Filomat 32:15 (2018), 5421–5431
https://doi.org/10.2298/FIL1815421B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Characterization of Unbounded Generalized Meromorphic Operators

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Abstract. In this paper, we study the class of unbounded generalized meromorphic operators $GM(E,\infty)$, where E is a finite subset of $\mathbb{C}$, which generalizes the notion of unbounded meromorphic operators. More precisely, we give a decomposition and some characterizing properties of these operators based on the punctured neighborhood theorem and the operational calculus for unbounded operators.

1. Introduction

In a Banach space X, denote by $C(X)$ (resp. L(X)) the set of all closed (resp. the algebra of all bounded) linear operators from X into X. For $A \in C(X)$, we write $D(A) \subset X$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset X$ for the range of A. We set $\alpha(A) = \dim N(A)$ and $\beta(A) = codim(R(A))$. Let $\sigma(A)$ (resp. $\rho(A)$) denote the spectrum (resp. the resolvent set) of A. For $A \in C(X)$, we define the set

$$\triangle(A) = \{n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow R(A^n) \cap N(A) \subset R(A^m) \cap N(A)\}$$

The degree of stable iteration of A is defined as $dis(A) = \inf \triangle(A)$, where $dis(A) = \infty$ if $\triangle(A) = \emptyset$.
We define the set of upper semi-Fredholm operators by

$$\Phi_+(X) = \{A \in C(X) \ \ such \ \ that \ \ \alpha(A) < \infty \ \ and \ \ R(A) \ \ is \ \ closed \ \ in \ \ X\}$$

and the set of lower semi-Fredholm operators by

$$\Phi_-(X) = \{A \in C(X) \ \ such \ \ that \ \ \beta(A) < \infty \ \ and \ \ R(A) \ \ is \ \ closed \ \ in \ \ X\}$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ will denote the set of Fredholm operators from X into X.
The index of a Fredholm operator A is defined by $i(A) = \alpha(A) - \beta(A)$.

Following [12, Definition 3.1.2], an operator $A \in C(X)$ is called quasi-Fredholm of degree $d \in \mathbb{N}$ if the following three conditions are fulfilled:

2010 *Mathematics Subject Classification*. Primary 47A53, ; Secondary 47B06
Keywords. B-Fredholm operator, unbounded generalized meromorphic operator, meromorphic operator, polynomially meromorphic operator
Received: 19 May 2018; Accepted: 01 August 2018
Communicated by Snežana Živković-Zlatanović
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(i) $dis(A) = d$;
(ii) $R(A^d) \cap N(A)$ is a closed and complemented subspace of X;
(iii) $R(A) + N(A^d)$ is a closed and complemented subspace of X.
We will denote by $QF(d)$ the set of quasi-Fredholm operators of degree d.
Note that, this definition is equivalent to the definition given in [5, Definition 2.2].

According to [5, Definition 2.4], an operator $A \in C(X)$ is called upper semi B-Fredholm (resp. lower semi B-Fredholm) if there exists an integer $d \in \mathbb{N}$ such that $A \in QF(d)$ and such that $N(A) \cap R(A^d)$ is of finite dimension (resp. $R(T) + N(T^d)$ is of finite codimension). These sets are denoted respectively by $\Phi_B^+(X)$ and $\Phi_B^-(X)$. A B-Fredholm operator is an upper and lower semi B-Fredholm operator. The set of these operators is denoted by $\Phi_B(X)$. Note that, this definition coincides with that given in [3]. In this case, we define the index of A as the integer: $ind(A) = \dim(N(A) \cap R(A^d)) - codim(R(A) + N(A^d))$.
For $A \in C(X)$, the ascent $a(A)$ of A is defined by:

$$a(A) = \inf\{n \in \mathbb{N} : N(A^n) = N(A^{n+1})\}$$

and the descent $d(A)$ of A is defined by:

$$d(A) = \inf\{n \in \mathbb{N} : R(A^n) = R(A^{n+1})\}$$

An operator $A \in C(X)$ is called Drazin invertible if $a(A)$ and $d(A)$ are both finite. This set is denoted by $DR(X)$. A is called left Drazin invertible (resp. right Drazin invertible) if $a(A)$ is finite and $R(A^{a(A)+1})$ is closed (resp. $d(A)$ is finite and $R(A^{d(A)})$ is closed). These sets are denoted respectively by $LD(X)$ and $RD(X)$.

