Filomat 32:15 (2018), 5319–5328
https://doi.org/10.2298/FIL1815319T

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Majorization Problems for Certain Classes of Multivalent Analytic Functions Related with the Srivastava-Khairnar-More Operator and Exponential Function

Huo Tang[a], **Guantie Deng**[b]

[a]*School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China*
[b]*School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China*

Abstract. In the present paper, we investigate several majorizaton problems for certain classes $M^{\lambda,\delta}_{\mu,p}(a,b,c;\eta)$ and $N^{\lambda,\delta}_{\mu,p}(a,b,c;\gamma)$ of multivalent analytic functions related to exponential function, which are defined through the Srivastava-Khairnar-More operator $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a,b,c)$ given by (1.4). Meanwhile, some special cases of our main results in form of corollaries are given.

1. Introduction

Let $\mathbb{C}$ be complex plane and $\mathcal{A}_p$ denote the class of analytic and p-valent functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \cdots\})$$

in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For convenience, we write $\mathcal{A}_1 = \mathcal{A}$.

In 1967, Macgregor [10] introduced the notion of majorization as follows.

Definition 1.1. Let f and g be analytic in $\mathbb{U}$. We say that f is majorized by g in $\mathbb{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}),$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C80
Keywords. Multivalent analytic function, subordination, majorization problem, exponential function, Srivastava-Khairnar-More operator
Received: 04 February 2018; Accepted: 02 January 2019
Communicated by Hari M. Srivastava
Research supported by the Natural Science Foundation of the People's Republic of China under Grants 11561001 and 11271045, the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region under Grant NJYT-18-A14, the Natural Science Foundation of Inner Mongolia of the People's Republic of China under Grant 2018MS01026, and the Higher School Foundation of Inner Mongolia of the People's Republic of China under Grants NJZY17300 and NJZY17301.
Email addresses: thth2009@163.com (Huo Tang), denggt@bnu.edu.cn (Guantie Deng)

if there exists a function $\varphi(z)$, analytic in $\mathbb{U}$, satisfying

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \ \ (z \in \mathbb{U}). \tag{1.1}$$

Later, Roberston [17] (see also [19]) gave the concept of quasi-subordination as below.

Definition 1.2. For two analytic functions f and g in $\mathbb{U}$, we say f is quasi-subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec_q g(z) \ \ (z \in \mathbb{U}),$$

if there exists two analytic functions $\varphi(z)$ and $\omega(z)$ in $\mathbb{U}$, such that $\frac{f(z)}{\varphi(z)}$ is analytic in $\mathbb{U}$ and

$$|\varphi(z)| \leq 1, \ \ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| \leq |z| < 1 \ \ (z \in \mathbb{U}),$$

satisfying

$$f(z) = \varphi(z)g(\omega(z)) \ \ (z \in \mathbb{U}). \tag{1.2}$$

Remark 1.1.

(i) For $\varphi(z) \equiv 1$ in (1.2), we have

$$f(z) = g(\omega(z)) \ \ (z \in \mathbb{U})$$

and say f is subordinate to g in $\mathbb{U}$, denoted by (see [21]; also see [18, 22, 23, 29])

$$f(z) \prec g(z) \ \ (z \in \mathbb{U}).$$

(ii) For $\omega(z) = z$ in (1.2), the quasi-subordination (1.2) reduces to the majorization (1.1).

In 1991, Ma and Minda [9] introduced the following function class $S^*(\phi)$, which is defined by using the above subordination principle:

$$S^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \ \ (z \in \mathbb{U}) \right\},$$

where $\phi(z)$ is analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $Re(\phi(z)) > 0$ for $z \in \mathbb{U}$.

We observe that, for choosing the appropriate function $\phi(z)$, the class $S^*(\phi)$ reduces to one of the well-known classes of functions. For example,

(i) If we put

$$\phi(z) = \frac{1 + Az}{1 + Bz} \ \ (-1 \leq B < A \leq 1; \ z \in \mathbb{U}),$$

then we get the class

$$S^*(A, B) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \ \ (-1 \leq B < A \leq 1; \ z \in \mathbb{U}) \right\},$$

which was introduced by Janowski [7]. In particular, for $A = 1 - 2\alpha$ and $B = -1$, we have the class $S^*(1 - 2\alpha, -1) = S^*(\alpha)$ of starlike function of order α $(0 \leq \alpha < 1)$. Further, for $A = 1$ and $B = -1$, we have the familiar class $S^*(1, -1) = S^*$ of starlike function in $\mathbb{U}$.

