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A Fixed Point Theorem for Generalized Contractive Type Set-valued Mappings with Application to Nonlinear Fractional Differential Inclusions

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Abstract. In this paper, we present some fixed point results for set-valued mappings of contractive type by using the concept of ω-distance. As an application, we prove the existence of solution of nonlinear fractional differential inclusion.

1. Introduction and Preliminaries

Banach's contraction principle is a forceful tool in nonlinear analysis, differential equation, inclusion and many other related areas of mathematics. Many authors generalized this principle to various directions [7–10, 12, 14, 19, 21, 31, 37]. In particular, in 2012, Samet et al. [39] introduced the concept of α-ψ-contractive type mappings and proved fixed point theorems for these mappings in complete metric spaces. After that, Hasanzade Asl et al. [13] extended the notion of α_*-ψ-contractive type for set-valued mappings and presented a fixed point result for such set-valued mappings. Recently, many generalization of the concept of α-ψ-contractive type mappings have been developed; see [22, 29, 32, 33, 35, 36, 38] and the references therein.

On the other hand in 1996, O. Kada [28] introduced the notion of ω-distance on a metric space. Using this new notion, Lakzian et al. [34] introduced the new concept of generalized α-ψ-contractive type mappings and investigated the existence and uniqueness of fixed points for these mappings.

In this paper, the concept of set-valued generalized (α, ψ, p)-contractive type mappings in the setting of ω-distances is introduced. The existence of fixed points for such mappings with ω-distances in a metric space is presented.

Nonlinear fractional differential equations and inclusions play a key role in the modeling of anomalous relaxation and diffusion processes. Fractional differential equations and inclusions appear naturally in various fields of science, such as physics, engineering, bio-physics, fluid mechanics, chemistry and biology [11, 17, 23, 25–27, 30]. Some standard fixed point theorems have a fundamental role for the study the existence of solutions for the fractional differential equations and inclusions; see [1–3, 5, 6, 15, 16, 18] and the references therein. Ahmad and Ntouyas [4] proved the existence of solutions for the fractional differential inclusions with integral boundary value problems in non-convex valued by applying a fixed

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point theorem for set-valued mapping due to Covitz and Nadler [20]. In this paper, as an application the existence of fixed point for (α, ψ, p)-contractive type mappings, we prove the existence of solution to a nonlinear fractional differential inclusion in of the form

$$^{C}D^{\beta}(x(t)) \in F(t, x(t)), \quad t \in J = [0,1] \; and \; \beta \in (1,2]$$

by integral boundary condition

$$x(0) = 0, \;\; x(1) = \int_0^{\eta} x(s)ds, \quad \eta \in (0,1)$$

where $x \in C(J, \mathbb{R})$ and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Caratheodory set-valued mapping.
Let us introduce some definitions and facts which will be used in the sequel. All topological spaces are assumed to be metric. We denote by $\mathcal{P}(X)$, $\mathcal{CB}(X)$ and $\mathcal{K}(X)$ the family of all nonempty subsets of X, the family of all nonempty closed and bounded subsets of X and the family of all nonempty compact subsets of X, respectively.
The set-valued mapping $T : X \to \mathcal{P}(Y)$ is said to be:

(i) upper semicontinuous, if for each closed set $B \subseteq Y$, $T^{-}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is closed in X.

(ii) lower semicontinuous if for each open set $V \subseteq Y$, $T^{-}(V) = \{x \in X : T(x) \cap V \neq \emptyset\}$ is open in X.

(iii) continuous if it is both upper and lower semicontinuous.

The concept of a ω-distance on a metric space was introduced in [28] as follows:

Definition 1.1. *A function $p : X \times X \to [0, \infty)$ is said to be ω-distance on metric space (X, d) if it satisfies the following properties:*

(p1) $p(x, z) \leq p(x, y) + p(y, z)$ *for any* $x, y, z \in X$;

(p2) *p is lower semicontinuous in its second variable; i.e., if* $x \in X$ *and* $y_n \to y \in X$, *then* $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$;

(p3) *for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $p(z, x) \leq \delta$ *and* $p(z, y) \leq \delta$ *imply* $d(x, y) \leq \varepsilon$.

Some example of ω-distance can be found in [28].
In order to prove of our main results, we need the following lemma:

Lemma 1.2. *([28]). Let (X, d) be a metric space and p be a ω-distance on X. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following assertions hold.*

(i) If $p(x_n, y) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for all* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $p(x, y) = p(x, z) = 0$, *then* $y = z$.