For $A \in C(X)$, we define respectively the B-Fredholm spectrum, Drazin spectrum, the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum, the left Drazin spectrum and the right Drazin spectrum of A as follows:

$$\sigma_{BF}(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin \Phi_B(X)\ \}$$

$$\sigma_D(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin DR(X)\ \}$$

$$\sigma_{BF^+}(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin \Phi_B^+(X)\ \}$$

$$\sigma_{BF^-}(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin \Phi_B^-(X)\ \}$$

$$\sigma_{LD}(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin LD(X)\ \}$$

$$\sigma_{RD}(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin RD(X)\ \}$$

The upper semi B-Fredholm, the lower semi B-Fredholm and the B-Fredholm resolvent of $A \in C(X)$ are defined respectively by

$$\rho_{BF^+}(A) = \mathbb{C} \backslash \sigma_{BF^+}(A)$$

$$\rho_{BF^-}(A) = \mathbb{C} \backslash \sigma_{BF^-}(A)$$

$$\rho_{BF}(A) = \rho_{BF^+}(A) \cap \rho_{BF^-}(A)$$

The purpose of this paper is to extend the concept of unbounded meromorphic operators introduced by S. R. Caradus in [8] to unbounded generalized meromorphic operators. It is well known that, this class of operators is a generalization of that of Riesz operators which includes also the class of compact operators as shown in [7]. We recall that, an operator $A \in L(X)$ is of Riesz-type if $\lambda I - A$ is a Fredholm operator and $a(\lambda I - A) = d(\lambda I - A) < \infty$, for each $\lambda \neq 0$ and is meromorphic, if every non-zero isolated point of its spectrum is a pôle of the resolvent of A. In the bounded case, the generalized meromorphic operators are studied by S. Č. Živković Zlatanović et al. in [17], they give some characterizing properties and they have established an equivalence between this class of operators and that of polynomially meromorphic operators (see Definition 3.8), which generalizes some of their results obtained in [16] on the structure of bounded linear generalized Riesz operators on Banach spaces. As an example of generalized meromorphic operators, polynomially compact more generally polynomially Riesz and generalized Riesz operators introduced in [11] (see Remark 3.4 ii). In Theorem 3.10, we generalize the equivalence stated as above to closed linear operators densely defined on a Banach space with a non-empty resolvent set. Among the main results of this paper, we have also Theorem 3.6 in which we characterize the unbounded generalized meromorphic operators by mean of B-Fredholm spectrum and the two decomposition Theorems 4.2 and 4.4 of these operators, which generalizes that obtained by S .R. Caradus in [8] on the structure of unbounded meromorphic operators.

We organize our paper in the following way: In Section 2, we establish a punctured neighborhood theorem for closed semi B-Fredholm operators with a non-empty resolvent set which plays an important role in all of this work. In Section 3, by applying the results obtained in Section 2, we gather some characterizing properties of unbounded generalized meromorphic operators which generalizes that obtained by P. Aiena [1] in the case of Riesz operators and B.P. Duggal et al. [17] in the case of bounded linear generalized meromorphic operators. Finally,

Section 4 is devoted to the decomposition of unbounded generalized meromorphic operators extending an earlier result of S. R. Caradus obtained in [8].

2. On Punctured Neighborhood Theorem

It is well known that the punctured neighborhood theorem for semi B-Fredholm operators has been established in [6] for bounded linear operators having topological uniform descent as a consequence of [10, Theorem 4.7]. The aim of this section is to generalize this result to closed linear operators with a non-empty resolvent set using another technique, since the previous reference is not applied in the case of unbounded operators.

Theorem 2.1. *Let $A \in C(X)$ such that $\rho(A) \neq \emptyset$. If A is an upper semi B-Fredholm (resp. lower semi B-Fredholm) operator, then there exists an open disc $D(0,\varepsilon)$ centered at 0 such that $\lambda I - A$ is an upper semi-Fredholm (resp. lower semi-Fredholm) operator, for each $\lambda \in D(0,\varepsilon)\backslash\{0\}$. Moreover, $i(\lambda I - A) = ind(A)$, for all $\lambda \in D(0,\varepsilon)$.*

Proof. Let A be an upper semi B-Fredholm (resp. lower semi B-Fredholm) operator. Then, using [5, Proposition 2.8], there exists $n \in \mathbb{N}$ such that $R(A^n)$ is a closed subspace of X and the restruction of A to $R(A^n)$ denoted by A_n is an upper semi-Fredholm (resp. lower semi-Fredholm) operator. So, by [13, Theorem 2.3], there exist $\varepsilon > 0$ and a nonnegative integer r such that, for all $0 < |\lambda| < \varepsilon$, we have