(ii) If we set

$$\phi(z) = e^z \ \ (z \in \mathbb{U}),$$

then we obtain the class

$$S_e^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z \ \ (z \in \mathbb{U}) \right\},$$

which was introduced and investigate by Mendiratta et al. [11] and implies that

$$f \in S_e^* \Longleftrightarrow \left| \log \frac{zf'(z)}{f(z)} \right| < 1 \ \ (z \in \mathbb{U}). \tag{1.3}$$

For the functions $f_j \in \mathcal{A}_p$, given by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \ \ (j = 1, 2; \ z \in \mathbb{U}),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 * f_1)(z).$$

Recently, Tang et al. [24] introduced a family of linear operators $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a, b, c) : \mathcal{A}_p \to \mathcal{A}_p$, which is the generalization of the Srivastava-Khairnar-More operator [20] (see also [30]), defined by

$$\mathcal{I}_{\mu,p}^{\lambda,\delta}(a, b, c) f(z) = f_{\mu,p}^{\lambda,\delta}(a, b, c)(z) * f(z) \tag{1.4}$$

$$(a, b \in \mathbb{C}; \ c \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}; \ \lambda > -p; \ \mu, \delta \geq 0; \ z \in \mathbb{U}),$$

where $f_{\mu,p}^{\lambda,\delta}(a, b, c)(z)$ is the function defined in terms of the Hadamard product (or convolution):

$$f_{\mu,p}^{\delta}(a, b, c)(z) * f_{\mu,p}^{\lambda,\delta}(a, b, c)(z) = \frac{z^p}{(1 - z)^{\lambda+p}} \ \ (\lambda > -p, \ \mu, \delta \geq 0)$$

and the function $f_{\mu,p}^{\delta}(a, b, c)(z)$ is given by

$$f_{\mu,p}^{\delta}(a, b, c)(z) = (1 - \mu + \delta) z^p \cdot {}_2F_1(a, b; c; z) + (\mu - \delta) z [z^p \cdot {}_2F_1(a, b; c; z)]' + \mu\delta z^2 [z^p \cdot {}_2F_1(a, b; c; z)]''$$

with the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$, defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \ \ (a, b \in \mathbb{C}; \ c \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mathbb{Z}_0^- = \{0, -1, -2, \cdots\})$$

and $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) given, in terms of Gamma function, by

$$(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1 & (k = 0; \ \nu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \nu(\nu + 1) \cdots (\nu + k - 1) & (k \in \mathbb{N}; \ \nu \in \mathbb{C}). \end{cases}$$

In particular, we find, from (1.4), that

$$\mathcal{I}_{0,p}^{\lambda,0}(a, \lambda + p, a) f(z) = f(z) \ \ \text{and} \ \ \mathcal{I}_{0,p}^{1,0}(a, p, a) f(z) = \frac{zf'(z)}{p}$$

and easily deduce that

$$z[\mathcal{I}_{\mu,p}^{\lambda,\delta}(a, b, c) f(z)]' = (\lambda + p) \mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a, b, c) f(z) - \lambda \mathcal{I}_{\mu,p}^{\lambda,\delta}(a, b, c) f(z) \tag{1.5}$$

and

$$z[\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)]' = a\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) - (a-p)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z). \tag{1.6}$$

We also notice that the operator $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as below:

(i) $\mathcal{I}_{\mu,1}^{\lambda,0}(a,b,c) = \mathcal{I}_{\mu}^{\lambda}(a,b,c)$, which is the Srivastava-Khairnar-More operator [20] (see also [30]);

(ii) $\mathcal{I}_{0,1}^{\lambda,0}(a,b,c) = \mathcal{I}_{\lambda}(a,b,c)$, which was introduced by Noor [14];

(iii) $\mathcal{I}_{0,p}^{\lambda,0}(a,1,c) = \mathcal{I}_{p}^{\lambda}(a,c)$, which is the Cho-Kwon-Srivastava operator [3];

(iv) $\mathcal{I}_{0,1}^{n,0}(a,n+1,a) = \mathcal{I}_{n}$, which is the Noor integral operator [13].

