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Quasihyperbolic Quasi-Isometry and Schwarz Lemma of Planar Flat Harmonic Mappings

Qingtian Shi[a], Yi Qi[a]

[a] *School of Mathematics and Systems Science & LMIB, Beihang University, Beijing, 100191, China*

Abstract. A sufficient condition of a flat harmonic quasiconformal mapping to be a quasihyperbolic quasi-isometry on any subdomain of $\mathbb{C}$ is given in this paper, which generalizes the corresponding results of Euclidean and $1/|\omega|^2$ harmonic mappings. As an application, Schwarz lemma of flat harmonic mapping is also investigated. Besides, properties and constructions of flat harmonic mapping are obtained at the same time.

1. Introduction

Let Ω and Ω' be two proper domains of complex plane $\mathbb{C}$ with conformal metric $ds^2 = \sigma(z)|dz|^2$ and $ds^2 = \rho(z)|dz|^2$ respectively. If f is a C^2 mapping from Ω into Ω', then f is said to be harmonic with respect to ρ (denoted ρ-harmonic mapping for short) if it satisfies the Euler-Lagrange equation

$$f_{z\bar{z}}(z) + (\log\rho)_{\omega} \circ f \cdot f_z(z) f_{\bar{z}}(z) = 0 \tag{1.1}$$

on Ω with $\omega = f(z)$. Specially, when ρ equals to a positive constant, then f is called a Euclidean harmonic mapping, which has been conducted extensive studies by many scholars, see [4], [27], [28] for more details.

It is well known that (1.1) holds true if and only if its Hopf differential $\Psi dz^2 = \rho(f) f_z \overline{f_{\bar{z}}} dz^2$ is a holomorphic quadratic differential on Ω. Moreover, Wan proved that f is a ρ-harmonic quasiconformal mapping if and only if its Hopf differential is bounded with respect to the Poincaré metric in [30]. Here, ρ-harmonic quasiconformal mapping on Ω refers to a ρ-harmonic mapping which satisfies those conditions of K-quasiconformal mapping on Ω , that is to say,

$$|f_{\bar{z}}(z)| \le k|f_z(z)| \ \text{ a.e. } \ z \in \Omega, \ \text{ with } k = \frac{K-1}{K+1} \in [0, 1). \tag{1.2}$$

For the theory of quasiconformal mapping, please refers to [2].

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Email address: shiqingtian2013@gmail.com (Qingtian Shi)

It needs to explain that (1.2) holds under the assumption that f is a sense preserving mapping on Ω, i.e. the Jacobian of f satisfies $J_f(z) := |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0$ for all $z \in \Omega$. Since Lewy ([18], [27]) proved that Euclidean harmonic mapping f is locally univalent in Ω if and only if $J_f(z) \neq 0$ for any $z \in \Omega$, which has been generated to an arbitrary ρ-harmonic mapping [25]. Thus, without loss of generality, we assume that f is sense preserving in this paper.

Let $ds^2 = \rho(\omega)|d\omega|^2$ be a conformal metric on $\Omega' \subset \mathbb{C}$. The Gaussian curvature of ρ is given by

$$K(\rho)(\omega) = -\frac{1}{2}\frac{\Delta \log \rho}{\rho}.$$

Distinguishingly, ρ is flat if $K(\rho)(\omega) = 0$, meanwhile f is called to be a flat harmonic mapping on Ω when $f : \Omega \to \Omega'$ is a ρ-harmonic mapping. It shown that $\Delta \log \rho = 0$ implies that $\rho(\omega) = |e^{g(\omega)}|$, where g is a holomorphic function on Ω' in [15]. Thus flat metric ρ is induced by non-vanishing holomorphic function $\varphi(\omega) = e^{g(\omega)}$ for $\omega \in \Omega'$ with $\rho = |\varphi|$. For convenience, flat harmonic mapping f is said to be φ-harmonic mapping. Therefore, (1.1) deduce to

$$f_{z\bar{z}}(z) + \frac{\varphi'(\omega)}{2\varphi(\omega)} f_z(z) f_{\bar{z}}(z) = 0, \quad \omega = f(z) \text{ and } z \in \Omega.$$

Consider the hyperbolic distance and quasihyperbolic distance as follows respectively. Let $\lambda_\Omega|dz|^2$ be the hyperbolic metric of domain $\Omega \subset \mathbb{C}$ with $K(\lambda_\Omega)(z) = -1$. For any given $z \in \Omega$, denote

$$d(z, \partial\Omega) := \inf\{|z - \omega| : \omega \in \partial\Omega\}.$$

If $z_1,\ z_2 \in \Omega$, then the hyperbolic distance and quasihyperbolic distance are defined by

$$d_h(z_1, z_2) = \inf_\gamma \int_\gamma \sqrt{\lambda_\Omega(z)}|dz|, \quad \kappa(z_1, z_2) = \inf_\gamma \int_\gamma \frac{1}{d(z, \partial\Omega)}|dz|.$$

Here, the infimum is taken over all rectifiable curves γ in Ω joining z_1 and z_2. It is well known that there exists a quasihyperbolic geodesic γ_0 in Ω connecting z_1 and z_2 [20]. Notice that when Ω is the upper half plane $\mathbb{H}$, then $d_h(z_1, z_2) = \kappa(z_1, z_2)$. When Ω is a general simply connected domain, then the link between $d_h(z_1, z_2)$ and $\kappa(z_1, z_2)$ is as follows [5], [13]

$$C_2\kappa(z_1, z_2) \leq d_h(z_1, z_2) \leq C_1\kappa(z_1, z_2), \tag{1.3}$$

for all $z_1, z_2 \in \Omega$, where C_1, C_2 are two universal constants.

