Filomat 32:15 (2018), 5385–5402
https://doi.org/10.2298/FIL1815385Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Schwarz Lemma and Boundary Schwarz Lemma for Pluriharmonic Mappings

Jian-Feng Zhu[a,b]

[a]*School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, People's Republic of China*
[b]*Department of Mathematics, Shantou University, Shantou, Guangdong 515063, People's Republic of China*

Abstract. In this paper, we first improve the boundary Schwarz lemma for holomorphic self-mappings of the unit ball $\mathbb{B}^n$, and then we establish the boundary Schwarz lemma for harmonic self-mappings of the unit disk $\mathbb{D}$ and pluriharmonic self-mappings of $\mathbb{B}^n$. The results are sharp and coincides with the classical boundary Schwarz lemma when $n = 1$.

1. Introduction

Let $\mathbb{C}^n$ be the complex Euclidean n-space. In this paper, we write a point $z \in \mathbb{C}^n$ as a column vector of the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

and the symbol T stands for the transpose of vectors or matrices. For two points $z = \{z_1, \ldots, z_n\}^T$ and $w = \{w_1, \ldots, w_n\}^T$ of $\mathbb{C}^n$ the standard Hermitian scalar product on $\mathbb{C}^n$ is given by $\langle z, w\rangle = \sum_{k=1}^{n} z_k\overline{w_k}$ and the Euclidean norm of z is given by $|z| = \langle z, z\rangle^{\frac{1}{2}}$. Throughout this paper, we denote by $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit ball of $\mathbb{C}^n$, and $\partial\mathbb{B}^n$ the boundary of $\mathbb{B}^n$.

For $n = 1$, the planar case, we let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. In this paper, we always use $w(z)$ stands for the harmonic (or pluriharmonic) mapping and $f(z)$ stands for the holomorphic mapping. We will first improve the boundary Schwarz lemma for holomorphic mappings and then establish similar results for harmonic mappings and pluriharmonic mappings.

The classical Schwarz lemma states that any holomorphic mapping f maps $\mathbb{D}$ into itself, with $f(0) = 0$, satisfies $|f(z)| \le |z|$ in $\mathbb{D}$.

2010 *Mathematics Subject Classification*. Primary 30C62; Secondary 30C20, 30F15
Keywords. Schwarz lemma, boundary Schwarz lemma, holomorphic function, pluriharmonic mapping
Received: 01 May 2018; Accepted: 17 May 2018
Communicated by Miodrag Mateljević
The research of the author was supported by NSFs of China (No. 11501220 and No. 11471128), NSFs of Fujian Province (No. 2016J01020), and the Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (ZQN-PY402).
Email address: flandy@hqu.edu.cn (Jian-Feng Zhu)

It is well known that the Schwarz lemma has become a crucial theme in many branches of mathematical research for more than a hundred years. One can refer to the references [1, 9, 11–15, 17, 20, 21] for generalizations and applications of this lemma.

The classical Schwarz lemma at the boundary is as follows:

Theorem A. ([3, Page 42]) *Suppose f is a holomorphic self-mapping of $\mathbb{D}$ with $f(0) = 0$, and, further, f is holomorphic at $z = 1$ with $f(1) = 1$. Then, the following two conclusions hold:*

1. $f'(1) \geq 1$.
2. $f'(1) = 1$ *if and only if* $f(z) \equiv z$.

Theorem A has the following generalization.

Theorem B. ([9, Theorem 1.1′]) *Suppose f is a holomorphic self-mapping of $\mathbb{D}$ with $f(0) = 0$, and, further, f is holomorphic at $z = \alpha \in \mathbb{T}$ with $f(\alpha) = \beta \in \mathbb{T}$. Then, the following two conclusions hold:*

1. $\overline{\beta} f'(\alpha)\alpha \geq 1$.
2. $\overline{\beta} f'(\alpha)\alpha = 1$ *if and only if* $f(z) \equiv e^{i\theta} z$, *where* $e^{i\theta} = \beta\alpha^{-1}$ *and* $\theta \in \mathbb{R}$.

We remark that, when $\alpha = \beta = 1$, Theorem B coincides with Theorem A.

The study on the boundary version of the Schwarz lemma has been attracted much attention. For more discussions in this line, see, e.g., [15] for functions with one complex variable, and [9–11, 17, 20] for functions with several complex variables.

In this paper, we first improve Theorem A as follows.

Theorem 1.1. *Let f be a holomorphic self-mapping of $\mathbb{D}$. If f is holomorphic at $z = 1$ with $f(1) = 1$, then*

$$f'(1) \geq \frac{2|1 - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}. \tag{1}$$

The above inequality is sharp with the extremal function

$$\varphi(z) = \frac{\beta A(z) + f(0)}{1 + \beta\overline{f(0)}A(z)}, \tag{2}$$

where $\beta = \frac{1 - f(0)}{1 - \overline{f(0)}} \in \mathbb{T}$ *and*

$$A(z) = z\frac{(1 - |f(0)|^2)z + |f'(0)|}{(1 - |f(0)|^2) + |f'(0)|z}.$$

Remark 1.2. *In Theorem* 1.1, *if* $f(0) = 0$, *then we have*

$$f'(1) \geq \frac{2}{1 + |f'(0)|} \geq 1.$$

The last equality holds if and only if $f(z) = e^{i\alpha}z$ where α is a real number. This improves the classical boundary Schwarz lemma.

Theorem 1.1 has the following generalization.

Theorem 1.3. *Let f be a holomorphic self-mapping of $\mathbb{D}$. If f is holomorphic at $z = \alpha \in \mathbb{T}$ with $f(\alpha) = \beta \in \mathbb{T}$, then*

$$\overline{\beta} f'(\alpha)\alpha \geq \frac{2|1 - \overline{\beta} f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}. \tag{3}$$

When $\alpha = \beta = 1$, then Theorem 1.1 *coincides with Theorem* 1.3.

Furthermore, if we fix points $a, b \in \mathbb{D}$ such that $f(a) = b$, then we can obtain the boundary Schwarz lemma for f as follows.

Theorem 1.4. *Let f be a holomorphic self-mapping of $\mathbb{D}$ satisfying $f(a) = b$, where $a, b \in \mathbb{D}$. If f is holomorphic at $z = \alpha \in \mathbb{T}$ with $f(\alpha) = \beta \in \mathbb{T}$, then*

$$\overline{\beta} f'(\alpha)\alpha \geq \frac{1-|a|^2}{1-|b|^2}\frac{|1-\bar{b}\beta|^2}{|1-\bar{a}\alpha|^2}\frac{2}{1+\frac{1-|a|^2}{1-|b|^2}|f'(a)|}. \tag{4}$$

When $a = b = 0$, then (4) *coincides with* (18).

For the higher dimensional case, we have some similar results which are given as follows.

Theorem 1.5. *Let f be a holomorphic self-mapping of $\mathbb{B}^n$. If f is holomorphic at $z = \alpha \in \partial\mathbb{B}^n$ with $f(\alpha) = \beta \in \partial\mathbb{B}^n$, then we have the followng inequality holds.*

$$\overline{\beta}^T J_f(\alpha)\alpha \geq \frac{2|1-\overline{\beta}^T f(0)|^2}{1-\left|\overline{\beta}^T f(0)\right|^2 + \|J_f(0)\|}, \tag{5}$$

where $\|J_f(0)\|$ is the norm of $J_f(0)$ and the norm $\|\cdot\|$ is defined in (13).

When $n = 1$, then Theorem 1.5 *coincides with Theorem* 1.3.