$$\alpha(\lambda I - A_n) = \alpha(A_n) - r \tag{2.1}$$

$$\beta(\lambda I - A_n) = \beta(A_n) - r \tag{2.2}$$

and

$$R(\lambda I - A_n) = R(\lambda I - A) \cap R(A^n) \ \ is \ \ closed$$

To show that $R(\lambda I - A)$ is closed, the polynomials z^n and $\lambda - z$ are prime, then there exist $u(z)$ and $v(z)$ two polynomials such that $z^n u(z) + (\lambda - z)v(z) = 1$. Thus,

$$A^n u(A) + (\lambda I - A)v(A) = I_{D(A^n)} \tag{2.3}$$

Since, $\rho(A) \neq \emptyset$, then from [13, Lemma 1.1], we have $X = D(A^n) + R(A^n)$. Hence, by using the equality 2.3, we obtain that $X = R(\lambda I - A) + R(A^n)$. On the other hand, since $R(\lambda I - A) + R(A^n)$ and $R(\lambda I - A) \cap R(A^n)$ are closed, then by Neubauer

Lemma [12, Proposition 2.1.1], we obtain that $R(\lambda I - A)$ is closed. By [13, Lemma 1.3], we have $\alpha(\lambda I - A_n) = \alpha(\lambda I - A)$ and $\beta(\lambda I - A_n) = \beta(\lambda I - A)$, hence, by using Eqs. 2.1 and 2.2, we get $\lambda I - A$ is an upper semi-Fredholm (resp. lower semi-Fredholm) operator, for all $0 < |\lambda| < \varepsilon$. So, we can deduce by using Eqs. 2.1, 2.2 and [3, Theorem 2.4], that $i(\lambda I - A) = i(A_n) = ind(A)$, for each $0 \leq |\lambda| < \varepsilon$. ■

From Theorem 2.1, we can deduce the following result:

Corollary 2.2. *Let $A \in C(X)$ such that $\rho(A) \neq \emptyset$. Then,*
(i) *$\rho_{BF^+}(A)$, $\rho_{BF^-}(A)$ and $\rho_{BF}(A)$ are open subsets of $\mathbb{C}$.*
(ii) *$ind(\lambda I - A)$ is constant on any component of $\rho_{BF^+}(A)$ (resp. $\rho_{BF^-}(A)$).*

Proof. (i) Let $\alpha_0 \in \rho_{BF^-}(A)$ resp. $(\rho_{BF^+}(A))$, then $\alpha_0 I - A$ is a lower semi B-Fredholm (resp. an upper semi B-Fredholm) operator. Since, $\rho(\alpha_0 I - A) \neq \emptyset$, from Theorem 2.1, there exists an $\varepsilon > 0$, such that $\lambda I - A$ is a lower semi-Fredholm (resp. an upper semi-Fredholm) operator, for each $\lambda \in D(\alpha_0, \varepsilon)\backslash\{\alpha_0\}$, which implies that $\rho_{BF^-}(A)$ resp. $(\rho_{BF^+}(A))$ is an open subset of $\mathbb{C}$.
Since, $\rho_{BF}(A) = \rho_{BF^+}(A) \cap \rho_{BF^-}(A)$, then we get $\rho_{BF}(A)$ is an open subset of $\mathbb{C}$.
(ii) Let Ω be a component of $\rho_{BF^+}(A)$ (*resp.* $\rho_{BF^-}(A)$), $\lambda_0 \in \Omega$ be a fixed point and $\lambda_1 \in \Omega$ be an arbitrary point that are connected by a polygonal line Γ contained in Ω. Hence, from the assertion (i) of this corollary, for each $\mu \in \Gamma$, there exits an open disc $D(\mu, \varepsilon)$, such that $ind(\mu I - A) = ind(\lambda I - A)$, for each $\lambda \in D(\mu, \varepsilon)$. By the Heine-Borel theorem, there exist a finite number of open discs that cover Γ, therefore, we deduce that $ind(\lambda_0 I - A) = ind(\lambda_1 I - A)$. ■

Remark 2.3. *Assertions (i) and (ii) of the above corollary are well-known for the unbounded Fredholm operators densely defined [14, Theorem 7.25].*

We finish this section by the following assertions which are proved in [2] for bounded linear operators. Based on the Theorem 2.1, we will prove that they remain also valid in the case of closed linear operators with a non-empty resolvent set.