A mapping f of Ω into Ω' is said to be a hyperbolic Lipschitz continuity if there is a positive constant C_1 such that

$$d_h(f(z_1), f(z_2)) \leq C_1 d_h(z_1, z_2)$$

holds for all $z_1, z_2 \in \Omega$. Moreover, if there exists another positive constant C_2 such that

$$C_2 d_h(z_1, z_2) \leq d_h(f(z_1), f(z_2)) \leq C_1 d_h(z_1, z_2), \tag{1.4}$$

then we say that f is hyperbolic bi-Lipschitz continuity or hyperbolic quasi-isometry on Ω. Researches on the hyperbolic quasi-isometry of quasiconformal mapping see [12] for more details.

Martio [19] first investigated the Euclidean Lipschitz and bi-Lipschitz continuities of Euclidean harmonic quasiconformal mappings. In 1992, Wan [30] further studies on hyperbolic quasi-isometry and finds that every Euclidean (hyperbolic) harmonic quasiconformal diffeomorphism of $\mathbb{D}$ onto itself is hyperbolic quasi-isometry. Later, Knežvić and Mateljević [17] retrieve Wan's result by using Ahlfors-Schwarz lemma and prove that every Euclidean harmonic quasiconformal mapping of $\mathbb{H}$ onto itself is also a hyperbolic quasi-isometry. Chen and Fang generated Wan's result to any convex domain in [8]. In 2014, Mateljević [24] further generated Wan's result to any simply connected proper domain for Euclidean harmonic mapping, see [22], [23], [24] for more details.

For a ρ-harmonic quasiconformal mapping, as the metric ρ and the characteristic of image domain are associated with its hyperbolic quasi-isometry, thus studying on hyperbolic bi-Lipschitz continuity is much more complicated relative to Euclidean case. By building a differential equation for hyperbolic metric of an angular range, Chen [6] obtained hyperbolic quasi-isometry of $1/|\omega|^2$-harmonic quasiconformal mapping from $\mathbb{D}$ onto an angular. We find that Chen's result can be generated to any simply connected domain in [29]. Besides, the hyperbolic bi-Lipschitz continuity for a class of ρ-harmonic quasiconformal mapping from $\mathbb{D}$ to a strongly hyperbolically convex ranges have been studied in [9]. In 2015, a sufficient condition of ρ-harmonic quasiconformal mapping to be a hyperbolic quasi-isometry is given as follows.

Theorem A.[10] *Let f be a ρ-harmonic K-quasiconformal mapping of $\mathbb{D}$ onto a simply connected domain Ω. If the pair of metric densities ρ and λ_Ω defined on Ω satisfies the inequality*

$$\frac{\left|(\log \lambda_\Omega)_{\omega\omega} - 2(\log \lambda_\Omega)_\omega (\log\rho)_\omega - (\log\rho)_{\omega\omega} + 2(\log\rho)^2_\omega\right| + \left|(\log\rho)_{\omega\overline{\omega}}\right|}{(\lambda_\Omega)^2} \le 1 \tag{1.5}$$

then f is hyperbolic K-Lipschitz. If f also satisfies that

$$\lambda_\Omega |f_z| \to +\infty \ as \ |z| \to 1^-, \tag{1.6}$$

then f is hyperbolic $(1/K, K)$-bi-Lipschitz.

Note that (1.5) and (1.6) depend on the metric ρ and λ_Ω which are not easy to verify. Recall that Euclidean and $1/|\omega|^2$ harmonic mappings are φ-harmonic mapping, thus it is natural to investigate the hyperbolic quasi-isometry and quasihyperbolic quasi-isometry of φ-harmonic quasiconformal mappings.

In 2006, Kalaj and Mateljević prove that every φ-harmonic mapping F can be decomposed into the form as $F = \phi \circ f$, where ϕ is a conformal mapping and f is a Euclidean harmonic mapping. Thus *every φ-harmonic quasiconformal mapping is hyperbolic quasi-isometry in any simply connected domain*. It shows that (1.5) and (1.6) are not necessary. Meanwhile, we find that φ-harmonic quasiconformal mapping is quasihyperbolic quasi-isometry if it satisfies the equation (1.7) as follows.

Theorem 1. *For any two subdomains Ω and Ω' of the complex plane $\mathbb{C}$. Let φ be an non-vanishing analytic function on Ω' and $\omega = f(z)$ be a φ-harmonic K-quasiconformal mapping from Ω onto Ω'. If φ satisfies*

$$\Re\left\{\left[\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)' - \frac{1}{2}\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)^2\right] f_z f_{\overline{z}}\right\} = 0, \quad \omega \in \Omega'. \tag{1.7}$$

Then for any $z_1, z_2 \in \Omega$, there exists a positive constant c such that

$$\frac{1}{c}\,\kappa_\Omega(z_1, z_2) \le \kappa_{\Omega'}(f(z_1), f(z_2)) \le c\,\kappa_\Omega(z_1, z_2). \tag{1.8}$$

In addition, if φ satisfies

$$\Re\left\{\left[\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)' - \frac{1}{2}\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)^2\right] f_z f_{\overline{z}}\right\} \le 0, \quad \omega \in \Omega'. \tag{1.7a}$$