Theorem 1.6. *Let f be a holomorphic self-mapping of $\mathbb{B}^n$ satisfying $f(a) = b$, where $a, b \in \mathbb{B}^n$. If f is holomorphic at $z = \alpha \in \partial\mathbb{B}^n$ with $f(\alpha) = \beta \in \partial\mathbb{B}^n$, then*

$$\overline{\beta}^T J_f(\alpha)\alpha \geq \frac{1-|a|^2}{1-|b|^2}\cdot\frac{|1-\overline{b}^T\beta|^2}{|1-\overline{a}^T\alpha|^2}\cdot\frac{2}{1+\frac{1-|a|^2}{1-|b|^2}\|J_f(a)\|}, \tag{6}$$

where $\|J_f(a)\|$ is the norm of $J_f(a)$ and the norm $\|\cdot\|$ is defined in (13).

When $n = 1$, then Theorem 1.6 *coincides with Theorem* 1.4.

The proofs of Theorem 1.1 ∼ Theorem 1.6 will be given in the section 3.

A twice continuously differentiable, complex-valued mapping w defined on $\Omega \subseteq \mathbb{C}$ is harmonic on Ω if it satisfies the following Laplace equation.

$$\Delta w(z) = 4w_{z\bar{z}}(z) = 0 \quad \text{for} \quad z \in \Omega.$$

For any $z = re^{i\theta} \in \mathbb{D}$ and $\alpha \in [0, 2\pi]$, the directional derivative of w is defined as follows

$$\partial_\alpha w(z) = \lim_{r\to 0^+}\frac{w(z+re^{i\alpha})-w(z)}{re^{i\alpha}} = e^{i\alpha}w_z(z) + e^{-i\alpha}w_{\bar{z}}(z). \tag{7}$$

Then

$$\max_{0\leq\alpha\leq 2\pi}|\partial_\alpha w(z)| = \Lambda_w(z) = |w_z(z)| + |w_{\bar{z}}(z)| \tag{8}$$

and

$$\min_{0\leq\alpha\leq 2\pi}|\partial_\alpha w(z)| = \lambda_w(z) = \left||w_z(z)| - |w_{\bar{z}}(z)|\right|. \tag{9}$$

According to [8], w is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian J_w satisfies the following condition: For every $z \in \mathbb{D}$,

$$J_w(z) = |w_z(z)|^2 - |w_{\bar{z}}(z)|^2 > 0.$$

In the second part of this paper, we will establish the boundary Schwarz lemma for harmonic mappings and pluriharmonic mappings. We first improve Heinz's result [4, Lemma] as follows.

Theorem 1.7. *Suppose that w is a harmonic self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$. Then we have the followng inequality holds.*

$$|w(z)| \le \frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|}\right) \quad \text{for} \quad z \in \mathbb{D}. \tag{10}$$

We remark here that since $\Lambda_w(0) \le \frac{4}{\pi}$, and for $0 < r < 1$ the function $\varphi(x) = \frac{r+\frac{\pi}{4}x}{1+\frac{\pi}{4}xr}$ is an increasing function of x, we know that the following inequality

$$\frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|}\right) \le \frac{4}{\pi} \arctan |z|,$$

holds for all $z \in \mathbb{D}$.

By using Theorem 1.7, we prove the following theorem which is the so-called boundary Schwarz lemma for harmonic mappings.

Theorem 1.8. *Suppose that w is a harmonic self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$. If w is differentiable at $z = 1$ with $w(1) = 1$, then we have the followng inequality holds.*

$$\mathrm{Re}[w_z(1) + w_{\bar{z}}(1)] \ge \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4}\Lambda_w(0)}. \tag{11}$$

The above inequality is sharp.

Remark 1.9. *According to [2, page 7], a harmonic mapping w of $\mathbb{D}$ has the representation $w = h + \bar{g}$, where h and g are holomorphic in $\mathbb{D}$. Furthermore, if $g(0) = 0$, then the representation is unique and is called the canonical representation of w.*

Under the hypotheses of Theorem 1.8, if $\varphi = h - g$ is holomorphic at $z = 1$, then

$$\mathrm{Im}[w_z(1)] = 0 = \mathrm{Im}[w_{\bar{z}}(1)],$$

*and the symbol "*Re*" in (11) can be removed. See Remark 4.1 for the proof.*

Example 1.10. *Let $\psi_1(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ and $\psi_2(z) = \frac{z-\beta}{1-\bar{\beta}z}$ be the holomorphic automorphism of $\mathbb{D}$, where $\alpha = \frac{1+i}{2}$ and $\beta = 1 - \frac{\sqrt{3}}{2} + \frac{i}{2}$. Consider the harmonic mapping*

$$w(z) = az\psi_1(z) + b\overline{z\psi_2(z)} \quad \text{for} \quad z \in \mathbb{D},$$

where $a, b \in \mathbb{C}$.

If $w(0) = 0$, $w(1) = 1$ and $|w(z)| \le 1$ for all $z \in \mathbb{D}$, then we have $a = \frac{i}{2}$ and $b = \frac{1-i\sqrt{3}}{4}$. Therefore, $w_z(1) = 1$ and $w_{\bar{z}}(1) = \frac{\sqrt{3}}{2}$ both are real numbers.

Theorem 1.11. *Suppose that w is a harmonic self-mapping of $\mathbb{D}$ satisfying $w(a) = 0$. If w is differentiable at $z = \alpha$ with $w(\alpha) = \beta$, where $\alpha, \beta \in \mathbb{T}$, then we have the followng inequality holds.*

$$\mathrm{Re}\left\{\overline{\beta}\left[w_z(\alpha)\alpha + w_{\bar{z}}(\alpha)\bar{\alpha}\right]\right\} \ge \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4}\Lambda_w(a)(1 - |a|^2)} \frac{1 - |a|^2}{|1 - \bar{a}\alpha|^2}. \tag{12}$$

When $\alpha = \beta = 1$ and $a = 0$, then Theorem 1.11 *coincides with Theorem* 1.8.

A continuous complex-valued function w defined on a domain $G \subseteq \mathbb{C}^n$ is said to be pluriharmonic if for each fixed $z \in G$ and $\theta \in \partial\mathbb{B}^n$, the function $w(z + \theta\zeta)$ is harmonic in $\{\zeta : |\zeta| < d_G(z)\}$ where $d_G(z)$ denotes the distance from z to the boundary ∂G of G. If G is simply connected, then a real-valued function u defined on G is pluriharmonic if and only if u is the real part of a holomorphic function on G. Clearly, a mapping $w : \mathbb{B}^n \to \mathbb{C}$ is pluriharmonic if and only if w has a representation $w = h + \overline{g}$, where h and g are holomorphic in $\mathbb{B}^n$. We refer to [5], [6], [7] for more details on pluriharmonic mappings.

For a complex-valued and differentiable function f from $\mathbb{B}^n$ into $\mathbb{C}$ we introduce

$$f_z = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right) \quad \text{and} \quad f_{\overline{z}} = \left(\frac{\partial f}{\partial \overline{z_1}}, \cdots, \frac{\partial f}{\partial \overline{z_n}}\right).$$

If $f : \mathbb{B}^n \to \mathbb{C}^m$ is differentiable, then we introduce

$$f_z = \left(\frac{\partial f_j}{\partial z_k}\right)_{m\times n} \quad \text{and} \quad f_{\overline{z}} = \left(\frac{\partial f_j}{\partial \overline{z_k}}\right)_{m\times n}.$$

A function $w = \{w_1, \cdots, w_m\}^T : \mathbb{B}^n \to \mathbb{C}^m$ is said to be pluriharmonic, if each component w_j $(j = 1, \cdots, m)$ is a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$. Let $w = h + \overline{g}$ be a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}^n$. Then, the real Jacobian determinant of w can be written in the following form

$$\det J_w = \det \begin{pmatrix} \partial h & \overline{\partial g} \\ \partial g & \overline{\partial h} \end{pmatrix}$$

and if h is locally biholomorphic, then the determinant of J_w can be written as follows

$$\det J_w = |\det \partial h|^2 \det(I_n - \partial g[\partial h]^{-1}\overline{\partial g[\partial h]^{-1}}).$$