Theorem 2.4. *Let $A \in C(X)$ such that $\rho(A) \neq \emptyset$. Then*
(i) *If Ω is a connected component of $\rho_{BF^+}(A)$, then $\lambda I - A \in LD(X)$, either for every point of Ω or for no point of Ω.*
(ii) *If Ω is a connected component of $\rho_{BF^-}(A)$, then $\lambda I - A \in RD(X)$, either for every point of Ω or for no point of Ω.*
(iii) *If Ω is a connected component of $\rho_{BF}(A)$, then $\lambda I - A \in DR(X)$, either for every point of Ω or for no point of Ω.*

Proof. (i) Let Ω be a connected component of $\rho_{BF^+}(A)$ and define the set

$$\Psi_1 = \{\lambda \in \Omega \ \ such \ \ that \ \ \lambda I - A \in LD(X)\}$$

Suppose there exists $\alpha_0 \in \Psi_1$, then $\alpha_0 I - A$ is an upper semi B-Fredholm operator, $a(\alpha_0 I - A) < \infty$ and $R(\alpha_0 I - A)^{p+1}$ is closed, where $p = a(\alpha_0 I - A)$. Since, $\rho(\alpha_0 I - A) \neq \emptyset$, then by using Theorem 2.1, we can find an $\varepsilon_1 > 0$ such that $\lambda I - A$ is an upper semi-Fredholm operator, for all $\lambda \in D(\alpha_0, \varepsilon_1)\backslash\{\alpha_0\}$. From [13, Lemma 2.5], there exists $\varepsilon_2 > 0$ such that $a(\lambda I - A) = 0$, for all $\lambda \in D(\alpha_0, \varepsilon_2)\backslash\{\alpha_0\}$. If we take $D(\alpha_0, \varepsilon) = D(\alpha_0, \varepsilon_1) \cap D(\alpha_0, \varepsilon_2)$, then we get $D(\alpha_0, \varepsilon) \cap \Omega \subset \Psi_1$. This shows that Ψ_1 is an open subset of Ω. To show that Ψ_1 is a closed subset of Ω, we consider $\alpha_0 \in \overline{\Psi_1} \cap \Omega$, where $\overline{\Psi_1}$ is the closure of Ψ_1. Hence, $\alpha_0 I - A$ is an upper semi B-Fredholm operator. Hence, it follows from Theorem 2.1, that there exists $\varepsilon > 0$ such that $\lambda I - A$ is an upper semi-Fredholm operator, for each $\lambda \in D(\alpha_0, \varepsilon)\backslash\{\alpha_0\}$. Since, $\alpha_0 \in \overline{\Psi_1}$, then $D(\alpha_0, \varepsilon) \cap \Psi_1 \neq \emptyset$. Let $\mu \in D(\alpha_0, \varepsilon) \cap \Psi_1$ and Γ a polygonal line contained in Ω connecting α_0 and μ. By the Heine-Borel theorem, there is a finite number of open discs that cover Γ. Therefore, we have $\alpha_0 \in \Psi_1$, which entails that Ψ_1 is closed in Ω. Then, by the connectivity of Ω, we conclude that $\Psi_1 = \Omega$ or $\Psi_1 = \emptyset$. This shows the assertion (i).
(ii) Let Ω be a connected component of $\rho_{BF^-}(A)$ and define the set

$$\Psi_2 = \{\lambda \in \Omega \ \ such \ \ that \ \ \lambda I - A \in RD(X)\}$$

Suppose there exists $\alpha_0 \in \Psi_2$, then $\alpha_0 I - A$ is a lower semi B-Fredholm operator, $d(\alpha_0 I - A) < \infty$ and $R(\alpha_0 I - A)^q$ is closed, where $q = d(\alpha_0 I - A)$. Thus, from [13, Lemma 2.4], there exists $\varepsilon > 0$ such that $d(\lambda I - A) = 0$, for all $\lambda \in D(\alpha_0, \varepsilon)\backslash\{\alpha_0\}$ and $R(\lambda I - A)^0 = X$ is closed, which implies that $D(\alpha_0, \varepsilon) \cap \Omega \subset \Psi_2$. Hence, Ψ_2 is an open set of Ω. To show that Ψ_2 is closed in Ω, we use the same proof as in part (i) and finally we conclude that $\Psi_2 = \Omega$ or $\Psi_2 = \emptyset$.
(iii) It is an immediate result from (i) and (ii). ■

3. Unbounded Generalized Meromorphic Operators

The aim of this section is to give some characterizing properties of unbounded generalized meromorphic operators by mean of B-Fredholm spectrum and polynomially meromorphic operators. First, we start with the following theorem proved in [4] in the case of Hilbert space. Based on the Kato decomposition theorem [12, Theorem 3.2.1], which is also true in the case of Banach spaces as shown by Labrousse in [12, p. 206], we can generalize this result to the case of Banach spaces.