Then there exists a positive constant c such that

$$\kappa_{\Omega'}(f(z_1), f(z_2)) \le c\,\kappa_\Omega(z_1, z_2). \tag{1.8a}$$

If φ satisfies

$$\Re\left\{\left[\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)' - \frac{1}{2}\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)^2\right] f_z f_{\overline{z}}\right\} \ge 0, \quad \omega \in \Omega'. \tag{1.7b}$$

Then there exists a positive constant c such that

$$\kappa_{\Omega'}(f(z_1), f(z_2)) \ge c\,\kappa_\Omega(z_1, z_2). \tag{1.8b}$$

Notice that Euclidean and $1/|\omega|^2$ harmonic mappings are satisfy the equation (1.7), thus Theorem 1 generalizes the corresponding results in [24] and [29]. In addition, Example 4.1 shows that (1.7) is not necessary and Example 4.2 implies that there exists a φ which is not the derivative of a Möbius transformation such that (1.7a) holds true but we don't know if the solutions of the equation (1.7) are only the derivative of a Möbius transformation.

In section 3, the Schwarz lemma of φ-harmonic mapping is investigated. Recall that the classical Schwarz lemma states that every analytic function f from $\mathbb{D}$ into itself has

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{1.9}$$

In 1989, Colonna [11] established an analogue Schwarz lemma for planar Euclidean harmonic mapping. The corresponding case of planar harmonic quasiregular mapping is obtained by Knežević and Mateljević as follows.

Theorem B.[17] *Let f be a K-quasiregular Euclidean harmonic mapping from $\mathbb{D}$ into itself, then for all $z \in \mathbb{D}$ we have*

$$|f_z(z)| + |f_{\bar{z}}(z)| \leq K\frac{1-|f(z)|^2}{1-|z|^2}. \tag{1.10}$$

It is not difficult to find that (1.9) is the special case of (1.10), as K-quasiregular Euclidean harmonic mapping degenerates to an analytic function when $K = 1$. Recently, we prove that (1.10) holds for $1/|\omega|^2$-harmonic K-quasiregular mapping in [29]. In the second part of this paper, We want to investigate the question that can the inequality (1.10) holds for φ-harmonic quasireguar mapping? We partially answer this question as follows.

Theorem 2. *Let φ be an non-vanishing analytic function on $\mathbb{D}$ and $\omega = f(z)$ which maps from $\mathbb{D}$ into itself be a φ-harmonic K-quasiregular mapping. If φ satisfies (1.7) and*

$$\left|\omega\varphi'(\omega)\right| \leq 2|\varphi(\omega)|. \tag{1.11}$$

Then

$$\Lambda_f(z) := |f_z(z)| + |f_{\bar{z}}(z)| \leq K\frac{1-|f(z)|^2}{1-|z|^2} \quad z \in \mathbb{D}. \tag{1.12}$$

Notice that Theorem 2 also contains the corresponding results of Euclidean and $1/|\omega|^2$-harmonic mapping and we find that (1.11) is not necessary through Example 4.1. The related researches of Schwarz lemma refer to [7], [14], [16] for more details.

2. The Proof of Theorem 1

Let $\omega = f(z)$ be a sense preserving mapping defined on $\Omega \subset \mathbb{C}$ with $f \neq 0$, then f is said to be a logharmonic mapping if there exists an analytic function $a(z)$ with $|a(z)| < 1$ such that

$$\overline{f_{\bar{z}}(z)} = a(z)\frac{\overline{f(z)}}{f(z)}f_z(z), \tag{2.1}$$

holds for all $z \in \Omega$. It is proved that f is a logharmonic mapping if and only if f is a $1/|\omega|^2$-harmonic mapping in [6] and [29]. The following conclusion shows that (2.1) can be generalized to arbitrary φ-harmonic mapping.

Proposition 2.1. *Let $\omega = f(z) \in C^2$ be a sense preserving mapping from domain Ω onto Ω' and φ be an non-vanishing analytic function defined on Ω'. Then f is a φ-harmonic mapping if and only if there exists an analytic function $a(z)$ with $|a(z)| < 1$ such that*

$$\overline{f_{\bar{z}}^2(z)} = a(z)\frac{\varphi(\omega)}{\overline{\varphi(\omega)}} f_z^2(z) \tag{2.2}$$

holds for all $z \in \Omega$.

Proof First, we prove the necessary part of this statement. Assume that $f(z)$ is a φ-harmonic mapping defined on Ω. Then we have

$$f_{z\bar{z}}(z) + \frac{\varphi'(\omega)}{2\varphi(\omega)} f_z(z) f_{\bar{z}}(z) = 0 \quad z \in \Omega.$$

Since $\varphi(\omega)$ is analytic on Ω', then we get

$$\frac{\partial}{\partial \bar{z}}\left(\varphi(\omega) f_z^2(z)\right) = f_z(z)\Big[2\varphi(\omega) f_{z\bar{z}}(z) + \varphi'(\omega) f_z(z) f_{\bar{z}}(z)\Big] = 0, \tag{2.3}$$

and

$$\frac{\partial}{\partial z}\left(\varphi(\omega) f_{\bar{z}}^2(z)\right) = f_{\bar{z}}(z)\Big[2\varphi(\omega) f_{z\bar{z}}(z) + \varphi'(\omega) f_z(z) f_{\bar{z}}(z)\Big] = 0. \tag{2.4}$$

Set

$$h(z) = \varphi(\omega) f_z^2(z) \text{ and } g(z) = \varphi(\omega) f_{\bar{z}}^2(z) \ \ z \in \Omega.$$

From the equalities (2.3) and (2.4), $h(z)$ and $\overline{g(z)}$ are analytic on Ω and $h(z) \neq 0$ (since $f(z)$ is sense preserving on Ω). Then

$$a(z) := \frac{\overline{g(z)}}{h(z)} = \frac{\overline{\varphi(\omega)}}{\varphi(\omega)} \frac{\overline{f_{\bar{z}}^2(z)}}{f_z^2(z)}$$

is also analytic on Ω and $|a(z)| < 1$. That is to say, there exists an analytic function $a(z)$ with $|a(z)| < 1$ such that (2.2) holds true for all $z \in \Omega$.