If w is planar harmonic mapping of $\mathbb{D}$, then its Jacobian is given as follows

$$\det J_w = |w_z|^2 - |w_{\overline{z}}|^2 = |h'|^2 - |g'|^2.$$

For an $n \times n$ complex matrix A, we introduce the operator norm

$$\|A\| = \sup_{z\neq 0} \frac{\|Az\|}{\|z\|} = \max\{\|A\theta\| : \theta \in \partial\mathbb{B}^n\}. \tag{13}$$

Theorem 1.12. *Let w be a pluriharmonic self-mapping of $\mathbb{B}^n$ satisfying $w(a) = 0$, where $a \in \mathbb{B}^n$. If $w(z)$ is differentiable at $z = \alpha \in \partial\mathbb{B}^n$ with $w(\alpha) = \beta \in \partial\mathbb{B}^n$, then we have the followng inequality holds.*

$$\begin{aligned} &\mathrm{Re}\left\{\overline{\beta}^T \left[w_z(\alpha)\frac{1 - \overline{a}^T\alpha}{1 - |a|^2}(\alpha - a) + w_{\overline{z}}(\alpha)\frac{1 - a^T\overline{\alpha}}{1 - |a|^2}(\overline{\alpha} - \overline{a})\right]\right\} \\ &\geq \frac{4}{\pi}\frac{1}{1 + \frac{\pi}{4}\Lambda_w(a)\left|\frac{\alpha - a}{1 - \overline{a}^T\alpha}\right|(1 - |a|^2)}, \end{aligned} \tag{14}$$

where $\Lambda_w(a) = \|w_z(a)\| + \|w_{\overline{z}}(a)\|$. If $a = 0$, then we have

$$\mathrm{Re}\left[\overline{\beta}^T(w_z(\alpha)\alpha + w_{\overline{z}}(\alpha)\overline{\alpha})\right] \geq \frac{4}{\pi}\frac{1}{1 + \frac{\pi}{4}\Lambda_w(0)}. \tag{15}$$

Remark 1.13. *Since the following inequalities*

$$\left|\frac{\alpha - a}{1 - \overline{a}^T\alpha}\right|(1 - |a|^2) \leq (1 + |a|)^2$$

and

$$\left|\frac{1 - a^T\overline{\alpha}}{1 - |a|^2}\right||\alpha - a| \leq \frac{1 + |a|}{1 - |a|},$$

hold, we obtain that

$$\begin{aligned}
&\frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)(1+|a|)^2} \\
&\leq \left|\overline{\beta}^T(w_z(\alpha)(\alpha-a))\right|\left|\frac{1-\overline{a}^T\alpha}{1-|a|^2}\right| \\
&+\left|\overline{\beta}^T(w_{\overline{z}}(\alpha)(\overline{\alpha}-\overline{a}))\right|\left|\frac{1-\overline{a}^T\alpha}{1-|a|^2}\right| \\
&\leq (\|w_z(\alpha)\|+\|w_{\overline{z}}(\alpha)\|)\left(\frac{1+|a|}{1-|a|}\right).
\end{aligned} \tag{16}$$

Therefore

$$\Lambda_w(\alpha) \geq \frac{4}{\pi}\left(\frac{1-|a|}{1+|a|}\right)\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)(1+|a|)^2}.$$

The proofs of Theorem 1.7 ∼ Theorem 1.12 will be given in the section 4.

2. Preliminaries

In this section, we shall introduce some necessary terminologies, recall several known results.

The following results are due to Osserman [15] which is the Schwarz lemma at the boundary.

Theorem C. ([15, Lemma 1 and Lemma 2]) *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-mapping of $\mathbb{D}$ satisfying $f(0) = 0$. Then*

$$|f(z)| \leq |z|\frac{|z|+|f'(0)|}{1+|f'(0)||z|} \quad \text{for} \quad |z|<1. \tag{17}$$

Moreover, if for some $b \in \mathbb{T}$, $f(z)$ extends continuously to b with $|f(b)| = 1$ and $f'(b)$ exists, then

$$|f'(b)| \geq \frac{2}{1+|f'(0)|}. \tag{18}$$

By using the classical Schwarz lemma for f, we know that $|f'(b)| > 1$ unless $f(z) = e^{i\theta}z$ and θ is a real number.

Theorem D. ([15, Lemma 4]) *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-mapping of $\mathbb{D}$. If for some $b \in \mathbb{T}$, $f(z)$ extends continuously to b with $|f(b)| = 1$ and $f'(b)$ exists, then*

$$|f'(b)| \geq \frac{2(1-|f(0)|)^2}{1-|f(0)|^2+|f'(0)|}. \tag{19}$$

The following lemma will be used in the proofs of Theorem 1.6 and Theorem 1.12.

Theorem E. ([18, Theorem 2.2.2]) *For given $a \in \mathbb{B}^n$, let $A = sI_n + \frac{a\overline{a}^T}{1+s}$, where $s = \sqrt{1-|a|^2}$ and I_n is the unit square matrix of order n. Then*

$$\varphi_a(z) = A\frac{a-z}{1-\overline{a}^Tz}$$

is a biholomorphic automorphism of $\mathbb{B}^n$ which interchanges 0 and a. Moreover, φ_a is biholomorphic in a neighborhood of $\overline{\mathbb{B}^n}$, and

$$A^2 = s^2I_n + a\overline{a}^T, \quad Aa = a, \quad \varphi_a^{-1} = \varphi_a, \quad J_{\varphi_a}(z) = A\left[-\frac{I_n}{1-\overline{a}^Tz}+\frac{(a-z)\overline{a}^T}{(1-\overline{a}^Tz)^2}\right].$$

In 2016, Tang et al proved the following theorem which is an improvement of Theorem B.

Theorem F. ([19, Theorem 3.1]) *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic mapping, and let $f(a) = a$ for some $a \in \mathbb{B}^n$. Then we have the following two conclusions:*

1. *If f is holomorphic at $z = \alpha \in \partial\mathbb{B}^n$ with $f(\alpha) = \beta \in \partial\mathbb{B}^n$, then*

$$\overline{\beta}^T J_f(\alpha)\alpha \geq \frac{|1-\overline{a}^T\beta|^2}{|1-\overline{a}^T\alpha|^2}.$$

2. *If there exist $\alpha_1, \cdots, \alpha_n \in \partial\mathbb{B}^n$ such that $\alpha_1 - a, \cdots, \alpha_n - a$ are linearly independent and f is holomorphic at $z = \alpha_k$ with $f(\alpha_k) = \beta_k \in \partial\mathbb{B}^n (k = 1, \cdots, n)$, then the following n equalities:*

$$\overline{\beta_k}^T J_f(\alpha_k)\alpha_k = \frac{|1-\overline{a}^T\beta_k|^2}{|1-\overline{a}^T\alpha_k|^2}(k = 1, \cdots, n)$$

hold if and only if

$$f(z) \equiv \varphi_a(U\varphi_a(z)),$$

where $U = (\varphi_a(\beta_1), \cdots, \varphi_a(\beta_n))(\varphi_a(\alpha_1), \cdots, \varphi_a(\alpha_n))^{-1}$ is a unitary square matrix of order n.

When $n = 1$ and $a = 0$, then Theorem F coincides with Theorem B.

In 1959, Heinz proved the following result ([4, Lemma]) which is the so-called Schwarz lemma for harmonic mappings: If w is a harmonic mapping of $\mathbb{D}$ into itself such that $w(0) = 0$, then for any $z \in \mathbb{D}$,

$$|w(z)| \leq \frac{4}{\pi}\arctan|z|.$$

Later, Pavlović improved the above result of Heinz and proved the following theorem.