Based on the above class S_e^* and by virtue of the operator $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)$, we now define the following classes $M_{\mu,p}^{\lambda,\delta}(a,b,c;\eta)$ and $N_{\mu,p}^{\lambda,\delta}(a,b,c;\gamma)$ of functions $f \in \mathcal{A}_p$.

Definition 1.3. Let $p \in \mathbb{N}$; $a,b \in \mathbb{C}$; $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\mathbb{Z}_0^- = \{0,-1,-2,\cdots\}$; $\lambda > -p$ and $\mu,\delta \geq 0$. A function $f \in \mathcal{A}_p$ is said to be in the class $M_{\mu,p}^{\lambda,\delta}(a,b,c;\eta)$ of multivalent analytic functions of order η $(0 \leq \eta < p)$, related with exponential function, if and only if

$$\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)} - \eta\right) \prec e^z. \tag{1.7}$$

Remark 1.2.

(i) For $\eta = 0$ in (1.7), we have the function class

$$M_{\mu,p}^{\lambda,\delta}(a,b,c) := M_{\mu,p}^{\lambda,\delta}(a,b,c;0) = \left\{ f \in \mathcal{A}_p : \frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)} \prec pe^z \ (p \in \mathbb{N}) \right\}.$$

(ii) For $p = 1$ in (1.7), we get the function class

$$M_{\mu}^{\lambda,\delta}(a,b,c;\eta) := M_{\mu,1}^{\lambda,\delta}(a,b,c;\eta) = \left\{ f \in \mathcal{A} : \frac{1}{1-\eta}\left(\frac{z(\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)f(z))'}{\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)f(z)} - \eta\right) \prec e^z \ (0 \leq \eta < 1) \right\}.$$

(iii) Further, for $\eta = p - 1 = 0$ in (1.7), we obtain the function class

$$M_{\mu}^{\lambda,\delta}(a,b,c) := M_{\mu,1}^{\lambda,\delta}(a,b,c;0) = \left\{ f \in \mathcal{A} : \frac{z(\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)f(z))'}{\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)f(z)} \prec e^z \ (z \in \mathbb{U}) \right\}.$$

Definition 1.4. Let $p \in \mathbb{N}$; $\gamma \in \mathbb{C}^*$; $a,b \in \mathbb{C}$; $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\mathbb{Z}_0^- = \{0,-1,-2,\cdots\}$; $\lambda > -p$ and $\mu,\delta \geq 0$. A function $f \in \mathcal{A}_p$ is said to be in the class $N_{\mu,p}^{\lambda,\delta}(a,b,c;\gamma)$ of multivalent analytic functions of complex order $\gamma \neq 0$, related with exponential function, if and only if

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)} - p\right) \prec e^z. \tag{1.8}$$

Remark 1.3.

(i) For $\gamma = 1$ in (1.8), we have the function class

$$N_{\mu,p}^{\lambda,\delta}(a,b,c) := N_{\mu,p}^{\lambda,\delta}(a,b,c;1) = \left\{ f \in \mathcal{A}_p : \frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a+1,b,c)f(z)} \prec (p-1+e^z)\ (p \in \mathbb{N}) \right\}.$$

(ii) For $p = 1$ in (1.8), we get the function class

$$N_{\mu}^{\lambda,\delta}(a,b,c;\gamma) := N_{\mu,1}^{\lambda,\delta}(a,b,c;\gamma) = \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}_{\mu,1}^{\lambda,\delta}(a+1,b,c)f(z))'}{\mathcal{I}_{\mu,1}^{\lambda,\delta}(a+1,b,c)f(z)} - 1 \right) \prec e^z\ (\gamma \in \mathbb{C}^*) \right\}.$$

(iii) Further, for $p = \gamma = 1$ in (1.8), we obtain the function class

$$N_{\mu}^{\lambda,\delta}(a,b,c) := N_{\mu,1}^{\lambda,\delta}(a,b,c;1) = \left\{ f \in \mathcal{A} : \frac{z(\mathcal{I}_{\mu,1}^{\lambda,\delta}(a+1,b,c)f(z))'}{\mathcal{I}_{\mu,1}^{\lambda,\delta}(a+1,b,c)f(z)} \prec e^z\ (z \in \mathbb{U}) \right\}.$$