(ii) If $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, y) \leq \beta_n$ *for all* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* y.

(iii) If $p(x_n, x_m) \leq \alpha_n$ *for all* $m, n \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$; for all $t > 0$, where ψ^n is the nth iterate of ψ.

It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$.

2. Main Results

In this section, first we introduce the concept of set-valued generalized (α, ψ, p)-contractive type mappings in the setting of ω-distances. Then we present new fixed point result for (α, ψ, p)-contractive type for set-valued mappings by using ω-distances in complete metric spaces.

Suppose that p is a ω-distance on X. For any $x \in X$ and two subsets $A, B \in \mathcal{P}(X)$, we define

$$D_p(x, A) = \inf\{p(x, y) : y \in A\}$$

and

$$H_p(A, B) = \max\{\sup_{x\in A} D_p(x, B), \sup_{y\in B} D_p(y, A)\}.$$

The maximum in this definition always exists; it can be finite or infinite.
Notice that if $p = d$ then $H_p(A, B)$ is Pompeiu-Hausdorff metric $H(A, B)$.

Definition 2.1. *Let (X, d) be a metric space with ω-distance p and $T : X \to \mathcal{CB}(X)$ be a set-valued mapping. We say that T is an (α, ψ, p)-contractive type mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$,*

$$\alpha(x, y)H_p(Tx, Ty) \leq \psi(p(x, y)). \tag{1}$$

Definition 2.2. *Let (X, d) be a metric space. A set-valued mapping $T : X \to \mathcal{CB}(X)$ is called an (α, ψ)-contractive type mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$,*

$$\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y)). \tag{2}$$

Definition 2.3. *Let $T : X \to \mathcal{CB}(X)$ be a set-valued mapping and $\alpha : X \times X \to [0, \infty)$ be a given mapping. Then T is called an α-admissible mapping if for each $x, y \in X$, $\alpha(x, y) \geq 1$ then $\alpha(z, w) \geq 1$ for all $z \in T(x)$ and $w \in T(y)$.*

Now, we present a fixed point theorem for set-valued mapping on a complete metric space endowed with a ω-distance.

Theorem 2.4. *Let p be a ω-distance on a complete metric space (X, d) and let $T : X \to \mathcal{K}(X)$ be a set-valued (α, ψ, p)-contractive type mapping and T be an α-admissible mapping. Suppose that there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$. Furthermore, let T satisfies one of the following hypotheses:*

(i) T is continuous;

(ii) for any sequence $\{x_n\}$ in X if $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;

(iii) for every $u \in X$ with $u \notin T(u)$, $\inf\{p(x, u) + D_p(x, T(x)) : x \in X\} > 0$.

Then there exists a $u \in X$ such that $u \in T(u)$.

Proof. By hypothesis, there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that

$$\alpha(x_0, x_1) \geq 1.$$

Since p is lower semicontinuous in its second variable and T is compact valued, by Lemma 3.4 of [22], there exists a point $x_2 \in T(x_1)$ such that $p(x_1, x_2) = D_p(x_1, T(x_1))$. Hence,

$$p(x_1, x_2) \leq H_p(T(x_0), T(x_1)).$$

By induction there exists a sequence $\{x_n\}$ in which $x_{n+1} \in T(x_n)$ and

$$p(x_n, x_{n+1}) \leq H_p(T(x_{n-1}), T(x_n)). \tag{3}$$

for all $n \in \mathbb{N}$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then $u = x_{n_0}$ is a fixed point of T. Thus, assume that for all $n \in \mathbb{N} \cup \{0\}$, $x_n \neq x_{n+1}$. Since T is an α-admissible mapping, then $\alpha(x_0, x_1) \geq 1$ yields $\alpha(x_1, x_2) \geq 1$. By mathematical induction, for each $n \in \mathbb{N}$, we have

$$\alpha(x_n, x_{n+1}) \geq 1. \tag{4}$$

Now, we want to show that $p(x_n, x_{n+1}) \to 0$. By inequalities (3), (1) and (4) we have

$$\begin{aligned} p(x_n, x_{n+1}) &\leq H_p(T(x_{n-1}), T(x_n)) \\ &\leq \alpha(x_{n-1}, x_n) H_p(T(x_{n-1}), T(x_n)) \\ &\leq \psi(p(x_{n-1}, x_n)), \end{aligned}$$

for each $n \in \mathbb{N}$. Iterate this process to deduce that

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)).$$

Condition (Ψ_2) implies that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Now, we prove that $\{x_n\}$ is a Cauchy sequence. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + ... + p(x_{m-1}, x_m) \\ &\leq \textstyle\sum_{i=n}^{m} \psi^i(p(x_0, x_1)) \\ &\leq \textstyle\sum_{i=n}^{\infty} \psi^i(p(x_0, x_1)) \to 0. \end{aligned} \tag{5}$$