Theorem G. ([16, Theorem 3.6.1]) *Suppose w is a harmonic mapping of $\mathbb{D}$ into itself. Then, the following statement holds:*

$$\left|w(z) - \frac{1-|z|^2}{1+|z|^2}w(0)\right| \leq \frac{4}{\pi}\arctan|z| \quad z \in \mathbb{D}.$$

3. Schwarz Lemma at the Boundary for Holomorphic Mappings

In this section we first prove the Theorem 1.1 which will be used in the proofs of Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.1

Let

$$F(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}.$$

Then we have $F : \mathbb{D} \to \mathbb{D}$ is a holomorphic self-mapping of $\mathbb{D}$ such that $F(0) = 0$ and $F(1) = \beta = \frac{1-f(0)}{1-\overline{f(0)}}$. By using (17), we have

$$|F(z)| \leq |z|\frac{|z| + |F'(0)|}{1 + |F'(0)||z|}, \tag{20}$$

where

$$F'(0) = \frac{f'(0)}{1 - |f(0)|^2}. \tag{21}$$

Consider the function $g(z) = \overline{\beta}F(z)$. Then we $g(0) = 0$, $g(1) = 1$ and $|g(z)| = |F(z)|$. Since $g(z)$ is holomorphic at $z = 1$, we see that

$$g(z) = 1 + g'(1)(z-1) + \circ(|z-1|).$$

This together with (20) show that

$$\left|1 + g'(1)(z-1) + \circ(|z-1|)\right|^2 \le \left(|z|\frac{|z| + |F'(0)|}{1 + |F'(0)||z|}\right)^2.$$

Therefore,

$$2\mathrm{Re}\left[g'(1)(1-z)\right] \ge 1 - \left(|z|\frac{|z| + |F'(0)|}{1 + |F'(0)||z|}\right)^2 + \circ(|1-z|). \tag{22}$$

Take $z = r \in (0,1)$ and letting $r \to 1^-$ we have

$$\begin{aligned} 2\mathrm{Re}\left[g'(1)\right] &\ge \lim_{r\to 1^-} \frac{1 - \left(r\frac{r+|F'(0)|}{1+|F'(0)|r}\right)^2}{1-r} \\ &= \frac{4}{1 + |F'(0)|}. \end{aligned} \tag{23}$$

Assume that $z = re^{i\theta} \in \mathbb{D}$, where $\theta \neq 0$. By letting $r \to 1^-$, it follows from (22) that

$$\begin{aligned} 2\mathrm{Re}\left[g'(1)\frac{1-e^{i\theta}}{|1-e^{i\theta}|}\right] &= 2\mathrm{Re}\left[g'(1)\frac{\sin(\theta/2)}{|\sin(\theta/2)|}(-i)e^{i\theta/2}\right] \\ &\ge \frac{\circ(|1-e^{i\theta}|)}{|1-e^{i\theta}|}. \end{aligned} \tag{24}$$

Letting $\theta \to 0^\pm$ leads to $2\mathrm{Re}\left[\mp i g'(1)\right] \ge 0$. This shows that

$$\mathrm{Im}\left[g'(1)\right] = 0. \tag{25}$$

Therefore we have

$$g'(1) \ge \frac{2}{1 + |F'(0)|}.$$

Note that $\beta = \frac{1-f(0)}{1-\overline{f(0)}} \in \mathbb{T}$ and $g(z) = \overline{\beta}F(z)$. Then we have

$$g'(1) = \overline{\beta}\frac{1-|f(0)|^2}{\left(1-\overline{f(0)}\right)^2}f'(1) = \frac{1-|f(0)|^2}{|1-f(0)|^2}f'(1).$$

According to (21), we have

$$f'(1) \ge \frac{2|1-f(0)|^2}{1-|f(0)|^2+|f'(0)|}.$$

This completes the proof of the lemma. □

Proof of Theorem 1.3

Let $g(\zeta) = \overline{\beta}f(\zeta\alpha)$, where $\alpha, \beta \in \mathbb{T}$ and $\zeta \in \mathbb{D}$. Then we have $g(\zeta)$ is differentiable at $z = 1$ with $g(1) = 1$. By using Theorem 1.1, we have

$$g'(1) \ge \frac{2|1-g(0)|^2}{1-|g(0)|^2+|g'(0)|}.$$

This implies that

$$\overline{\beta}f'(\alpha)\alpha \ge \frac{2|1-\overline{\beta}f(0)|^2}{1-|f(0)|^2+|f'(0)|} \ge \frac{2(1-|f(0)|)^2}{1-|f(0)|^2+|f'(0)|}. \tag{26}$$

The proof of the theorem is complete. □

Proof of Theorem 1.4

Consider the automorphisms $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ and $\varphi_b(z) = \frac{b-z}{1-\bar{b}z}$ which interchange 0 and a, 0 and b respectively. Then we have $\varphi_a(\alpha) = p \in \mathbb{T}$ and $\varphi_b(\beta) = q \in \mathbb{T}$. Moreover we have

$$\varphi_a'(p) = \frac{-(\bar{a}\alpha - 1)^2}{1-|a|^2} \quad , \quad \varphi_a'(0) = -1 + |a|^2 \tag{27}$$

and

$$\varphi_b'(\beta) = \frac{-1+|b|^2}{(1-\bar{b}\beta)^2} \quad , \quad \varphi_b'(b) = \frac{-1}{1-|b|^2}. \tag{28}$$

Let $g(z) = \varphi_b \circ f \circ \varphi_a(z)$ for $z \in \mathbb{D}$. Then we have

$$g(0) = \varphi_b \circ f(a) = \varphi_b(b) = 0$$

and

$$g(p) = \varphi_b \circ f(\alpha) = \varphi_b(\beta) = q.$$

According to Theorem 1.3 we see that

$$\bar{q}g'(p)p \geq \frac{2}{1+|g'(0)|}.$$

Elementary calculations lead to the following inequalities

$$\begin{aligned} \bar{q}\varphi_b'(\beta) &= \overline{\left(\frac{b-\beta}{1-\bar{b}\beta}\right)} \cdot \frac{-1+|b|^2}{(1-\bar{b}\beta)^2} \\ &= \bar{\beta}\frac{1-|b|^2}{|1-\bar{b}\beta|^2}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \varphi_a'(p)p &= \frac{-(\bar{a}\alpha-1)^2}{1-|a|^2} \cdot \frac{a-\alpha}{1-\bar{a}\alpha} \\ &= \alpha\frac{|1-\bar{a}\alpha|^2}{1-|a|^2}, \end{aligned} \tag{30}$$

and

$$g'(0) = \varphi_b'(b)f'(a)\varphi_a'(0) = \frac{1-|a|^2}{1-|b|^2}f'(a).$$

These show that

$$\begin{aligned} \bar{q}g'(p)p &= \bar{q}\varphi_b'(\beta) \cdot f'(\alpha) \cdot \varphi_a'(p) \cdot p \\ &= \frac{1-|b|^2}{|1-\bar{b}\beta|^2}\bar{\beta}f'(\alpha)\alpha\frac{|1-\bar{a}\alpha|^2}{1-|a|^2} \\ &\geq \frac{2}{1+|g'(0)|} \\ &= \frac{2}{1+\frac{1-|a|^2}{1-|b|^2}|f'(a)|}. \end{aligned} \tag{31}$$

Therefore,

$$\bar{\beta}f'(\alpha)\alpha \geq \frac{1-|a|^2}{1-|b|^2}\frac{|1-\bar{b}\beta|^2}{|1-\bar{a}\alpha|^2}\frac{2}{1+\frac{1-|a|^2}{1-|b|^2}|f'(a)|}.$$