A majorization problem for the normalized class of starlike functions has been investigated by MacGregor [10] and Altintas et al. [1] (see also [2]). Recently, many researchers have studied several majorization problems for univalent and multivalent functions, which are all subordinate to certain function $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, involving various different operators, for instance, see [5, 8, 16, 27, 28]. More recently, Goyal and Goswami [6], Tang et al. [25], and Panigrahi and El-Ashwah [15] have considered majorization problems for meromorphic and multivalent meromorphic functions. Nevertheless, only a few articles deal with the above-mentioned problems associated with exponential function (see[26]). Here, in the present paper, we aim to investigate the problems of majorization of the classes $M_{\mu,p}^{\lambda,\delta}(a,b,c;\eta)$ and $N_{\mu,p}^{\lambda,\delta}(a,b,c;\gamma)$ defined by the operator $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)$, which are related with exponential function.

2. Majorization Problem for the Class $S_{p,q,s}^{\lambda,\mu,m}[\eta; A, B]$

Firstly, we give and prove majorization property for the class $M_{\mu,p}^{\lambda,\delta}(a,b,c;\eta)$.

Theorem 2.1. Let the function $f \in \mathcal{A}_p$ and assume that $g \in M_{\mu,p}^{\lambda,\delta}(a,b,c;\eta)$ with $e|p-\eta| \le |\lambda+\eta|$. If $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)$ is majorized by $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)$ in $\mathbb{U}$, that is, that

$$\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) \ll \mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z) \quad (z \in \mathbb{U}),$$

then, for $|z| \le r_1$, we have

$$\left|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)\right| \le \left|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z)\right|,$$

where $r_1 = r_1(p,\lambda,\eta)$ is the smallest positive root of the equation

$$|p-\eta| r^2 e^r - |\lambda+\eta| r^2 - |p-\eta| e^r - 2r + |\lambda+\eta| = 0 \quad (p \in \mathbb{N};\ \lambda > -p;\ 0 \le \eta < p). \tag{2.1}$$

Proof. Since $g \in M_{\mu,p}^{\lambda,\delta}(a,b,c;\eta)$, we see, from (1.7), that

$$\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)} - \eta \right) = e^{\omega(z)}, \tag{2.2}$$

where $\omega(z) = c_1 z + c_2 z^2 + \cdots$ is bounded and analytic in $\mathbb{U}$, satisfying (see, for details, Goodman [4])

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| \le |z| \quad (z \in \mathbb{U}). \tag{2.3}$$

From (2.2), we easily obtain

$$\frac{z(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z))'}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)} = \eta + (p-\eta)e^{\omega(z)}. \tag{2.4}$$

Now, by virtue of (1.5) and (2.4) and making simple computations, we have

$$\frac{\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z)}{\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)} = \frac{\lambda + \eta + (p-\eta)e^{\omega(z)}}{\lambda + p},$$

which, using (2.3), yields the inequality

$$|\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)| \leq \frac{\lambda + p}{|\lambda+\eta| - |p-\eta|e^{|z|}} |\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z)|. \tag{2.5}$$

Also, because $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)$ is majorized by $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)$ in $\mathbb{U}$, so we find, from (1.1), that

$$\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z) = \varphi(z)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z). \tag{2.6}$$

Differentiating (2.6) on both sides with respect to z and multiplying by z, we obtain

$$z\left(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)\right)' = z\varphi'(z)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z) + z\varphi(z)\left(\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)\right)'. \tag{2.7}$$

By using (1.5) in (2.7), together with (2.6), we have

$$\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z) = \frac{1}{\lambda+p} z\varphi'(z)\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z) + \varphi(z)\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z). \tag{2.8}$$

On the other hand, noting that the Schwarz function φ satisfies the inequality (see, e.g. Nehari [12])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z \in \mathbb{U}), \tag{2.9}$$

and in view of (2.5) and (2.9) in (2.8), we get

$$|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)| \leq \left[|\varphi(z)| + \frac{|z|(1-|\varphi(z)|^2)}{(1-|z|^2)\left(|\lambda+\eta| - |p-\eta|e^{|z|}\right)}\right] |\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z)|,$$

which, by letting

$$|z| = r, \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

becomes the inequality

$$|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)| \leq \Phi_1(r,\rho)|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z)|,$$

where

$$\Phi_1(r,\rho) = \frac{r(1-\rho^2)}{(1-r^2)\left(|\lambda+\eta| - |p-\eta|e^r\right)} + \rho.$$