From Lemma 1.2, we obtain that $\{x_n\}$ is a Cauchy sequence in (X, d). Since X is a complete metric space, $\{x_n\}$ converges to $u \in X$. It is enough to show that u is a fixed point of T.

If T is continuous, since $x_{n+1} \in T(x_n)$ and T is compact valued, we obtain $u \in T(u)$.

If (ii) holds, we have $\alpha(x_n, u) \geq 1$, for any $n \in \mathbb{N}$. Property (p2) of Definition 1.1 follows,

$$p(x_n, u) \leq \liminf_{m\to\infty} p(x_n, x_m) = \alpha_n$$

for each $n \in \mathbb{N}$. Then by inequality (5), $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} p(x_n, u) = 0$. Since function p is lower semicontinuous in its second variable and T is compact valued, then for each $n \in \mathbb{N}$, there exists $w_n \in T(u)$ such that $p(x_{n+1}, w_n) = D_p(x_{n+1}, T(u))$. Hence, for each $n \in \mathbb{N}$, there exist the following inequalities

$$\begin{aligned} p(x_{n+1}, w_n) &= D_p(x_n, T(u)) \leq H_p(T(x_n), T(u)) \\ &\leq \alpha(x_n, u) H_p(T(x_n), T(u)) \leq \psi(p(x_n, u)) \\ &\leq p(x_n, u), \end{aligned}$$

then $\lim_{n\to\infty} p(x_{n+1}, w_n) = 0$. By (P1) we have,

$$p(x_n, w_n) \leq p(x_n, x_{n+1}) + p(x_{n+1}, w_n),$$

and so

$$\lim_{n\to\infty} p(x_n, w_n) = 0.$$

Since mapping T is compact valued, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that it is convergent to $w \in T(u)$. It follows from Proposition 1 and Lemma 3 of [41] that $\lim_{n\to\infty} d(x_n, w_n) = 0$. Therefore, one can conclude that $u \in T(u)$.

Let (iii) hold. Assume, on the contrary, that $u \notin T(u)$. Therefore hypothesis implies that following

$$\begin{aligned} 0 &< \inf\{p(x, u) + D_p(x, T(x)) : x \in X\} \\ &\leq \inf\{p(x_n, u) + D_p(x_n, T(x_n)) : n \in \mathbb{N}\} \\ &\leq \inf\{p(x_n, u) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} \\ &= 0. \end{aligned}$$

But, this would be a contradiction and so $u \in T(u)$. □

As a consequence of Theorem 2.4, we obtain the existence of a fixed point for set-valued (α, ψ)-contractive type mappings.

Corollary 2.5. *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a set-valued (α, ψ)-contractive type mapping and T be an α-admissible mapping. Suppose that there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$. Furthermore, let T satisfies one of the following hypothesis:*

(i) T is continuous;

(ii) for any sequence $\{x_n\}$ in X, if $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;

(iii) for every $u \in X$ with $u \notin T(u)$, $\inf\{d(x, u) + D(x, T(x)) : x \in X\} > 0$.

Then there exists a $u \in X$ such that $u \in T(u)$.

Now, we present some examples that satisfy in conditions in Theorem 2.4.

Example 2.6. *Let $X = [0, \infty)$ with usual norm and $p(x, y) = y$, for all $x, y \in X$. It can be easily seen that p is a ω-distance on (X, d). Let $T : X \to \mathcal{K}(X)$ be defined by*

$$T(x) = \begin{cases} \frac{1}{2}x^2 & x \in [0, 1], \\ \\ [2, 3] & x \notin [0, 1]. \end{cases} \tag{6}$$