The proof of the theorem is complete. $\square$

Proof of Theorem 1.5

Assume that f is a holomorphic self-mapping of $\mathbb{B}^n$ and holomorphic at $z = \alpha$ with $f(\alpha) = \beta$, where $\alpha, \beta \in \partial\mathbb{B}^n$. Let $g(\zeta) = \overline{\beta}^T f(\alpha\zeta)$ for $\zeta \in \mathbb{D}$. Then g is a holomorphic self-mapping of $\mathbb{D}$ satisfying $g(0) = \overline{\beta}^T f(0)$ and $g(1) = \overline{\beta}^T f(\alpha) = 1$. By using (1), we know that

$$g'(1) \geq \frac{2|1 - g(0)|^2}{1 - |g(0)|^2 + |g'(0)|}.$$

It follows from $g'(0) = \overline{\beta}^T J_f(0)\alpha$ and $g'(1) = \overline{\beta}^T J_f(\alpha)\alpha$ that

$$\begin{aligned} g'(1) &= \overline{\beta}^T J_f(\alpha)\alpha \\ &\geq \frac{2|1 - g(0)|^2}{1 - |g(0)|^2 + |g'(0)|} \\ &\geq \frac{2\left|1 - \overline{\beta}^T f(0)\right|^2}{1 - \left|\overline{\beta}^T f(0)\right|^2 + \|J_f(0)\|} \\ &\geq \frac{2\left(1 - \left|\overline{\beta}^T f(0)\right|\right)^2}{1 - \left|\overline{\beta}^T f(0)\right|^2 + \|J_f(0)\|} \end{aligned} \tag{32}$$

where $\|J_f(0)\|$ is the operator norm defined in (13). Let $a = 1 + \|J_f(0)\| \geq 1$ and $\phi(x) := \frac{2(1-x)^2}{a-x^2}$. Then we have $\phi(x)$ is decreasing for $x \in (0, 1)$. Therefore

$$\overline{\beta}^T J_f(\alpha)\alpha \geq \frac{2(1 - |f(0)|)^2}{1 - |f(0)|^2 + \|J_f(0)\|}.$$

The proof of the theorem is complete. □

The following Lemma 3.1 will be used in proving Theorem 1.6.

Lemma 3.1. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic mapping with $f(0) = 0$. If f is holomorphic at $z = \alpha \in \partial\mathbb{B}^n$ with $f(\alpha) = \beta \in \partial\mathbb{B}^n$, then*

$$\overline{\beta}^T J_f(\alpha)\alpha \geq \frac{2}{1 + \|J_f(0)\|}. \tag{33}$$

Proof. Let

$$g(\zeta) = \overline{\beta}^T f(\zeta\alpha), \quad \zeta \in \mathbb{D}.$$

Then $g : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $g(0) = 0$ and g is holomorphic at $\zeta = 1$ with

$$g(1) = \sum_{j=1}^{n} \overline{\beta_j} f_j(\alpha) = \sum_{j=1}^{n} \overline{\beta_j}\beta_j = 1.$$

By applying (1), we know that

$$
\begin{aligned}
g'(1) &= \overline{\beta}^T J_f(\alpha)\alpha \\
&\geq \frac{2}{1+|g'(0)|} \\
&= \frac{2}{1+\left|\overline{\beta}^T J_f(0)\alpha\right|} \\
&\geq \frac{2}{1+\|J_f(0)\|}.
\end{aligned} \tag{34}
$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 1.6

Let $\varphi_a(z) = A\frac{a-z}{1-\overline{a}z}$ be the holomorphic automorphism of $\mathbb{B}^n$ where $A = s_a I_n + \frac{a\overline{a}'}{1+s_a}$, $s_a = \sqrt{1-|a|^2}$. Similarly, let $\varphi_b(z) = B\frac{b-z}{1-\overline{b}z}$ be the holomorphic automorphism of $\mathbb{B}^n$ where $B = s_b I_n + \frac{b\overline{b}'}{1+s_b}$, $s_b = \sqrt{1-|b|^2}$.

Let $g(z) = \varphi_b \circ f \circ \varphi_a(z) : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic self-mapping of $\mathbb{D}$. Then

$$g(0) = \varphi_b \circ f \circ \varphi_a(0) = \varphi_b \circ f(a) = \varphi_b(b) = 0.$$

Assume that $\varphi_a(\alpha) = p \in \partial\mathbb{B}^n$ and $\varphi_b(\beta) = q \in \partial\mathbb{B}^n$. Then we have

$$g(p) = \varphi_b \circ f \circ \varphi_a(p) = \varphi_b \circ f(\alpha) = \varphi_b(\beta) = q.$$

By using the above Lemma 3.1, we have

$$\frac{2}{1+\|J_g(0)\|} \leq \overline{q}^T J_g(p)p = \overline{\varphi_b(\beta)}^T J_{\varphi_b}(\beta) J_f(\alpha) J_{\varphi_a}(p)\varphi_a(\alpha).$$

According to Lemma E and the proof of [19, (3.2) and (3.3)] we know that

$$
\begin{aligned}
\overline{\varphi_b(\beta)}^T J_{\varphi_b}(\beta) &= \frac{\overline{\alpha}^T - \overline{\beta}^T}{1-\overline{\beta}^T b} B^2 \left[-\frac{I_n}{1-\overline{b}^T\beta} + \frac{(b-\beta)\overline{b}^T}{(1-\overline{b}^T\beta)^2} \right] \\
&= \frac{1}{|1-\overline{b}^T\beta|^2} [(1-\overline{\beta}^T b)\overline{b}^T - s^2\overline{\beta}^T] \left[-I_n + \frac{(b-\beta)\overline{b}^T}{1-\overline{b}^T\beta} \right] \\
&= \frac{1-|b|^2}{|1-\overline{b}^T\beta|^2}\overline{\beta}^T
\end{aligned} \tag{35}
$$

and

$$
\begin{aligned}
J_{\varphi_a}(\varphi_a(\alpha))\varphi_a(\alpha) &= [J_{\varphi_a}(\alpha)]^{-1}\varphi_a(\alpha) \\
&= \frac{1-\overline{a}^T a}{1-|a|^2}(\alpha - a).
\end{aligned} \tag{36}
$$

Let $\gamma_2, \cdots, \gamma_n$ be a standard orthogonal basis of $\partial\mathbb{B}^n$ such that $\overline{\alpha}^T\gamma_k = 0$, $(k = 2, \cdots, n)$. Then there are $\mu_1, \cdots, \mu_n \in \mathbb{C}$ such that

$$\frac{1-\overline{a}^T\alpha}{1-|a|^2}(\alpha - a) = \mu_1\alpha + \mu_2\gamma_2 + \cdots + \mu_n\gamma_n. \tag{37}$$

This gives

$$\mu_1 = \overline{\alpha}^T \frac{1-\overline{a}^T\alpha}{1-|a|^2}(\alpha - a) = \frac{|1-\overline{a}^T\alpha|^2}{1-|a|^2}. \tag{38}$$