Theorem 3.1. *Let $A \in C(X)$ be a densely defined linear operator such that $\rho(A) \neq \emptyset$. Let $p(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)^{m_i}$ be a polynomial of degree s with complex coefficients and $\lambda_i \neq \lambda_j$. Then, $p(A)$ is a B-Fredholm operator if and only if $A - \lambda_i I$ is a B-Fredholm operator, for all $1 \leq i \leq n$.*

Proof. Since, $\lambda_i \neq \lambda_j$, then the polynomials $p_i(\lambda) = (\lambda - \lambda_i)^{m_i}$, $1 \leq i \leq n$, are relatively prime, and hence there exist polynomials $q_i(\lambda)$ and $q_j(\lambda)$ such that $q_i(\lambda)p_i(\lambda) + q_j(\lambda)p_j(\lambda) = 1$, with $1 \leq i \neq j \leq n$. Then, we have for sufficiently large m

$$q_i(A)p_i(A) + q_j(A)p_j(A) = I_{D(A^m)} \tag{3.1}$$

Using the identity 3.1, we can deduce that

$$R[p(A)]^m = R[p_i(A)]^m \cap R[p_j(A)]^m, \forall m \geq 1 \tag{3.2}$$

$$N[p(A)]^m = N[p_i(A)]^m + N[p_j(A)]^m, \forall m \geq 1 \tag{3.3}$$

$'' \Longleftarrow ''$ Assume that $A - \lambda_i I$ is a B-Fredholm operator, for all $1 \leq i \leq n$. Then, by using [3, Theorem 3.6], we obtain that $(A - \lambda_i I)^{m_i}$ is a B-Fredholm operator, for all $1 \leq i \leq n$. Hence, $(A - \lambda_i I)^{m_i}$ is a Quasi-Fredholm operator of degree $d_i = dis((A - \lambda_i I)^{m_i})$, $\dim[N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d_i})] < \infty$ and $codim[N((A - \lambda_i I)^{m_i d_i}) + R((A - \lambda_i I)^{m_i})] < \infty$, for all $1 \leq i \leq n$. Set $d = \max_{1 \leq i \leq n}(d_i)$. Since, $R((A - \lambda_i)^{m_i})^m$ is closed, for each $m \geq d_i$, then from the formula 3.2, we have $R[p(A)]^m$ is closed, for each $m \geq d$. We have from [4, Lemma 3.2], that

$$N[p(A)] \cap R[p(A)]^d = \sum_{i=1}^{n} N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d})$$

Since, the polynomials $(A - \lambda_i I)^{m_i}_{1 \leq i \leq n}$ are relatively prime, then we have

$$\dim[N[p(A)] \cap R[p(A)]^d] = \sum_{i=1}^{n} \dim[N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d})]$$

Since, $N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d}) \subset N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d_i})$ and using the same reasoning as in [4, Theorem 3.3], we get $\dim[N[p(A)] \cap R[p(A)]^d] < \infty$.
On the other hand, the fact that $R(p(A^*)^d)$ and $N(p(A)^*))$ are closed, this implies that $\dim[R(p(A)) + N(p(A)^d]^{\perp} = \dim[R(p(A^*)^d) \cap N(p(A)^*))] = \sum_{i=1}^{n} \dim[N((A^* - \lambda_i I)^{m_i}) \cap R((A^* - \lambda_i I)^{m_i d})]$, where A^* is the adjoint operator of A. Since, $(A - \lambda_i I)^{m_i}$ is a B-Fredholm operator, for all $1 \leq i \leq n$, then it follows from [3, Proposition 2.5], that $(A^* - \lambda_i I)^{m_i}$ is also a B-Fredholm operator, for all $1 \leq i \leq n$. Therefore, we use the same arguments as above, we obtain $codim[R(p(A)) + N(p(A)^d] = \dim[R(p(A)) + N(p(A)^d]^{\perp} < \infty$. Finally, using [3, Proposition 2.2], we can deduce that $p(A)$ is a B-Fredholm operator.
$'' \Longrightarrow ''$ If $p(A)$ is a B-Fredholm operator, then it is a Quasi-Fredholm operator of degree $d = dis(p(A))$,

$\dim(N[p(A)] \cap R[p(A)]^d) < \infty$ and $codim(R(p(A)) + N(p(A))^d) < \infty$. Set $d_i = dis((A - \lambda_i I)^{m_i})$, then we have $d = \max\limits_{1 \le i \le n}(d_i)$. Using the same technique as above and [3, Proposition 2.2], we get $(A - \lambda_i I)^{m_i}$ is a B-Fredholm operator, for all $1 \le i \le n$. Since

$$N(A - \lambda_i I) \cap R((A - \lambda_i I)^{m_i d_i}) \subset N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d_i})$$

$$R((A - \lambda_i I)^{m_i}) + N((A - \lambda_i I)^{m_i d_i}) \subset N((A - \lambda_i I)^{m_i d_i}) + R(A - \lambda_i I)$$