Conversely, if there is an analytic function $a(z)$ with $|a(z)| < 1$ such that (2.2) holds for all $z \in \Omega$, then differentiate with respect to $\bar{z}$ on both side of (2.2), we get that

$$\overline{\varphi'(\omega) f_z f_{\bar{z}}^2 + 2\varphi(\omega) f_{\bar{z}} f_{z\bar{z}}} = a(z)\varphi'(\omega) f_z^2 f_{\bar{z}} + 2a(z)\varphi(\omega) f_z f_{z\bar{z}},$$

which implies that

$$|f_{\bar{z}}| \left|\varphi'(\omega) f_z f_{\bar{z}} + 2\varphi(\omega) f_{z\bar{z}}\right| = |a(z)||f_z| \left|\varphi'(\omega) f_z f_{\bar{z}} + 2\varphi(\omega) f_{z\bar{z}}\right|. \tag{2.5}$$

Denoted $E = \{z \in \Omega : f_{\bar{z}}(z) = 0\}$ and $E^c = \Omega - E$. Then E is a discrete set, since $a(z)$ is analytic on Ω.

Case 1 If $z \in E$, it is obvious that $\varphi' f_z f_{\bar{z}} + 2\varphi f_{z\bar{z}} = 0$ holds for $z \in E$.

Case 2 If $z \in E^c$, we obtain that $\varphi' f_z f_{\bar{z}} + 2\varphi f_{z\bar{z}} = 0$ holds true for all $z \in E^c$ from (2.5).

Combined with the above analysis, we conclude that $2\varphi f_{z\bar{z}} + \varphi' f_z f_{\bar{z}} = 0$ holds for all $z \in \Omega$, which implies that f is a φ-harmonic mapping on Ω. The proof of this Proposition is finished. □

Notice that φf_z^2 and $\overline{\varphi f_{\bar{z}}^2}$ are analytic on Ω is consistent with Lemma 1 and Lemma 2 in [25].

Given a sense preserving Euclidean harmonic mapping f on Ω, it is well known that $f \circ \phi$ preserves harmonicity but $\phi \circ f$ may not, if ϕ is analytic on the corresponding domains. However, Kalaj and Mateljević [15] prove that if ϕ is a conformal mapping, then $\phi \circ f$ is a φ-harmonic mapping. Meanwhile, they also find that every φ-harmonic mappings can be decomposed into the form as $f = \phi \circ f_1$, where ϕ is a conformal mapping and f_1 is a Euclidean harmonic mapping. Applying Proposition 2.1, this conclusion also can be obtained easily.

Corollary 2.2. *For domains $D,\ \Omega,\ \Omega' \subset \mathbb{C}$. Let f be an sense preserving Euclidean harmonic mapping of D onto Ω and let g be a conformal mapping of Ω onto Ω'. Then $F = g \circ f$ is a φ-harmonic mapping, where*

$$\varphi(\omega) = 1/\left(g' \circ g^{-1}(\omega)\right)^2 \quad \omega \in \Omega'. \tag{2.6}$$

Proof Differential on the both sides of $\omega = F = g \circ f$ on z and $\overline{z}$ respectively, we have

$$F_z = g' \cdot f_z, \quad F_{\overline{z}} = g' \cdot f_{\overline{z}}.$$

Therefore,

$$\frac{\overline{F_{\overline{z}}^2}}{F_z^2} = a(z)^2 \frac{\overline{g'^2}}{g'^2},$$

where $a(z) = \frac{\overline{f_{\overline{z}}}}{f_z}$ is analytic and $|a(z)| < 1$ in Ω since f is a Euclidean harmonic mapping. Let $\varphi(\omega) = 1/g'(\xi)^2$ with $\omega = g(\xi)$ and $\xi = f(z)$, then it easy to get that F is a φ-harmonic mapping from Proposition 2.1. □

Corollary 2.3. *If F is a φ-harmonic mapping from D onto Ω', then there exist an Euclidean harmonic mapping f which maps from D onto $f(D) = \Omega$ and a conformal mapping g which maps Ω onto Ω' such that $F = g \circ f$.*

Proof Let $\varphi = \varphi_1^2$. As F is a φ-harmonic mapping on D, then there is an analytic function $a(z)$ with $|a(z)| < 1$ such that

$$\frac{\overline{F_{\overline{z}}}}{F_z} = a(z)\frac{\varphi_1}{\overline{\varphi_1}}.$$

Let g_1 be a conformal mapping on Ω' satisfying $g_1'(\omega) = \varphi_1(\omega)$. Consider the composite function $f = g_1 \circ F : D \to g_1(\Omega') = \Omega$. Then we have

$$f_z = g_1' F_z, \quad f_{\overline{z}} = g_1' F_{\overline{z}}.$$

Therefore,

$$\frac{\overline{f_{\overline{z}}}}{f_z} = \frac{\overline{\varphi_1}}{\varphi_1}\frac{\overline{F_{\overline{z}}}}{F_z} = a(z)$$

holds for all $z \in D$, which implies that f is an Euclidean harmonic mapping by differentiating on the both sides of above equation on $\overline{z}$. Let $g = g_1^{-1}$ and $f = g_1 \circ F$, then the proof of this corollary is complete. □

In order to proof Theorem 1, the following two lemmas should be given at first.