Suppose that f is differentiable in a neighborhood V of α. Then $f(\partial\mathbb{B}^n \cap V)$ and $\partial\mathbb{B}^n$ are tangent at β. This means that the tangent space and holomorphic tangent space to $f(\partial\mathbb{B}^n \cap V)$ at $f(\alpha) = \beta$ are contained in

$$T_\beta(\partial\mathbb{B}^n) = \{w \in \mathbb{C}^n : \mathrm{Re}\overline{\beta}^T w = 0\}$$

and

$$T_\beta^{(1,0)}(\partial\mathbb{B}^n) = \{w \in \mathbb{C}^n : \overline{\beta}^T w = 0\}$$

respectively. Note that for any $\xi \in T_\alpha(\partial\mathbb{B}^n)$, $J_f(\alpha)\xi$ is a tangent vector of $f(\partial\mathbb{B}^n \cap V)$ at β. Then

$$J_f(\alpha)\xi \in T_\beta(\partial\mathbb{B}^n).$$

This shows that $J_f(\alpha)(T_\alpha^{(1,0)}(\partial\mathbb{B}^n)) \subseteq T_\beta(\partial\mathbb{B}^n)$. Using

$$< J_f(\alpha)\xi, \beta >= 0 =< \overline{\beta}^T J_f(\alpha)\xi, 1 >,$$

we see that $\overline{\beta}^T J_f(\alpha)\xi = 0$ holds for any $\xi \in T_\alpha(\partial\mathbb{B}^n)$. So there exists $\lambda \in \mathbb{R}$ such that $\overline{\beta}^T J_f(\alpha) = \lambda\overline{\alpha}^T$. That is

$$\overline{\beta}^T J_f(\alpha)\alpha = \lambda. \tag{39}$$

(See [10, Theorem 3.1] for more details). Therefore we have

$$\begin{aligned} \overline{q}^T J_g(p)p &= \overline{\varphi_b(\beta)}^T J_{\varphi_b}(\beta) J_f(\alpha) J_{\varphi_a}(\varphi_a(\alpha))\varphi_a(\alpha) \\ &= \frac{1-|b|^2}{|1-\overline{b}^T\beta|^2}\overline{\beta}^T J_f(\alpha)\left(\frac{|1-\overline{a}^T\alpha|^2}{1-|a|^2}\alpha + \mu_2\gamma_2 + \cdots + \mu_n\gamma_n\right) \\ &= \frac{1-|b|^2}{|1-\overline{b}^T\beta|^2}\lambda\overline{\alpha}^T\left(\frac{|1-\overline{a}^T\alpha|^2}{1-|a|^2}\alpha + \mu_2\gamma_2 + \cdots + \mu_n\gamma_n\right) \\ &= \frac{1-|b|^2}{|1-\overline{b}^T\beta|^2}\lambda\frac{|1-\overline{a}^T\alpha|^2}{1-|a|^2}. \end{aligned} \tag{40}$$

On the other hand, since

$$J_{\varphi_b}(b) = \frac{-B}{1-|b|^2} \quad \text{and} \quad J_{\varphi_a}(0) = A(-1+|a|^2)$$

we see that

$$\begin{aligned} J_g(0) &= J_{\varphi_b}(b) J_f(a) J_{\varphi_a}(0) \\ &= \frac{1-|a|^2}{1-|b|^2} B \cdot J_f(a) \cdot A. \end{aligned} \tag{41}$$

Thus

$$\|J_g(0)\| \le \frac{1-|a|^2}{1-|b|^2}\|J_f(a)\|.$$

This shows that

$$\overline{\beta}^T J_f(\alpha)\alpha \ge \frac{1-|a|^2}{1-|b|^2}\frac{|1-\overline{b}^T\beta|^2}{|1-\overline{a}^T\alpha|^2} \cdot \frac{2}{1+\frac{1-|a|^2}{1-|b|^2}\|J_f(a)\|},$$

as required. □

4. Schwarz Lemma at the Boundary for Harmonic Mappings

In this section, we shall prove Theorem 1.7 ∼ Theorem 1.12. We start with the proof of Theorem 1.7.

Proof of Theorem 1.7

Let $w(z) = u(z) + iv(z)$ be a harmonic self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$. For any $\theta \in \mathbb{R}$, let $f(z) = \xi(z) + i\eta(z)$ be the function which is holomorphic in $\mathbb{D}$ and satisfies the relations $f(0) = 0$, $|f(z)| < 1$ for all $z \in \mathbb{D}$ and

$$\mathrm{Re} f(z) = u(z)\cos\theta + v(z)\sin\theta. \tag{42}$$

Then we have

$$\xi(z) = \mathrm{Re} w(z)e^{-i\theta}$$

and

$$\begin{aligned} f'(z) &= \xi_x - i\xi_y = 2\xi_z \\ &= 2\frac{\partial}{\partial z} Rew(z)e^{-i\theta} \\ &= w_z(z)e^{-i\theta} + \overline{w_{\bar z}(z)}e^{i\theta}. \end{aligned} \tag{43}$$

Consider the holomorphic mapping

$$g(z) = \frac{e^{\frac{\pi}{2}if(z)} - 1}{e^{\frac{\pi}{2}if(z)} + 1}.$$

Then we have $g(0) = 0$ and $g'(0) = \frac{\pi i}{4} f'(0)$. According to the assumption we know that $|Ref| = |\xi| < 1$. This implies that

$$\mathrm{Re} e^{\frac{\pi}{2}if} = e^{\frac{-\pi\eta}{2}} \cos\frac{\pi}{2}\xi > 0.$$

Therefore, $|g(z)| < 1$. Applying the Schwarz lemma (17) we obtain

$$|g(z)| \le |z| \frac{|z| + \frac{\pi}{4}|f'(0)|}{1 + \frac{\pi}{4}|f'(0)||z|}. \tag{44}$$

On the other hand, the following elementary inequality holds

$$\left| \frac{e^{i\zeta} - 1}{e^{i\zeta} + 1} \right| \ge \tan\frac{1}{2}|\mathrm{Re}\zeta| \quad \text{for} \quad |\mathrm{Re}\zeta| \le \frac{\pi}{2}. \tag{45}$$

The inequalities (44) together with (45) show that

$$\tan\frac{\pi}{4}|\mathrm{Re} f| \le |g(z)| \le |z| \frac{|z| + \frac{\pi}{4}|f'(0)|}{1 + \frac{\pi}{4}|f'(0)||z|}.$$

For $0 < r < 1$ the function $\varphi(x) = \frac{r + \frac{\pi}{4}x}{1 + \frac{\pi}{4}xr}$ is an increasing function of x. Also (43) shows that $|f'(0)| = |w_z(0)e^{-i\theta} + \overline{w_{\bar z}(0)}e^{i\theta}| \le \Lambda_w(0)$. Thus

$$|\mathrm{Re} f| \le \frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|} \right).$$

Applying (42) we obtain the estimate

$$|u(z)\cos\theta + v(z)\sin\theta| \le \frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|} \right).$$

Since the above inequality holds for every $\theta \in \mathbb{R}$, the inequality (10) follows, which proves Theorem 1.7. □

Proof of Theorem 1.8

According to Theorem 1.7 we see that

$$|w(z)| \leq \frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|}\right) := M(z) \quad \text{for} \quad z \in \mathbb{D}. \tag{46}$$

Since w is differentiable at $z = 1$, we know that

$$w(z) = 1 + w_z(1)(z-1) + w_{\bar{z}}(1)(\bar{z}-1) + \circ(|z-1|).$$

This together with (46) show that

$$\left|1 + w_z(1)(z-1) + w_{\bar{z}}(1)(\bar{z}-1) + \circ(|z-1|)\right|^2 \leq (M(z))^2 .$$

Therefore,

$$2\mathrm{Re}\left[w_z(1)(1-z) + w_{\bar{z}}(1)(1-\bar{z})\right] \geq 1 - (M(z))^2 + \circ(|z-1|). \tag{47}$$

Take $z = r \in (0,1)$ and letting $r \to 1^-$, it follows from $M(1) = 1$ that

$$\begin{aligned} 2\mathrm{Re}\left[w_z(1) + w_{\bar{z}}(1)\right] &\geq \lim_{r\to 1^-} \frac{1 - M(r)^2}{1-r} \\ &= \frac{4}{\pi} \frac{2}{1 + \frac{\pi}{4}\Lambda_w(0)}. \end{aligned} \tag{48}$$

Therefore we have

$$\mathrm{Re}[w_z(1) + w_{\bar{z}}(1)] \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4}\Lambda_w(0)}$$

as required.