Then, we have $\dim[N(A - \lambda_i I) \cap R(A - \lambda_i I)^{m_i d_i}] < \infty$ and $codim[N(A - \lambda_i I)^{m_i d_i} + R(A - \lambda_i I)] < \infty$, which implies from [3, Proposition 2.2], that $A - \lambda_i I$ is a B-Fredholm operator, for all $1 \le i \le n$. ■

It is well known that the spectral mapping theorem, holds for the usual spectrum [9]. Here, using Theorem 3.1, we can generalize it for the B-Fredholm spectrum of closed linear operators densely defined with a non-empty resolvent set.

Theorem 3.2. *Let $A \in C(X)$ be a densely defined linear operator such that $\rho(A) \neq \emptyset$ and p be a polynomial with complex coefficients. Then,*

$$\sigma_{BF}(p(A)) = p(\sigma_{BF}(A))$$

Proof. Let $\lambda \notin \sigma_{BF}(p(A))$ and we show that $\lambda \notin p(\sigma_{BF}(A))$.
If there exists an $\alpha_0 \in \sigma_{BF}(A)$ such that $\lambda = p(\alpha_0)$, then we have $p(\alpha_0) - p(z) = (\alpha_0 - z)^k H(z)$, with $H(\alpha_0) \neq 0$. Since, $\lambda - p(A)$ is a B-Fredholm operator, then from Theorem 3.1, we get $\alpha_0 I - A$ is a B-Fredholm operator, which contradicts the fact that $\alpha_0 \in \sigma_{BF}(A)$. Hence, we conclude that $\lambda \notin p(\sigma_{BF}(A))$.
Conversely, assume that $\lambda \notin p(\sigma_{BF}(A))$ and set $Q(z) = \lambda - p(z) = c\prod_{i=1}^{n}(z - \lambda_i)^{m_i}$, with c is non-zero constant and m_i, $1 \le i \le n$ are positive integers. Suppose that there exists $1 \le i_0 \le n$ such that $\lambda_{i_0} \in \sigma_{BF}(A)$, then the fact that $Q(\lambda_{i_0}) = 0$, shows that $\lambda = p(\lambda_{i_0})$, this contradicts the fact that $\lambda \notin p(\sigma_{BF}(A))$. Hence, $\lambda_i \notin \sigma_{BF}(A)$, for all $1 \le i \le n$. Thus, from Theorem 3.1, we can deduce that $Q(A)$ is a B-Fredholm operator and finally $\lambda \notin \sigma_{BF}(p(A))$ which finish the proof of this theorem. ■

In the sequel, we introduce a new class of operators which generalizes the notion of unbounded meromorphic operators introduced by S. R. Caradus in [8].

Definition 3.3. *Let $A \in C(X)$ be an unbounded linear operator. We say that A is a generalized meromorphic operator if there exists a finite subset E of $\mathbb{C}$ such that the only allowable points of accumulation of $\sigma(A)$ are $\{\infty\}$ and the points of E and every isolated point $\lambda \in \sigma(A)\backslash E$ is a pole of the resolvent of A. We will denote this set by $GM(E, \infty)$. If $\lambda \in E$ (resp. $\lambda = \infty$) is the only allowable point of accumulation of $\sigma(A)$, we will write $GM(E)$ (resp. $GM(\infty)$) to denote the corresponding class of operators. We will denote by $GM_f(E, \infty)$ (resp. $GM_f(\infty)$) the set of all operators $A \in GM(E, \infty)$ such that every $\lambda \in \sigma(A)\backslash E$ is an eigenvalue of finite multiplicity and the allowable accumulation points of $\sigma(A)$ are ∞ and the points of E (resp. the set of all operators $A \in GM(E, \infty)$ such that every $\lambda \in \sigma(A)\backslash E$ is an eigenvalue of finite multiplicity and ∞ is the only point of accumulation of its spectrum).*
Note that,

$$GM_f(E) \subseteq GM(E) \subseteq GM(E, \infty),$$

$$GM_f(\infty) \subseteq GM(\infty) \subseteq GM(E, \infty)$$

Remark 3.4. (*i*) *If $E := \{0\}$, then we find the class of unbounded meromorphic operators introduced by S. R. Caradus in [8].*
(*ii*) *We recall that an operator $A \in L(X)$ is called a generalized Riesz operator [11, Definition 1.2] if there exists a finite subset E of $\mathbb{C}$ such that*
(*a*) *For all $\lambda \in \mathbb{C}\backslash E$, $(\lambda I - A)$ is a Fredholm operator on X.*
(*b*) *For all $\lambda \in \mathbb{C}\backslash E$, $(\lambda I - A)$ has finite ascent and finite descent.*
(*c*) *All $\lambda \in \sigma(A)\backslash E$ are eigenvalues of finite multiplicity, and have no accumulation point except possibly points of E.*
We note that, a generalized Riesz operator is a generalized meromorphic operator.