In order to determine r_1, we must choose

$$\begin{aligned} r_1 &= \max\{r \in [0,1) : \Phi_1(r,\rho) \leq 1, \forall\, \rho \in [0,1]\} \\ &= \max\{r \in [0,1) : \Psi_1(r,\rho) \geq 0, \forall\, \rho \in [0,1]\}, \end{aligned}$$

where

$$\Psi_1(r,\rho) = (1-r^2)\left(|\lambda+\eta| - |p-\eta|e^r\right) - r(1+\rho).$$

Clearly, for $\rho = 1$, the function $\Psi_1(r, \rho)$ takes its minimum value, namely,

$$\min\{\Psi_1(r,\rho) : \rho \in [0,1]\} = \Psi_1(r,1) := \psi_1(r),$$

where

$$\psi_1(r) = (1 - r^2)(|\lambda + \eta| - |p - \eta|e^r) - 2r.$$

Further, because $\psi_1(0) = |\lambda + \eta| - |p - \eta| > 0$ and $\psi_1(1) = -2 < 0$, so there exists r_1, such that $\psi_1(r) \geq 0$ for all $r \in [0, r_1]$, where $r_1 = r_1(p, \lambda, \eta)$ is the smallest positive root of the equation (2.1). This completes the proof of Theorem 2.1.

3. Majorization Problem for the Class $I^{\lambda,\mu,m}_{p,q,s}[\alpha, b; A, B]$

Next, we discuss majorization property for the class $N^{\lambda,\delta}_{\mu,p}(a, b, c; \gamma)$.

Theorem 3.1. Let the function $f \in \mathcal{A}_p$ and assume that $g \in N^{\lambda,\delta}_{\mu,p}(a, b, c; \gamma)$ with $e|\gamma| \leq |a - \gamma|$. If $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)f(z)$ is majorized by $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z)$ in $\mathbb{U}$, that is, that

$$\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)f(z) \ll \mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z) \quad (z \in \mathbb{U}),$$

then, for $|z| \leq r_2$, we have

$$\left|\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)f(z)\right| \leq \left|\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)g(z)\right|, \tag{3.1}$$

where $r_2 = r_2(a, \gamma)$ is the smallest positive root of the equation

$$|\gamma|r^2e^r - |a - \gamma|r^2 - |\gamma|e^r - 2r + |a - \gamma| = 0 \quad (\gamma \in \mathbb{C}^*;\ a \in \mathbb{C}). \tag{3.2}$$

Proof. Because $g \in N^{\lambda,\delta}_{\mu,p}(a, b, c; \gamma)$, so, from (1.8), we show that

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z))'}{\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z)} - p\right) = e^{\omega(z)}, \tag{3.3}$$

where $\omega(z)$ is defined as (2.3).

From (3.3), it follows that

$$\frac{z(\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z))'}{\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z)} = p - \gamma + \gamma e^{\omega(z)}. \tag{3.4}$$

Now, using (1.6) in (3.4) and making simple calculations, we get

$$\frac{\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)g(z)}{\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z)} = \frac{a - \gamma + \gamma e^{\omega(z)}}{a},$$

which, in terms of (2.3), yields the inequality

$$|\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z)| \leq \frac{|a|}{|a - \gamma| - |\gamma|e^{|z|}}|\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)g(z)|. \tag{3.5}$$

Again, since $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)f(z)$ is majorized by $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z)$ in $\mathbb{U}$, then, applying the same process of (2.6) and (2.7) of Theorem 2.1, we verify, from (1.6), that

$$\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)f(z) = \frac{1}{a}z\varphi'(z)\mathcal{I}^{\lambda,\delta}_{\mu,p}(a+1, b, c)g(z) + \varphi(z)\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)g(z). \tag{3.6}$$