Lemma 2.4. *Let $\omega = f(z)$ be a φ-harmonic mapping from Ω onto Ω'. Then $\log |f_z(z)|$ is a real-valued harmonic mapping on Ω if and only if φ satisfies the equality (1.7) for all $\omega \in \Omega'$.*

Proof Based on the proof of the necessary part of Proposition 2.1, we obtain that $\varphi(\omega) f_z^2(z)$ is analytic about the variable z on Ω, which deduce that

$$\Delta_z \log |\varphi(\omega)| + \Delta_z \log |f_z(z)|^2 = 0, \tag{2.7}$$

where $\Delta_z := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ z = x + iy \in \Omega$.

Since

$$\begin{aligned}\Delta_z \log |\varphi(\omega)| &= 2\left(\frac{\varphi'}{\varphi} f_z + \frac{\overline{\varphi'}}{\overline{\varphi}} \overline{f_{\overline{z}}}\right)_{\overline{z}} \\ &= 2\left[\frac{\varphi''\varphi - \varphi'^2}{\varphi^2} f_z f_{\overline{z}} + \frac{\overline{\varphi''}\,\overline{\varphi} - \overline{\varphi'}^2}{\overline{\varphi}^2} \overline{f_z f_{\overline{z}}} + \frac{\varphi'}{\varphi} f_{z\overline{z}} + \frac{\overline{\varphi'}}{\overline{\varphi}} \overline{f_{z\overline{z}}}\right],\end{aligned} \tag{2.8}$$

substitute in (2.8) with the equation $f_{z\overline{z}} = -\frac{\varphi'}{2\varphi} f_z f_{\overline{z}}$ to find

$$\Delta_z \log |\varphi(\omega)| = 4\Re\left\{\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2\right] f_z f_{\overline{z}}\right\}. \tag{2.9}$$

Together with (2.7) and (2.9),

$$\Delta_z \log|f_z(z)| = 0 \Longleftrightarrow \Re\left\{\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2\right] f_z f_{\bar{z}}\right\} = 0,$$

Therefore, we finished the proof of this Lemma. □

Note that if φ satisfies

$$\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 = 0, \tag{2.10}$$

then the relation (1.7) obviously holds true. Set $\Phi(\omega) = \int_0^\omega \varphi(\omega)d\omega$, then $\Phi(\omega)$ is analytic on Ω. From the relation (2.10), we obtain that $S_\Phi(\omega) = 0$ for all $\omega \in \Omega'$. Here S_Φ is the Schwarzian derivative of Φ. Thus,

$$\int_0^\omega \varphi(\omega)d\omega = \Phi(\omega) = \frac{A\omega + B}{C\omega + D}, \tag{2.11}$$

where $A, B, C, D \in \mathbb{C}$ are constants which satisfy $AD - BC \neq 0$. Differentiate with respect to ω on both side of (2.11) to find

$$\varphi(\omega) = \frac{AD - BC}{(C\omega + D)^2}, \quad \omega \in \Omega'.$$

Thus by the remark of Proposition 2.1, Lemma 2.4 implies the following.

Corollary 2.5 *For $C, D \in \mathbb{C}$ with $|C| + |D| \neq 0$. Let $\varphi(\omega) = \frac{1}{(C\omega+D)^2}$ and $\omega = f(z)$ be a φ-harmonic mapping from Ω onto Ω', then $\Delta_z \log|f_z(z)| = 0$ holds for all $z \in \Omega$.*

Lemma 2.6 (Astala-Gehring) [3] *Suppose that D and D' are domains in $\mathbb{R}^n$ $(n \geq 2)$, if $f : D \to D'$ is a K-quasiconformal mapping, then there exists a positive constant $c := c(K, n)$ such that*

$$\frac{1}{c}\frac{d\,(f(z), \partial D')}{d\,(z,\ \partial D)} \leq \alpha_{f,D}(z) \leq c\frac{d\,(f(z),\ \partial D')}{d\,(z,\ \partial D)}.$$

where

$$\alpha_{f,D}(x) = \exp\left\{\frac{1}{n|\mathrm{B}_x|}\int_{\mathrm{B}_x} \log J_f(z)dz\right\} \tag{2.12}$$

for all $x \in D$, here $\mathrm{B}_x := \mathrm{B}(x, d(x, \partial D))$ is a ball and $|\mathrm{B}_x|$ stands for the volume of the ball B_x.