To check the sharpness of (11), let

$$U(z) = \frac{2}{\pi} \arctan \frac{2x}{1 - x^2 - y^2},$$

where $z = x + iy \in \mathbb{D}$. Then, we have that U is harmonic in $\mathbb{D}$ with $U(0) = 0$, $U(1) = 1$, and for any $z = r \in (-1, 1)$,

$$U(r) = \frac{4}{\pi} \arctan r.$$

Elementary computations show that

$$\Lambda_U(0) = \frac{4}{\pi}$$

and

$$\frac{\partial U}{\partial r}(1) = \frac{2}{\pi} = \frac{\partial U}{\partial z}(1) + \frac{\partial U}{\partial \bar{z}}(1).$$

This shows that (11) is sharp, and hence, the proof of the theorem is complete. □

Remark 4.1. *Since w is harmonic in $\mathbb{D}$, according to [2, page 7], we know that w has the canonical representation $w = h + \bar{g}$, where $h = u_h + iv_h$, $g = u_g + iv_g$ are holomorphic in $\mathbb{D}$. If $g(0) = 0$, then the representation is unique.*

We point out that under the assumptions of Theorem 1.8, if $\varphi = h - g$ is holomorphic at $z = 1$, then

$$\mathrm{Im}[w_z(1)] = 0 = \mathrm{Im}[w_{\bar{z}}(1)]. \tag{49}$$

We first prove the following equality holds:

$$\mathrm{Im}[w_z(1) - w_{\bar{z}}(1)] = 0. \tag{50}$$

For the proof of (50), assume that $z = re^{i\theta} \in \mathbb{D}$, where $\theta \neq 0$. By letting $r \to 1^-$, it follows from (47) that

$$2Re\left[w_z(1)\frac{1-e^{i\theta}}{|1-e^{i\theta}|} + w_{\bar{z}}(1)\frac{1-e^{-i\theta}}{|1-e^{i\theta}|}\right] \geq \frac{\circ(|1-e^{i\theta}|)}{|1-e^{i\theta}|}.$$

Letting $\theta \to 0^{\pm}$ leads to

$$2\mathrm{Re}\left[w_z(1)(\mp i) + w_{\bar{z}}(1)(\pm i)\right] \geq 0,$$

which guarantees the validity of (50).

Secondly, we prove the following equality holds:

$$\mathrm{Im}[w_z(1) + w_{\bar{z}}(1)] = 0. \tag{51}$$

To prove (51), note that $w(0) = 0$ implies $\varphi(0) = 0$, since $g(0) = 0$. The condition $w(1) = 1$ ensure that

$$\varphi(1) = 1 - \overline{g(1)} - g(1) = 1 - 2u_g(1) \in \mathbb{R}$$

and

$$|\varphi(z)|^2 = |w(z)|^2 - 4u_h(z)u_g(z), \quad \textit{for} \quad z \in \mathbb{D}. \tag{52}$$

Thus, we have

$$\varphi(1)^2 = 1 - 4u_h(1)u_g(1). \tag{53}$$

Since φ is holomorphic in $\mathbb{D}$ with $\varphi(0) = 0$, we know that $\varphi(1) \neq 0$. If φ is holomorphic at $z = 1$, then

$$\varphi(z) = \varphi(1) + \varphi'(1)(z-1) + \circ(|z-1|). \tag{54}$$

Hence,

$$2\varphi(1)\mathrm{Re}[\varphi'(1)(z-1)] = |w(z)|^2 - 4u_h(z)u_g(z) - \varphi(1)^2 - \circ(|z-1|). \tag{55}$$

Assume that $z = re^{i\theta} \in \mathbb{D}$, where $\theta \neq 0$. By letting $r \to 1^-$, it follows from (53) and (55) that

$$2\varphi(1)\mathrm{Re}\left[\varphi'(1)\frac{1-e^{i\theta}}{|1-e^{i\theta}|}\right] = \frac{\circ(|1-e^{i\theta}|)}{|1-e^{i\theta}|}.$$

Letting $\theta \to 0^{\pm}$ leads to

$$2\varphi(1)\mathrm{Re}\left[\varphi'(1)(\mp i)\right] = 0,$$

which shows that $\mathrm{Im}[\varphi'(1)] = 0$, and hence, (51) holds true.

The equality (49) holds obviously from (50) and (51), thus we can rewrite (11) as follows

$$w_z(1) + w_{\bar{z}}(1) \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(0)}. \tag{56}$$

Proof of Theorem1.11

Let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ be the automorphism of $\mathbb{D}$. Then we have $\varphi_a(z)$ interchanges 0 and a. For $\alpha \in \mathbb{T}$, let $p = \varphi_a(\alpha) \in \mathbb{T}$. Then we have

$$\varphi_a(p) = \alpha$$

and

$$\varphi_a'(p) = \frac{-(1-\bar{a}\alpha)^2}{1-|a|^2}, \quad \varphi_a'(0) = -1 + |a|^2.$$

For $\beta \in \mathbb{T}$, let $G(\zeta) := \bar{\beta} w \circ \varphi_a(\zeta p)$ where $\zeta \in \mathbb{D}$. Then we have $G(\zeta)$ is a harmonic self-mapping of $\mathbb{D}$ and satisfies the following conditions

$$G(0) = \overline{\beta} w(\varphi_a(0)) = \overline{\beta} w(a) = 0$$

and

$$G(1) = \overline{\beta} w(\varphi_a(p)) = \overline{\beta} w(\alpha) = |\beta|^2 = 1.$$

Note that

$$G_\zeta(\zeta) = \overline{\beta} w_z(\varphi_a(\zeta p))\varphi_a'(\zeta p)p$$

and

$$G_{\overline{\zeta}}(\zeta) = \overline{\beta} w_{\bar{z}}(\varphi_a(\zeta p))\overline{\varphi_a'(\zeta p)}\bar{p}.$$

By using Theorem 1.8, we see that

$$\begin{aligned}
\mathrm{Re}\Big(G_\zeta(1) + G_{\overline{\zeta}}(1)\Big) &= \mathrm{Re}\Big(\overline{\beta}\left[w_z(\alpha)\varphi_a'(p)p + w_{\bar{z}}(\alpha)\overline{\varphi_a'(p)}\bar{p}\right]\Big) \\
&= \mathrm{Re}\left(\overline{\beta}\left[w_z(\alpha)\alpha\frac{|1-\bar{a}\alpha|^2}{1-|a|^2} + w_{\bar{z}}(\alpha)\overline{\alpha}\frac{|1-\bar{a}\alpha|^2}{1-|a|^2}\right]\right) \\
&\geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_G(0)} \\
&= \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}(|G_\zeta(0)| + |G_{\bar{\zeta}}(0)|)} \\
&= \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}(|w_z(a)| + |w_{\bar{z}}(a)|)(1-|a|^2)} \\
&= \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)(1-|a|^2)}.
\end{aligned} \tag{57}$$

Thus

$$\mathrm{Re}\Big(\overline{\beta}\left[w_z(\alpha)\alpha + w_{\bar{z}}(\alpha)\bar{\alpha}\right]\Big) \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)(1-|a|^2)}\frac{1-|a|^2}{|1-\bar{a}\alpha|^2}.$$

If $a = 0$, then

$$\mathrm{Re}\Big(\overline{\beta}\left[w_z(\alpha)\alpha + w_{\bar{z}}(\alpha)\bar{\alpha}\right]\Big) \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(0)}. \tag{58}$$