Next, in view of (2.9) as well as (3.5) in (3.6), and just as the proof of Theorem 2.1, we have

$$|\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)| \leq \left[|\varphi(z)| + \frac{|z|(1-|\varphi(z)|^2)}{(1-|z|^2)\left(|a-\gamma|-|\gamma|e^{|z|}\right)}\right]|\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)|,$$

which, by putting

$$|z| = r, \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

reduces to the inequality

$$|\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)| \leq \frac{\Phi_2(\rho)}{(1-r^2)\left(|a-\gamma|-|\gamma|e^{r}\right)}|\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)|, \tag{3.7}$$

where the function $\Phi_2(\rho)$ given by

$$\Phi_2(\rho) = -r\rho^2 + (1-r^2)\left(|a-\gamma|-|\gamma|e^{r}\right)\rho + r$$

takes its maximum value at $\rho = 1$ with $r_2 = r_2(a,\gamma)$ defined by (3.2). Furthermore, if $0 \leq \sigma \leq r_2(a,\gamma)$, then the function

$$\Psi_2(\rho) = -\sigma\rho^2 + (1-\sigma^2)\left(|a-\gamma|-|\gamma|e^{\sigma}\right)\rho + \sigma$$

increases on the interval $0 \leq \rho \leq 1$, therefore $\Psi_2(\rho)$ does not exceed

$$\Psi_2(1) = (1-\sigma^2)\left(|a-\gamma|-|\gamma|e^{\sigma}\right) \quad (0 \leq \sigma \leq r_2(a,\gamma)).$$

Hence, from this fact and (3.7), we conclude that the inequality (3.1) holds true. We complete the proof of Theorem 3.1.

4. Corollaries and Concluding Remarks

As a special case of Theorem 2.1, when $\eta = 0$, we get the following result.

Corollary 4.1. Let the function $f \in \mathcal{A}_p$ and $g \in M_{\mu,p}^{\lambda,\delta}(a,b,c)$ with $ep \leq |\lambda|$. If $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)f(z)$ is majorized by $\mathcal{I}_{\mu,p}^{\lambda,\delta}(a,b,c)g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_3$, we have

$$\left|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)f(z)\right| \leq \left|\mathcal{I}_{\mu,p}^{\lambda+1,\delta}(a,b,c)g(z)\right|,$$

where $r_3 = r_1(p,\lambda,0)$ is the smallest positive root of the equation

$$pr^2e^r - |\lambda|r^2 - pe^r - 2r + |\lambda| = 0 \quad (p \in \mathbb{N};\ \lambda > -p).$$

Setting $p = 1$ and $\eta = p - 1 = 0$ in Theorem 2.1, respectively, we obtain the following corollaries.

Corollary 4.2. Let the function $f \in \mathcal{A}$ and $g \in M_{\mu}^{\lambda,\delta}(a,b,c;\eta)$ with $e|1-\eta| \leq |\lambda+\eta|$. If $\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)f(z)$ is majorized by $\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_4$, we get

$$\left|\mathcal{I}_{\mu,1}^{\lambda+1,\delta}(a,b,c)f(z)\right| \leq \left|\mathcal{I}_{\mu,1}^{\lambda+1,\delta}(a,b,c)g(z)\right|,$$

where $r_4 = r_1(1,\lambda,\eta)$ is the smallest positive root of the equation

$$|1-\eta|r^2e^r - |\lambda+\eta|r^2 - |1-\eta|e^r - 2r + |\lambda+\eta| = 0 \quad (\lambda > -1;\ 0 \leq \eta < 1).$$

Corollary 4.3. Let the function $f \in \mathcal{A}$ and $g \in M_{\mu}^{\lambda,\delta}(a,b,c)$ with $e \leq |\lambda|$. If $\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)f(z)$ is majorized by $\mathcal{I}_{\mu,1}^{\lambda,\delta}(a,b,c)g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_5$, we obtain

$$\left|\mathcal{I}_{\mu,1}^{\lambda+1,\delta}(a,b,c)f(z)\right| \leq \left|\mathcal{I}_{\mu,1}^{\lambda+1,\delta}(a,b,c)g(z)\right|,$$

where $r_5 = r_1(1, \lambda, 0)$ is the smallest positive root of the equation

$$r^2e^r - |\lambda|r^2 - e^r - 2r + |\lambda| = 0 \ \ (\lambda > -1).$$

Putting $\gamma = 1$ in Theorem 3.1, we have the following result.