Also we define $\alpha : X \times X \to [0, \infty)$ as $\alpha(x, y) = 1$ whenever $x, y \in [0, 1]$ and $\alpha(x, y) = 0$ whenever $x \notin [0, 1]$ or $y \notin [0, 1]$. It is clear that T is an α-admissible mapping. Let $\psi : [0, \infty) \to [0, \infty)$ be the function $\psi(t) = \frac{1}{2}t$ for $t \in [0, \infty)$. Then, T is an (α, ψ, p)-contractive type mapping. Indeed, if $x, y \in [0, 1]$ then $\alpha(x, y) = 1$ and

$$\alpha(x, y)H_p(Tx, Ty) = p(\frac{1}{2}x^2, \frac{1}{2}y^2) = \frac{1}{2}y^2 \leq \frac{1}{2}y = \psi(p(x, y)).$$

Otherwise, if $x \notin [0, 1]$ or $y \notin [0, 1]$ then $\alpha(x, y) = 0$ so

$$0 = \alpha(x, y)H_p(Tx, Ty) \leq \psi(p(x, y)).$$

Moreover, suppose that there exist $x_0 \in [0, 1]$ and $x_1 \in T(x_0)$, then we have $\alpha(x_0, x_1) \geq 1$. Now let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and x_n convergence to $x \in X$. By the definition of the function α, we have $x_n \in [0, 1]$ for all $n \in \mathbb{N}$. Therefore $x \in [0, 1]$ and so $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Thus, all the hypothesis of Theorem 2.4 are satisfied and T has infinitely many fixed points.

Example 2.7. *Let $X = [0, 1]$ with usual norm and $p(x, y) = x + y$, for all $x, y \in X$. Clearly, p is a ω-distance on (X, d). Let $T : X \to \mathcal{K}(X)$ be defined by*

$$T(x) = \begin{cases} \frac{1}{2}x^2 & x \in [0, 1] \setminus \{\frac{1}{2}\}, \\ \\ \{\frac{1}{4}, \frac{1}{8}\} & x = \frac{1}{2}. \end{cases} \tag{7}$$

Also we define $\alpha : X \times X \to [0, \infty)$ as $\alpha(x, y) = 1$. It is clear that T is an α-admissible mapping. Let $\psi : [0, \infty) \to [0, \infty)$ be the function $\psi(t) = \frac{1}{2}t$ for $t \in [0, \infty)$. Now, we show that T is an (α, ψ, p)-contractive type mapping. If $x, y \neq \frac{1}{2}$ then

$$\begin{aligned} H_p(Tx, Ty) &= p(Tx, Ty) = p(\frac{1}{2}x^2, \frac{1}{2}y^2) \\ &= \frac{1}{2}(x^2 + y^2) \leq \frac{1}{2}(x + y) \\ &= \psi(p(x, y)). \end{aligned}$$

If $x = \frac{1}{2}$ and $y \neq \frac{1}{2}$, then

$$D_p(\frac{1}{4}, Ty) = p(\frac{1}{4}, \frac{1}{2}y^2) = \frac{1}{2}(\frac{1}{2} + y^2) \leq \frac{1}{2}(\frac{1}{2} + y) = \psi(p(\frac{1}{2}, y)),$$

and similarly $D_p(\frac{1}{8}, Ty) \leq \psi(p(\frac{1}{2}, y))$ and hence $\sup_{z\in Tx} D_p(z, Ty) \leq \psi(p(\frac{1}{2}, y))$. Also

$$D_p(\frac{1}{2}y^2, Tx) = \min\{p(\frac{1}{2}y^2, \frac{1}{4}), p(\frac{1}{2}y^2, \frac{1}{8})\} \leq \frac{1}{2}(\frac{1}{2} + y) = \psi(p(\frac{1}{2}, y)),$$

therefore, for each $x, y \in [0, 1]$, we have

$$\alpha(x, y)H_p(Tx, Ty) = H_p(Tx, Ty) \leq \psi(p(x, y)).$$

Thus, T is an (α, ψ, p)-contractive type mapping.
Finally for $u \notin T(u)$, that is, for $u \in (0, 1]$

$$\begin{aligned} \inf\{p(x, u) + D_p(x, T(x)) : x \in [0, 1]\} &\geq \inf\{p(x, u) : x \in [0, 1]\} \\ &= \inf\{x + u : x \in [0, 1]\} \\ &> u > 0. \end{aligned}$$

Therefore, all the hypothesis of Theorem 2.4 are satisfied and so $u = 0$ is a fixed point of T.

3. Application to Differential Inclusion

In this section, we extend results obtained by Ahmad and Ntouyas [4] for the existence of solution of nonlinear fractional differential inclusion with non-convex valued right hand side. First, we provide some definition and preliminaries that are required in this section.