This completes the proof of the theorem. $\square$

Proof of Theorem 1.12

Let $\varphi_a(z) = A\frac{a-z}{1-\bar{a}z}$ be the holomorphic automorphism of $\mathbb{B}^n$ where $A = s_a I_n + \frac{a\bar{a}'}{1+s_a}$, $s_a = \sqrt{1-|a|^2}$. Assume that $\varphi_a(\alpha) = p \in \partial\mathbb{B}^n$. Let $G(\zeta) = \overline{\beta}^T w \circ \varphi_a(\zeta p)$ be a harmonic mapping satisfying

$$G(0) = \overline{\beta}^T w \circ \varphi_a(0) = \overline{\beta}^T w(a) = 0.$$

Then

$$G(1) = \overline{\beta}^T w \circ \varphi_a(p) = \overline{\beta}^T w(\alpha) = 1.$$

Applying Theorem 1.8 we have

$$\begin{aligned}
\frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_G(0)} &\leq Re[G_z(1) + G_{\bar{z}}(1)] \\
&= Re\Big[\overline{\beta}^T\Big(w_z(\alpha)J_{\varphi_a}(p)p + w_{\bar{z}}(\alpha)\overline{J_{\varphi_a}(p)\bar{p}}\Big)\Big].
\end{aligned} \tag{59}$$

By using lemma E and [19, (3.3)], we obtain

$$\begin{aligned} J_{\varphi_a}(p)p &= J_{\varphi_a}(\varphi_a(\alpha))\varphi_a(\alpha) \\ &= [J_{\varphi_a}(\alpha)]^{-1}\varphi_a(\alpha) \\ &= \left[-\frac{I_n}{1-\overline{a}^T\alpha} + \frac{(a-\alpha)\overline{a}^T}{(1-\overline{a}^T\alpha)^2}\right]^{-1} A^{-1}A\left(\frac{a-\alpha}{1-\overline{a}^T\alpha}\right) \\ &= \left[-(1-\overline{a}^T\alpha)I_n - \frac{1-\overline{a}^T\alpha}{1-|a|^2}(a-\alpha)\overline{a}^T\right]\frac{a-\alpha}{1-\overline{a}^T\alpha} \\ &= \frac{1-\overline{a}^T\alpha}{1-|a|^2}(\alpha-a) \end{aligned} \tag{60}$$

and

$$\begin{aligned} J_{\varphi_a}(0)p &= J_{\varphi_a}(\varphi_a(a))\varphi_a(\alpha) \\ &= [J_{\varphi_a}(a)]^{-1}\varphi_a(\alpha) \\ &= \left[-\frac{I_n}{1-|a|^2}\right]^{-1} A^{-1}A\left(\frac{a-\alpha}{1-\overline{a}^T\alpha}\right) \\ &= \frac{\alpha-a}{1-\overline{a}^T\alpha}(1-|a|^2). \end{aligned} \tag{61}$$

Therefore,

$$\begin{aligned} \Lambda_G(0) &= |G_\zeta(0)| + |G_{\overline{\zeta}}(0)| \\ &= \left|\overline{\beta}^T w_\zeta(a)J_{\varphi_a}(0)p\right| + \left|\overline{\beta}^T w_{\overline{\zeta}}(a)\overline{J_{\varphi_a}(0)}\overline{p}\right| \\ &= \left(\left|\overline{\beta}^T w_\zeta(a)(\alpha-a)\right| + \left|\overline{\beta}^T w_{\overline{\zeta}}(a)(\overline{\alpha}-\overline{a})\right|\right)\frac{1-|a|^2}{|1-\overline{a}^T\alpha|} \\ &\le \Lambda_w(a)\left|\frac{\alpha-a}{1-\overline{a}^T\alpha}\right|(1-|a|^2), \end{aligned} \tag{62}$$

where $\Lambda_w(a) = \|w_z(a)\| + \|w_{\overline{z}}(a)\|$, and the norm $\|.\|$ is defined in (13). This yields that

$$\begin{aligned} &\frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)\left|\frac{\alpha-a}{1-\overline{a}^T\alpha}\right|(1-|a|^2)} \\ &\le Re\left[\overline{\beta}^T\left(w_z(\alpha)J_{\varphi_a}(p)p + w_{\overline{z}}(\alpha)\overline{J_{\varphi_a}(p)}\overline{p}\right)\right] \\ &= Re\left[\overline{\beta}^T\left(w_z(\alpha)\frac{1-\overline{a}^T\alpha}{1-|a|^2}(\alpha-a) + w_{\overline{z}}(\alpha)\frac{1-a^T\overline{\alpha}}{1-|a|^2}(\overline{\alpha}-\overline{a})\right)\right] \end{aligned} \tag{63}$$

If $a = 0$, then we have

$$Re\left[\overline{\beta}^T\left(w_z(\alpha)\alpha + w_{\overline{z}}(\alpha)\overline{\alpha}\right)\right] \ge \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(0)}.$$

This completes the proof of the theorem. □

References

[1] D. M. Burns, S. G. Krantz, Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary, Journal of the American Mathematical Society 7 (1994), 661–676.

[2] P. Duren, Harmonic Mappings in the Plane, Cambridge University Press, New York, 2004.
[3] J. B. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.
[4] E. Heinz, On one-to-one harmonic mappings, Pacfic Journal of Mathematics 9 (1959), 101–105.
[5] D. Kalaj, On harmonic functions and the Schwarz lemma, Proceedings of the American Mathematical Society 140 (2012), 161–165.
[6] D. Kalaj, On harmonic functions on surface with positive Gauss curvature and Schwarz lemma, Rocky Mountain Journal of Mathematics 44 (2014), 1585–1594.
[7] M. Knežević, M. Mateljević, On the quasi-isometries of harmonic quasiconformal mappings, Journal of Mathematical Analysis and Applications 334 (2007), 404–413.
[8] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bulletin of the American Mathematical Society 42 (1936), 689–692.
[9] T. Liu, X. Tang, A new boundary rigidity theorem for holomorphic self-mappings of the unit ball in C^n, Pure and Applied Mathematics Quarterly 11 (2015), 115–130.
[10] T. Liu, J. Wang, X. Tang, Schwarz lemma at the boundary of the unit ball in C^n and its applications, Journal of Geometric Analysis 25 (2015), 1890–1914.
[11] T. Liu, X. Tang, Schwarz lemma at the boundary of strongly pseudoconvex domain in C^n, Mathematische Annalen 366 (2016), 655–666.
[12] Y. Liu, S. Dai, Y. Pan, Boundary Schwarz lemma for pluriharmonic mappings between unit balls, Journal of Mathematical Analysis and Applications 433 (2016), 487–495.
[13] M. Mateljević, Ahlfors-Schwarz lemma, Hyperbolic geometry, The Carathéodory and Kobayashi Metrics, Zbornik radova, VII Simpozijum Matematika i primene, Beograd, November 2017.
[14] M. Mateljević, Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions, Journal of Mathematical Analysis and Applications (2018), https://doi.org/10.1016/j.jmaa.2018.03.069.
[15] R. Osserman, A sharp Schwarz inequality on the boundary, Proceedings of the American Mathematical Society 128 (2000), 3513–3517.
[16] M. Pavlović, Introduction to function space on the disk, Matematički Institut SANU, Belgrade, 2004.
[17] B. Rodin, Schwarz's lemma for circle packings, Inventiones mathematicae 89 (1987), 271–289.
[18] W. Rudin, Function theorey of the unit ball in $\mathbb{C}^n$, Springer Verlag, New York, 1980.
[19] X. Tang, T. Liu and W. Zhang, Schwarz lemma at the boundary and rigidity property for holomorphic mappings on the unit ball of $\mathbb{C}^n$, Proceedings of the American Mathematical Society 20 (2016), 1709–1716.
[20] H. Wu, Normal families of holomorphic mappings, Acta Mathematica 119 (1967), 192–233.
[21] Z. Xu, Schwarz lemma for pluriharmonic functions, Indagationes Mathematicae 27 (2016), 923–929.