Corollary 4.4. Let the function $f \in \mathcal{A}_p$ and $g \in N^{\lambda,\delta}_{\mu,p}(a, b, c)$ with $e \leq |a - 1|$. If $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a + 1, b, c)f(z)$ is majorized by $\mathcal{I}^{\lambda,\delta}_{\mu,p}(a + 1, b, c)g(z)$ in $\mathbb{U}$, then, for $|z| \leq r_6$, we obtain

$$\left|\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)f(z)\right| \leq \left|\mathcal{I}^{\lambda,\delta}_{\mu,p}(a, b, c)g(z)\right|,$$

where $r_6 = r_2(a, 1)$ is the smallest positive root of the equation

$$r^2e^r - |a - 1|r^2 - e^r - 2r + |a - 1| = 0 \ \ (a \in \mathbb{C}). \tag{4.1}$$

Taking $p = 1$ and $p = \gamma = 1$ in Theorem 3.1, respectively, we state the following corollaries.

Corollary 4.5. Let the function $f \in \mathcal{A}$ and $g \in N^{\lambda,\delta}_{\mu}(a, b, c; \gamma)$ with $e|\gamma| \leq |a - \gamma|$. If $\mathcal{I}^{\lambda,\delta}_{\mu,1}(a + 1, b, c)f(z)$ is majorized by $\mathcal{I}^{\lambda,\delta}_{\mu,1}(a + 1, b, c)g(z)$ in $\mathbb{U}$, then,

$$\left|\mathcal{I}^{\lambda,\delta}_{\mu,1}(a, b, c)f(z)\right| \leq \left|\mathcal{I}^{\lambda,\delta}_{\mu,1}(a, b, c)g(z)\right| \ \ (|z| \leq r_2),$$

where r_2 is given by (3.2).

Corollary 4.6. Let the function $f \in \mathcal{A}$ and $g \in N^{\lambda,\delta}_{\mu}(a, b, c)$ with $e \leq |a-1|$. If $\mathcal{I}^{\lambda,\delta}_{\mu,1}(a+1, b, c)f(z)$ is majorized by $\mathcal{I}^{\lambda,\delta}_{\mu,1}(a + 1, b, c)g(z)$ in $\mathbb{U}$, then,

$$\left|\mathcal{I}^{\lambda,\delta}_{\mu,1}(a, b, c)f(z)\right| \leq \left|\mathcal{I}^{\lambda,\delta}_{\mu,1}(a, b, c)g(z)\right| \ \ (|z| \leq r_6),$$

where r_6 is given by (4.1).

Concluding Remarks. By choosing the suitable parameters $p, \lambda, \mu, \delta, a, b$ and c in all results of this paper, we easily get the corresponding majorization results for the previously studied familiar operators $\mathcal{I}^{\lambda}_{\mu}(a, b, c)$, $\mathcal{I}_{\lambda}(a, b, c)$, $\mathcal{I}^{\lambda}_{p}(a, c)$ and $\mathcal{I}_n$, which are mentioned in the introduction.

References

[1] O. Altintas, O. Ozkan, H. M.Srivastava, Majorization by starlike functions of complex order, Complex Variables Theory Appl. 46 (2001) 207–218.

[2] O. Altintas, H. M. Srivastava, Some majorization problems associated with p-valently starlike and convex functions of complex order, East Asian Math. J. 17 (2) (2001) 207–218.

[3] N. E. Cho, O. S. Kwon, H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl. 292 (2004) 470–483.

[4] A. W. Goodman, Univalent Functions, Mariner Publishing Company, Tampa, Florida, 1983.

[5] P. Goswami, M. K. Aouf, Majorization properties for certain classes of analytic functions using the Salagean operator, Appl. Math. Lett. 23 (11) (2010) 1351–1354.

[6] S. P. Goyal, P. Goswami, Majorization for certain classes of meromorphic functions defined by integral operator, Ann. Univ. Mariae Curie Sklodowska Lublin-Polonia, 2 (2012) 57–62.

[7] W. Janowski, Some extremal problems for certain families of analytic functions, Ann. Polon. Math. 28 (1973) 297–326.

[8] S.-H. Li, H. Tang, Ao En, Majorization properties for certain new classes of analytic functions using the Salagean operator, J. Inequal. Appl. 2013, 2013: 86, doi:10.1186/1029-242X-2013-86.