Proof of Theorem 1 As f is a sense preserving φ-harmonic mapping on Ω, by Proposition 2.1, there exists an analytic function $a(z)$ with $|a(z)| < 1$ on Ω such that (2.2) holds on Ω. So the Jacobian $J_f(z)$ of $f(z)$ can be represented as

$$J_f(z) = \Lambda_f(z)\lambda_f(z) = (1 - |a(z)|)|f_z(z)|^2, \ z \in \Omega,$$

where

$$\Lambda_f(z) = |f_z(z)| + |f_{\bar{z}}(z)|, \quad \lambda_f(z) = |f_z(z)| - |f_{\bar{z}}(z)|.$$

Thus, the quantity $\alpha_{f,\Omega}(z)$ defined in (2.12) has the following form

$$\alpha_{f,\Omega}(z) = \exp\left\{\frac{1}{2|\mathrm{B}(z,r)|}\int_{\mathrm{B}(z,r)} \left[\log(1 - |a(\xi)|) + \log|f_\xi(\xi)|^2\right] dxdy\right\}, \tag{2.13}$$

where $\xi = x + iy$. Since $\varphi(\omega)$ satisfies the condition (1.7), using Lemma 2.4, then $\log|f_z(z)|$ is a real-valued harmonic function on Ω. Therefore, by mean value theorem,

$$\log|f_z(z)| = \frac{1}{2|\mathrm{B}(z,r)|}\int_{\mathrm{B}(z,r)} \log|f_\xi(\xi)|^2 dxdy \tag{2.14}$$

holds for every $z \in \Omega$ and every disk $B(z,r) \subset \Omega$ centered at z with radius r. Together with (2.13) and (2.14), for $z \in \Omega$

$$\alpha_{f,\Omega}(z) = |f_z(z)| \exp\left\{\frac{1}{2|B(z,r)|}\int_{B(z,r)} \log\left(1 - |a(\xi)|\right) dxdy\right\}. \tag{2.15}$$

Since $f(z)$ is a K-quasicomformal mapping, by (2.2) we have

$$|a(z)| \le k^2 = \left(\frac{K-1}{K+1}\right)^2 < 1$$

Therefore,

$$\sqrt{1-k^2}|f_z(z)| \le \alpha_{f,\Omega}(z) \le |f_z(z)|, \ \ z \in \Omega,$$

which derived directly

$$\frac{1}{\sqrt{K}}\Lambda_f(z) \le \alpha_{f,\Omega}(z) \le \frac{K+1}{2}\lambda_f(z), \ \ z \in \Omega. \tag{2.16}$$

Applying Lemma 2.6, (2.16) implies

$$C_2\Lambda_f(z) \le \frac{d(f(z),\ \partial\Omega')}{d(z,\ \partial\Omega)} \le C_1\lambda_f(z), \ \ z \in \Omega,$$

here $C_1 = \frac{K+1}{2}c$, $C_2 = \frac{1}{\sqrt{K}}c$ and c is the constant appeared in Lemma 2.6.

For any z_1 and $z_2 \in \Omega$. By a result of [20], there exists a quasihyperbolic geodesic γ_0 in Ω connecting z_1 and z_2. So

$$\begin{aligned}
\kappa_\Omega(z_1,z_2) = \int_{\gamma_0} \frac{1}{d(z,\partial\Omega)}|dz| &\ge \int_{\gamma_0} \frac{1}{d(z,\partial\Omega)}\frac{1}{\Lambda_f(z)}\left|f_z dz + f_{\bar{z}} d\bar{z}\right| \\
&\ge \int_{f(\gamma_0)} \frac{C_2}{d(\omega,\partial\Omega')}|d\omega| \\
&\ge C_2\kappa_{\Omega'}\left(f(z_1), f(z_2)\right).
\end{aligned} \tag{2.17}$$

Similarly, there exists a geodesic γ_0' in Ω' joining $f(z_1)$ and $f(z_2)$, and thus

$$\begin{aligned}
\kappa_{\Omega'}(f(z_1), f(z_2)) = \int_{\gamma_0'} \frac{1}{d(w,\partial\Omega')}|dw| &\ge \int_{f^{-1}(\gamma_0')} \frac{1}{d(f(z),\ \partial\Omega')}\lambda_f(z)|dz| \\
&\ge \int_{f^{-1}(\gamma_0)} \frac{1}{C_1 d\left(z,\partial\Omega\right)}|dz| \\
&\ge \frac{1}{C_1}\kappa_\Omega(z_1,z_2).
\end{aligned} \tag{2.18}$$

Therefore, (1.8) comes from (2.17) and (2.18). Analogously, (1.8a) and (1.8b) can be deduced from the properties of subharmonic and superharmonic functions respectively. Hence, the proof of Theorem 3.1 is complete. □

Remark: Example 4.1 shows that there exists a φ_0-harmonic quasiconformal mapping F_0 such that φ_0 is not the solution of the equation (1.7) but F_0 is bi-Lipschitz with respect to hyperbolic metric, therefore F_0 is quasihyperbolic quasi-isometry on $\mathbb{D}$. That is to say, (1.7) is not necessary in Theorem 1.

3. Schwarz Lemma of Flat Harmonic Quasiregular Mapping

To prove Theorem 2, the following Ahlfors-Schwarz lemma is needed.

Lemma 3.1. [1] *Let $\rho(z)$ be the density of a Riemann metric of the unit disk $\mathbb{D}$ with its Gaussian curvature $K(\rho)(z) \leq -1$. Then $\rho(z) \leq \lambda(z)$ for all $z \in \mathbb{D}$, where $\lambda(z)$ is the hyperbolic density of $\mathbb{D}$ with $K(\lambda)(z) = -1$.*

