\
&\mathrm{tr}(Z(w+1)^T Z(w)) = \mathrm{tr}((A_1^T R_1(w+1)B_1^T + C_1^T R_1(w+1)D_1^T + F_1 R_1(w+1)^T E_1\\
&+A_2^T R_2(w+1)B_2^T + C_2^T R_2(w+1)D_2^T + F_2 R_2(w+1)^T E_2 - \eta(w)Z(w))^T Z_2(w))\\
&= \mathrm{tr}((A_1^T R_1(w+1)B_1^T + C_1^T R_1(w+1)D_1^T + F_1 R_1(w+1)^T E_1\\
&+A_2^T R_2(w+1)B_2^T + C_2^T R_2(w+1)D_2^T + F_2 R_2(w+1)^T E_2)^T Z(w))\\
&-\mathrm{tr}(Z(w)^T(A_1^T R_1(w+1)B_1^T + C_1^T R_1(w+1)D_1^T + F_1 R_1(w+1)^T E_1\\
&+A_2^T R_2(w+1)B_2^T + C_2^T R_2(w+1)D_2^T + F_2 R_2(w+1)^T E_2) = 0,\\
&\mathrm{tr}(W_1(w+1)^T W_1(1)) + \mathrm{tr}(W_2(w+1)^T W_2(1))\\
&= \mathrm{tr}((A_1 S(w+1)B_1 + C_1 S(w+1)D_1 + E_1 S(w+1)^T F_1 - \gamma(w)W_1(w))^T W_1(w))\\
&+\mathrm{tr}((A_2 S(w+1)B_2 + C_2 S(w+1)D_2 + E_2 S(w+1)^T F_2 - \gamma(w)W_2(w))^T W_2(w))\\
&= \mathrm{tr}((A_1 S(w+1)B_1 + C_1 S(w+1)D_1 + E_1 S(w+1)^T F_1)^T W_1(w))\\
&+\mathrm{tr}((A_2 S(w+1)B_2 + C_2 S(w+1)D_2 + E_2 S(w+1)^T F_2)^T W_2(w))\\
&-\sum_{i=1}^{2} \mathrm{tr}(W_i(w)^T(A_i S(w+1)B_i + C_i S(w+1)D_i + E_i S(w+1)^T F_i)) = 0.
\end{aligned}$$

From

$$\mathrm{tr}(Z(w)^T Z(u)) = 0, \quad \mathrm{tr}(R_1(w+1)^T W_1(w)) + \mathrm{tr}(R_2(w+1)^T W_2(w)) = 0,$$

and (4.3), it could be concluded that

$$\mathrm{tr}(Z(w+1)^T Z(u)) = 0.$$

Also from

$$\mathrm{tr}(W_1(w)^T W_1(u)) + \mathrm{tr}(W_2(w)^T W_2(u)) = 0, \quad \mathrm{tr}(S(w+1)^T Z(w)) = 0,$$

and (4.4), one can easily verify

$$\mathrm{tr}(W_1(w+1)^T W_1(u)) + \mathrm{tr}(W_2(w+1)^T W_2(u)) = 0.$$

Hence Lemma 1 holds by the principle of induction.

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