Based on the Theorem 2.4, we can show the following important result, which is a generalization of that obtained in [1] in the case of Riesz operators acting on a Banach space.

Proposition 3.5. *Let A be a closed linear operator with a non-empty resolvent set and E a finite subset of $\mathbb{C}$. Then the following assertions are equivalent:*
(i) $\lambda I - A \in \Phi_B(X)$, for all $\lambda \in \mathbb{C}\backslash E$.
(ii) $\lambda I - A \in DR(X)$, for all $\lambda \in \mathbb{C}\backslash E$.
(iii) $\lambda I - A \in LD(X)$, for all $\lambda \in \mathbb{C}\backslash E$.
(iv) $\lambda I - A \in \Phi_B^+(X)$, for all $\lambda \in \mathbb{C}\backslash E$.
(v) $\lambda I - A \in RD(X)$, for all $\lambda \in \mathbb{C}\backslash E$.
(vi) $\lambda I - A \in \Phi_B^-(X)$, for all $\lambda \in \mathbb{C}\backslash E$.

Proof. ($i \Longrightarrow ii$) Let $\lambda_0 \in \mathbb{C}\backslash E$ and set $E' = E\backslash\rho(A)$, then $\lambda_0 \in \mathbb{C}\backslash E'$ and $\lambda_0 I - A \in \Phi_B(X)$. Since, $\mathbb{C}\backslash E'$ is a connected component of $\rho_{BF}(A)$, then applying Theorem 2.4 (*iii*), we obtain that $\lambda I - A \in DR(X)$ either for every point of $\mathbb{C}\backslash E'$ or for no point of $\mathbb{C}\backslash E'$. The fact that $\mathbb{C}\backslash E'$ contains $\rho(A)$ which is non-empty, this entails that $\lambda I - A \in DR(X)$ for every point of $\mathbb{C}\backslash E'$ in particular for $\lambda = \lambda_0$. Thus, we obtain the assertion (*ii*).
By using Theorem 2.4 (*i*) and (*ii*) and the same proof as in ($i \Longrightarrow ii$), we can prove the assertions ($iv \Longrightarrow iii$) and ($vi \Longrightarrow v$).
Since, a left Drazin invertible (resp. right Drazin invertible) operator is an upper semi B-Fredholm (resp. lower semi B-Fredholm) one, then we can deduce the following assertions ($ii \Longrightarrow i$), ($iii \Longrightarrow iv$) and ($v \Longrightarrow vi$).
($iii \Longrightarrow ii$) If $\lambda I - A \in LD(X)$, for all $\lambda \in \mathbb{C}\backslash E$, then it is left Drazin invertible operator, for all $\lambda \in \mathbb{C}\backslash E'$, where $E' = E\backslash\rho(A)$.
Let

$$\Omega_1 = \{\lambda \in \mathbb{C}\backslash E' \ \ such \ \ that \ \ d(\lambda I - A) < \infty\}$$