Proof of Theorem 2 Assume that $\omega = f(z)$ is a φ-harmonic K-quasiregular mapping which maps the unit disk $\mathbb{D}$ into itself. Without loss of generality, f is sense preserving, i.e. $|f_z(z)| > |f_{\bar{z}}(z)|$ for all $z \in \mathbb{D}$. Consequently, there exists an analytic function $a(z)$ with $|a(z)| < 1$ such that (2.2) holds in $\mathbb{D}$ and $\varphi(\omega)f_z^2(z)$ is an analytic function about the variable z from Proposition 2.1. Furthermore, $|a(z)| \leq k^2 = (\frac{K-1}{K+1})^2$, since f is a K-quasiregular mapping on $\mathbb{D}$. Let

$$\sigma(z) = (1-k)^2 \lambda(f(z))|f_z(z)|^2 \quad z \in \mathbb{D}, \tag{3.1}$$

where

$$\lambda(z) = \frac{4}{(1-|z|^2)^2} \quad z \in \mathbb{D}. \tag{3.2}$$

Through a series of calculations, we obtain

$$\begin{aligned}\Delta \log \sigma(z) &= \Delta \log \lambda(f(z)) - \Delta_z \log |\varphi(\omega)| \\ &= \frac{8}{(1-|f|^2)^2}\left\{|f_z|^2 + |f_{\bar{z}}|^2 + 2\Re f_z f_{\bar{z}} \bar{f}\left[\bar{f} - (1-|f|^2)\frac{\varphi'}{2\varphi}\right]\right\} \\ &\quad - 4\Re\left\{\left[\left(\frac{\varphi'}{\varphi}\right) - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2\right] f_z f_{\bar{z}}\right\}\end{aligned} \tag{3.3}$$

Thus, if φ satisfies the equality (1.7), then (3.3) becomes to the following form

$$\Delta \log \sigma(z) = \frac{8}{(1-|f|^2)^2}\left\{|f_z|^2 + |f_{\bar{z}}|^2 + 2\Re f_z f_{\bar{z}} \bar{f}\left[\bar{f} - (1-|f|^2)\frac{\varphi'}{2\varphi}\right]\right\}. \tag{3.4}$$

Moreover, when $|\omega\varphi'| \leq 2|\varphi|$ for all $\omega = f(z) \in \mathbb{D}$ and $f \neq 0$ on $\mathbb{D}$, then from (3.4) we get that

$$\begin{aligned}\Delta \log \sigma(z) &= \frac{8}{(1-|f|^2)^2}\left\{|f_z|^2 + |f_{\bar{z}}|^2 + 2\Re f_z f_{\bar{z}} \frac{\bar{f}}{f}\left[|\omega|^2 - (1-|\omega|^2)\frac{\omega\varphi'}{2\varphi}\right]\right\} \\ &\geq \frac{8|f_z|^2}{(1-|f|^2)^2}\left\{1 + |a(z)| - 2|a(z)|^{1/2}\right\} \\ &\geq \frac{8(1-k)^2|f_z|^2}{(1-|f|^2)^2} = 2\sigma(z)\end{aligned}$$

It is not difficult to find that $\Delta \log \sigma(z) \geq 2\sigma(z)$ also holds true when $f = 0$ on $\mathbb{D}$ from (3.4). That is to say, we prove that

$$K(\sigma)(z) \leq -1, \quad z \in \mathbb{D}. \tag{3.5}$$

which deduced to the fact that

$$\sigma(z) \leq \lambda(z) \quad z \in \mathbb{D}, \tag{3.6}$$

by Lemma 3.1. Together with (3.1), (3.2) and (3.6) to get that

$$|f_z(z)| \leq \frac{1}{1-k}\frac{1-|f|^2}{1-|z|^2} \quad z \in \mathbb{D},$$

which directly implies that

$$|f_z(z)| + |f_{\bar{z}}(z)| \leq K\frac{1-|f(z)|^2}{1-|z|^2} \quad z \in \mathbb{D}.$$

□

From Theorem 2 the following corollary is obtained, which illustrate the fact that the conditions (1.7) and (1.11) are non empty in Theorem 2.

Corollary 3.2 *Given $C, D \in \mathbb{C}$ with $|C| + |D| \neq 0$. Let $\varphi(\omega) = \frac{1}{(C\omega+D)^2}$ on $\mathbb{D}$ and $\omega = f(z)$ be a φ-harmonic K-quasiregular mapping of $\mathbb{D}$ into $\mathbb{D}$. If $D = 0$ or $|D| \geq 2|C|$, then (1.12) holds for all $z \in \mathbb{D}$.*

Notice that when $D = 0$ and $C \neq 0$, f is a $1/|\omega|^2$-harmonic mapping. When $C = 0$ and $D \neq 0$, f is a Euclidean harmonic mapping. Thus, Theorem 2 improves the relative results of Euclidean and $1/|\omega|^2$ harmonic quasiregular mapping. Meanwhile, Example 4.1 shows that (1.11) is also not necessary.

4. Auxiliary Example

In this section, we first give a special φ-harmonic mapping which shows that the relations (1.7) and (1.11) are not necessary in Theorem 1 and Theorem 2 respectively.

Example 4.1 Consider the function

$$\omega = f(z) = \frac{(K+1)|z|^2 - 2z - (K-1)}{(K-1)|z|^2 + 2\bar{z} - (K+1)} : \mathbb{D} \xrightarrow{\text{onto}} \mathbb{D}, \tag{4.1}$$

which is the composite mapping $f = \psi \circ A_K \circ \phi$, where ψ, A_K and ϕ are defined as

$$\psi(z) = \frac{z-i}{z+i}, \quad A_K(z) = x + iKy, \quad \phi(z) = i\frac{1+z}{1-z}, \quad z = x + iy.$$

Thus it is not difficult to obtain that $f|_{\partial\mathbb{D}} = id$ and

$$\begin{aligned} f_z(z) &= \frac{1+K}{2}\psi'\left(A_K(\phi)\right)\phi'(z), \\ f_{\bar{z}}(z) &= \frac{1-K}{2}\psi'\left(A_K(\phi)\right)\overline{\phi'(z)}. \end{aligned} \quad z \in \mathbb{D} \tag{4.2}$$