Definition 3.1. *A set-valued mapping $F : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Caratheodory mapping provided that*

(i) $F(t, .) : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is upper semicontinuous for a.e. $t \in [a, b]$; and

(ii) $F(., x) : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is measurable for every $x \in \mathbb{R}$.

Theorem 3.2. *(Kuratowskiĭ Ryll-Nardzewski selection theorem)[40] Let $(\Omega, \mathcal{A})$ be a measurable space and Y be a separable complete space. Suppose that $F : \Omega \to \mathcal{CB}(Y)$ is a set-valued mapping. If F is measurable, then it has a measurable selection.*

For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order β is defined as

$${}^C D^\beta(g(t)) = \frac{1}{\Gamma(n - \beta)} \int_0^t (t - s)^{n-\beta-1} g^{(n)}(s) ds, \qquad (\beta \in (n - 1, n), n = [\beta] + 1),$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is a gamma function.
We consider nonlinear fractional differential inclusion:

$${}^C D^\beta(x(t)) \in F(t, x(t)), \quad t \in J = [0, 1] \text{ and } \beta \in (1, 2] \tag{8}$$

by integral boundary condition

$$x(0) = 0, \;\; x(1) = \int_0^\eta x(s) ds, \quad \eta \in (0, 1)$$

where $x \in C(J,\mathbb{R})$ and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Caratheodory set-valued mapping.
We define set valued mapping $T : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ as

$$\begin{aligned} T(x) &= \{h \in C(J,\mathbb{R}) : \ h(t) = \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)ds \\ &- \tfrac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^t (1-s)^{\beta-1} g(s)ds \\ &+ \tfrac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta (\int_0^s (s-m)^{\beta-1} g(m)dm)ds\}, \end{aligned} \tag{9}$$

where, $g \in S_{F,x} = \{v \in L^1(J,\mathbb{R}) : \ v \in F(t,x(t)) \ \ for \ \ t \in J\}$. Since F is a Caratheodory mapping, the set $S_{F,x}$ is nonempty.

Now, we are in position to prove our main result in this section.

Theorem 3.3. *Let set-valued mapping $F : J \times \mathbb{R} \to \mathcal{K}(\mathbb{R})$ be compact valued. Suppose that $F(.,x) : J \to \mathcal{K}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$ and there exists $\psi \in \Psi$ and function $l \in L^1(J,[0,\infty))$ such that*

$$H(F(t,x),F(t,y)) \le l(t)\psi(|x-y|) \tag{10}$$

for each $t \in J$ and for each $x, y \in \mathbb{R}$. Moreover, $F(t,0) \subset l(t)\overline{B}(0,1)$ for $t \in J$ where $B(0,1)$ is open unit ball in $\mathbb{R}$. Then nonlinear fractional differential inclusion (8) has at least one solution if

$$M = (\frac{\|l\|}{\Gamma(\beta)} \sup_{t\in J}(\int_0^t |t-s|^{\beta-1} ds + \frac{2t}{(2-\eta^2)} \int_0^1 |1-s|^{\beta-1} ds + \frac{2t}{(2-\eta^2)} \int_0^\eta \int_0^s |s-m|^{\beta-1} dmds)) \le 1.$$

Proof. Let T be defined as (9). It is easy to check, fixed points of T are solutions of problem (8). Suppose that $x_1, x_2 \in C(J,\mathbb{R})$ and $h_1 \in T(x_1)$ so there exists $g_1 \in S_{F,x_1}$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_1(s)ds \\ &- \tfrac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} g_1(s)ds \\ &+ \tfrac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta (\int_0^s (s-m)^{\beta-1} g_1(m)dm)ds. \end{aligned} \tag{11}$$

From inequality (10), for each $t \in J$, we have

$$H(F(t,x_1(t)),F(t,x_2(t))) \le l(t)\psi(|x_1(t)-x_2(t)|).$$

Since set-valued mapping F is compact valued then there is $w(t) \in F(t,x_2(t))$ such that for each $t \in J$,

$$|g_1(t) - w(t)| \le l(t)\psi(|x_1(t)-x_2(t)|).$$

We define set-valued mapping $K : [0,1] \to \mathcal{P}(\mathbb{R})$ as

$$K(t) = \{w \in \mathbb{R} : \ |g_1(t) - w| \le l(t)\psi(|x_1(t)-x_2(t)|)\},$$

for each $t \in J$. Since g_1 is measurable, K is measurable. It follows from Proposition 19.3 in [24] that set-valued mapping $G(t) = K(t) \cap F(t,x_2(t))$ is measurable. Hence, by the Kuratowskiï Ryll-Nardzewski selection theorem, G has a measurable selection g_2. Therefore, $g_2(t) \in F(t,x_2(t))$ and for each $t \in J$,