[9] W. Ma, D. Minda, An internal geometric characterization of strongly starlike functions, Ann. Univ. Mariae Curie Sklodowska Lublin-Polonia, 45 (1991) 89–97.

[10] T. H. MacGregor, Majorization by univalent functions, Duke Math J. 34 (1967) 95–102.

[11] R. Mendiratta, S. Nagpal, V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, Bull. Malays. Math. Sci. Soc. 38 (1) (2015) 365–386.

[12] Z. Nehari, Conformal Mapping, MacGraw-Hill Book Company, New York, Toronto and London, 1955.

[13] K. I. Noor, On new classes of integral operators, J. Nat. Geom. 16 (1999) 71–80.

[14] K. I. Noor, Integral operators defined by convolution with hypergeometric functions, Appl. Math. Comput. 182 (2006) 1872–1881.

[15] T. Panigrahi, R. El-Ashwah, Majorization for subclasses of multivalent meromorphic functions defined through iterations and combinations of the Liu-Srivastava operator and a meromorphic analogue of the Cho-Kwon-Srivastava operator, Filomat, 31 (20) (2017) 6357–6365.

[16] J. K. Prajapat, M. K. Aouf, Majorization problem for certain class of p-valently analytic functions defined by generalized fractional differintegral operator, Comput. Math. Appl. 63 (2012) 42–47.

[17] M. S. Roberston, Quasi-subordination and coefficient conjectures, Bull. Amer. Math. Soc. 76 (1970) 1–9.

[18] H. M. Srivastava, M. K. Aouf, A. O. Mostafa, H. M. Zayed, Certain subordination-preserving family of integral operators associated with p-valent functions, Appl. Math. Inform. Sci. 11 (2017) 951–960.

[19] H. M. Srivastava, S. Hussain, A. Raziq, M. Raza, The Fekete-Szego functional for a subclass of analytic functions associated with quasi-subordination, Carpathian J. Math. 34 (2018) 103–113.

[20] H. M. Srivastava, S. M. Khairnar and M. More, Inclusion properties of a subclass of analytic functions defined by an integral operator involving the Gauss hypergeometric function, Appl. Math. Comput. 218 (2011) 3810–3821.

[21] H. M. Srivastava, S. Owa, Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.

[22] H. M. Srivastava, A. Prajapati, P. Gochhayat, Third-order differential subordination and differential superordination results for analytic functions involving the Srivastava-Attiya operator, Appl. Math. Inform. Sci. 12 (2018) 469–481.

[23] H. M. Srivastava, D. Raducanu, P. Zaprawa, A certain subclass of analytic functions defined by means of differential subordination, Filomat, 30 (2016) 3743–3757.

[24] H. Tang, M. K. Aouf, G.-T. Deng, Inclusion and argument properties for the Srivastava-Khairnar-More operator, Filomat, 28 (8) (2014) 1603–1618.

[25] H. Tang, M. K. Aouf, G.-T. Deng, Majorization problems for certain subclasses of meromorphic multivalent functions associated with the Liu-Srivastava operator, Filomat, 29 (4) (2015) 763–772.

[26] H. Tang, G.-T. Deng, Majorization problems for two subclasses of analytic functions connected with the Liu-Owa integral operator and exponential function, J. Inequal. Appl. 2018, 2018: 277.

[27] H. Tang, G.-T. Deng, S.-H. Li, Majorization properties for certain classes of analytic functions involving a generalized differential operator, J. Math. Res. Appl. 33 (5) (2013) 578–586.

[28] H. Tang, S.-H. Li, G.-T. Deng, Majorization properties for a new subclass of θ-spiral functions of order γ, Math. Slovaca, 64 (1) (2014) 39–50.

[29] H. Tang, H. M. Srivastava, G.-T. Deng, Some families of analytic functions in the upper half-plane and their associated differential subordination and differential superordination properties and problems, Appl. Math. Inform. Sci. 11 (2017) 1247–1257.

[30] Z.-G. Wang, G.-W. Zhang, F.-H. Wen, Properties and characteristics of the Srivastava-Khairnar-More integral operator, Appl. Math. Comput. 218 (7) (2012) 7747–7758.