Therefore,

$$\left(\frac{\overline{f_{\bar{z}}}}{f_z}\right)^2 = \left(\frac{1-K}{1+K}\right)^2 \frac{\overline{\psi'\left(A_K(\phi)\right)^2}}{\psi'\left(A_K(\phi)\right)^2} = \left(\frac{1-K}{1+K}\right)^2 \frac{\left(A_K(\phi)+i\right)^4}{\left(\overline{A_K(\phi)+i}\right)^4}. \tag{4.3}$$

Set $\varphi(\omega) = \frac{-2}{(1-\omega)^4}$, which deduce to

$$\frac{\varphi(\omega)}{\overline{\varphi(\omega)}} = \frac{(1-\overline{\omega})^4}{(1-\omega)^4} = \frac{\left(A_K(\phi)+i\right)^4}{\left(\overline{A_K(\phi)+i}\right)^4} \quad \omega \in \mathbb{D}. \tag{4.4}$$

Combined (4.3) and (4.4), by applying Proposition 2.1, we obtain that f is a φ-harmonic K-quasiconformal mapping of $\mathbb{D}$ onto itself. In addition, by (2.6), φ can be obtained by the form with

$$\varphi(\omega) = \frac{1}{\left(\psi'(\psi^{-1})\right)^2} \quad \omega \in \mathbb{D}.$$

Recall that f is a hyperbolical quasi-isometry on $\mathbb{D}$ by the fact that Euclidean harmonic mapping is hyperbolical quasi-isometry on simply connected domain. However, from the first equation in (4.2), we have

$$\begin{aligned}\Delta \log |f_z| &= -\Delta \log |A_K(\phi) + i|^2 - \Delta \log |1 - z|^2 \\ &= -(1-K^2)|\phi'(z)| \frac{\left(A_K(\phi) + i\right)^2 + \left(\overline{A_K(\phi)} + i\right)^2}{|A_K(\phi) + i|^4} \\ &\neq 0,\end{aligned}$$

that is to say, (1.7) is not necessary in Theorem 1.

Meanwhile, we find that the partial derivatives of f satisfies

$$\begin{aligned}|f_z(z)| + |f_{\bar z}(z)| &= \frac{2}{|1-z|^2}\frac{2K}{|A_K(\phi)+i|^2} \\ &\leq \frac{2K}{|1-z|^2}\frac{2K}{|A_K(\phi)+i|^2} \\ &\leq \frac{2K}{|A_K(\phi)+i|^2}\frac{i\left(\overline{A_K(\phi)} - A_\phi(\phi)\right)}{1-|z|^2} \\ &= K\frac{1 - \left|\dfrac{A_K(\phi)-i}{A_K(\phi)+i}\right|^2}{1-|z|^2} \\ &= K\frac{1-|f|^2}{1-|z|^2},\end{aligned}$$

for all $z \in \mathbb{D}$, which shows that (1.11) is also not necessary.

Next, the following example shows that there is a φ, which is not the derivative of Möbius transformation, such that (1.7a) holds true. But we are not sure if the solution of (1.7) are just the derivative of certain Möbius transformations in the view of φ.

Example 4.2 Consider the function

$$\omega = f(z) = \frac{1}{2z + \bar z + 2} : D \to f(D), \tag{4.5}$$

where $D = \{z = x + iy : |y| < |2 + 3x|, x \in \mathbb{R}, y \in \mathbb{R}\}$ is a subdomain of $\mathbb{C}$ containing two angular domains.

Direct computation show that

$$f_z(z) = \frac{2}{(2z + \bar z + 2)^2}, \quad f_{\bar z}(z) = \frac{-1}{(2z + \bar z + 2)^2}.$$

Hence, we obtain

$$\frac{\overline{f_{\bar z}^2}}{f_z^2} = \frac{1}{4}\frac{(2z + \bar z + 2)^4}{\left(\overline{2z + \bar z + 2}\right)^4} = \frac{1}{4}\frac{\overline{\omega}^4}{\omega^4},$$

which implies that f is a $1/|\omega|^4$-harmonic mapping according to Proposition 2.1, that is, $\varphi(\omega) = 1/\omega^4$ here, is not a derivative of a Möbius transformation.

Meanwhile, it is not difficult to verify that f is univalent and its Beltrami coefficient is bounded away from 1 on D. Thus f is a φ-harmonic quasiconformal mapping on D. Next, we illustrate the fact that f is not the solution of the equation (1.7).

Since $\varphi(\omega) = \frac{1}{\omega^4}$, then

$$\left[\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)' - \frac{1}{2}\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)^2\right] f_z(z) f_{\bar{z}}(z) = 8\omega^2.$$

Then

$$\Re\left\{\left[\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)' - \frac{1}{2}\left(\frac{\varphi'(\omega)}{\varphi(\omega)}\right)^2\right] f_z f_{\bar{z}}\right\} = 8\Re\left\{\frac{1}{(2z+\bar{z}+2)^2}\right\} = \frac{8}{\left|2z+\bar{z}+2\right|^4}\Re\left\{(2\bar{z}+z+2)^2\right\} > 0$$

by the fact that $z \in D = \{z = x + iy : |y| < |2 + 3x|, \ x \in \mathbb{R}, \ y \in \mathbb{R}\}$. $\Box$

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