$$|g_1(t) - g_2(t)| \le l(t)\psi(|x_1(t)-x_2(t)|).$$

Now, we define

$$\begin{aligned} h_2(t) &= \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_2(s)ds \\ &- \tfrac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} g_2(s)ds \\ &+ \tfrac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta (\int_0^s (s-m)^{\beta-1} g_2(m)dm)ds, \end{aligned} \tag{12}$$

so we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &= |\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g_1(s)ds \\
&- \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} g_1(s)ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta (\int_0^s (s-m)^{\beta-1} g_1(m)dm)ds \\
&- \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g_2(s)ds \\
&- \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} g_2(s)ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta (\int_0^s (s-m)^{\beta-1} g_2(m)dm)ds| \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}|g_1(s) - g_2(s)|ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 |1-s|^{\beta-1}|g_1(s) - g_2(s)|ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \int_0^s |s-m|^{\beta-1}|g_1(m) - g_2(m)|dmds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1} l(s)\psi(|x_1(s) - x_2(s)|) \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 |1-s|^{\beta-1} l(s)\psi(|x_1(s) - x_2(s)|) \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta (\int_0^s |s-m|^{\beta-1} l(m)\psi(|x_1(m) - x_2(m)|)dm)ds \\
&\le \psi(\|x_1 - x_2\|_\infty) \times \frac{\|l\|}{\Gamma(\beta)} \sup_{t\in(0,1)} (\int_0^t |t-s|^{\beta-1}ds \\
&+ \frac{2t}{(2-\eta^2)}\int_0^1 |1-s|^{\beta-1}ds + \frac{2t}{(2-\eta^2)}\int_0^\eta \int_0^s |s-m|^{\beta-1}dmds) \\
&\le \psi(\|x_1 - x_2\|_\infty).
\end{aligned}$$

Then

$$\|h_1 - h_2\|_\infty \le \psi(\|x_1 - x_2\|_\infty).$$

By similar way as above and by interchanging the roles of x_1 and x_2, we deduce that $H(T(x_1), T(x_2)) \le \psi(\|x_1 - x_2\|_\infty)$. Then for each $x, y \in X$ we have

$$H(T(x), T(y)) \le \psi(\|x - y\|_\infty). \tag{13}$$

As $\psi \in \Psi$, T is continuous mapping. Now, we consider the function $\alpha : X\times X \to [0, \infty)$ defined by $\alpha(x, y) = 1$. Therefore by inequality (13), we have

$$\alpha(x, y)H(T(x), T(y)) \le \psi(\|x - y\|_\infty),$$

for each $x, y \in X$. Consequently, T is (α, ψ)-contractive type mapping. It can be easily checked that T is α-admissible mapping and there exist points $x_0 \in C(J, \mathbb{R})$ and $y_0 \in T(x_0)$ such that $\alpha(x_0, y_0) \le 1$.

Moreover, since F is compact valued and $F(t,0) \subset l(t)\overline{B}(0,1)$, we can prove that T has compact values too. Thus, Corollary 2.5 can be applied for T and so there exists $x_* \in X$ such that $x_* \in T(x_*)$ and x_* is a solution of problem (8). $\square$

Example 3.4. *Consider the problem*

$$\begin{cases} {}^{C}D^2(x(t)) \in F(t,x(t)) & 0 \le t \le 1, \\ x(0) = 0 & x(1) = \int_0^{\frac{1}{2}} x(s)ds. \end{cases} \tag{14}$$

Also we consider the set-valued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *given by*

$$F(t,x) = [0, \frac{7}{46}(t+2)\sin x + 1].$$

We set $\psi(t) = \frac{1}{2}t$ *and* $l(t) = \frac{7}{23}(t+2)$ *for each* $t \in [0,\infty)$*. Then*

$$H(F(t,x), F(t,\overline{x})) \le l(t)\psi(|x-\overline{x}|)$$

and

$$d(0, F(t,0)) = 0 \le l(t).$$

Therefore, $\|l\| = \frac{21}{23}$ *and* $M = 1$*. Hence by Theorem 3.3 the problem (14) has a solution.*

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