The set $\Omega_1 \neq \emptyset$, since $\rho(A)$ is non-empty. Let $\alpha_0 \in \Omega_1$, then $\alpha_0 \in \mathbb{C}\backslash E'$ and $d(\alpha_0 I - A) < \infty$. From [13, Lemma 2.4], there exists an $\varepsilon > 0$ such that $d(\lambda I - A) = 0$, for all $\lambda \in D(\alpha_0, \varepsilon)\backslash\{\alpha_0\}$. Hence, Ω_1 is an open set of $\mathbb{C}\backslash E'$. To show that Ω_1 is closed in $\mathbb{C}\backslash E'$, we consider $\alpha_0 \in \overline{\Omega_1} \cap \mathbb{C}\backslash E'$. Then, $\alpha_0 I - A \in LD(X)$. So, it is an upper semi B-Fredholm operator. Hence, by using Theorem 2.1, there exists $\varepsilon > 0$ such that $\lambda I - A$ is an upper semi-Fredholm operator, for each $\lambda \in D(\alpha_0, \varepsilon)\backslash\{\alpha_0\}$. Since, $\alpha_0 \in \overline{\Omega_1}$, then we have $D(\alpha_0, \varepsilon) \cap \Omega_1 \neq \emptyset$. Let $\mu \in D(\alpha_0, \varepsilon) \cap \Omega_1$ and Γ a polygonal line contained in $\mathbb{C}\backslash E'$ connecting μ and α_0. By the Heine-Borel theorem, there exist a finite number of open discs that cover Γ. Thus, Ω_1 is a closed subset of $\mathbb{C}\backslash E'$ and finally $\Omega_1 = \mathbb{C}\backslash E'$. Since, $a(\lambda I - A) < \infty$ and $d(\lambda I - A) < \infty$, for each $\lambda \in \mathbb{C}\backslash E'$, this entails that $\lambda I - A \in DR(X)$, for each $\lambda \in \mathbb{C}\backslash E'$ and so it is Drazin invertible for every $\lambda \in \mathbb{C}\backslash E$ which shows the assertion (*ii*).
($v \Longrightarrow ii$) We use Theorem 2.1 and [13, Lemma 2.5] and the same technique as in ($iii \Longrightarrow ii$), we can obtain the assertion (*ii*). Since, a Drazin invertible operator is both left and right Drazin invertible, then we can deduce the assertions ($ii \Longrightarrow iii$) and ($ii \Longrightarrow v$).
The assertions ($i \Longrightarrow iv$) and ($i \Longrightarrow vi$) are immediate from the fact that a B-Fredholm operator is both upper and lower semi B-Fredholm operator. ■

In the following theorem, by applying Proposition 3.5, we give a characterization of unbounded generalized meromorphic operators by mean of B-Fredholm spectrum.

Theorem 3.6. *Let $A \in C(X)$ with a non-empty resolvent set and E a finite subset of $\mathbb{C}$. Then, $A \in GM(E, \infty)$ if and only if $\sigma_{BF}(A) \subset E$.*

Proof. Suppose that $A \in GM(E, \infty)$ and let $\lambda \in \mathbb{C}\backslash E$, then λ is a pôle of the resolvent of A. This, implies that $\sigma_D(A) \subset E$ and by Proposition 3.5, we get $\sigma_{BF}(A) \subset E$. Conversely, suppose that $\sigma_{BF}(A) \subset E$ and let $\lambda_0 \in \sigma(A)\backslash E$ be an isolated point. Then, from Proposition 3.5, we have $\lambda_0 I - A$ is a Drazin invertible operator, which implies by using [13, Theorem 2.1], that λ_0 is a pôle of the resolvent of A. Hence, $\sigma(A)$ is a discrete set, for which the points of E and ∞ are the only possible points of accumulation. Finally, we conclude that $A \in GM(E, \infty)$. ■

The following theorem extend [8, Corollary, p. 747] on the structure of unbounded meromorphic operators to the case of unbounded generalized meromorphic operators.

Theorem 3.7. *Let $A \in C(X)$ be an unbounded operator such that $0 \in \rho(A)$. Then, A is a generalized meromorphic operator if and only if A^{-1} is a bounded generalized meromorphic operator.*

Proof. If A^{-1} is a bounded generalized meromorphic operator, then it follows from Theorem 3.6, that $\sigma_{BF}(A^{-1})$ is a finite subset of $\mathbb{C}$. Then, from [5, Theorem 3.6], we have

$$\sigma_{BF}(A) = \{\lambda^{-1} | \lambda \in \sigma_{BF}(A^{-1}) \backslash \{0\}\} \tag{3.4}$$

Set $\{\lambda_1, .., \lambda_n\}$ the non-zero elements of $\sigma_{BF}(A^{-1})$. Then

$$\sigma_{BF}(A) = \{\lambda_1^{-1}, ..., \lambda_n^{-1}\} \tag{3.5}$$

By using Theorem 3.6, we obtain that A is a generalized meromorphic operator.
Conversely, suppose that A is a generalized meromorphic operator, then there exists a finite subset E of $\mathbb{C}$ such that $A \in GM(E, \infty)$. Thus, it follows from Theorem 3.6, that $\sigma_{BF}(A) \subset E$. Hence, we get

$$\sigma_{BF}(A^{-1}) \subset \{0\} \cup \{\lambda^{-1}, \lambda \in \sigma_{BF}(A)\} \tag{3.6}$$

This entails that, $\sigma_{BF}(A^{-1})$ is also a finite subset of $\mathbb{C}$ which prove from Theorem 3.6, that A^{-1} is a bounded generalized meromorphic operator. ■

We present now the following definition:

Definition 3.8. *We say that an operator $A \in C(X)$ is polynomially meromorphic if there exists a non-zero polynomial $p(z)$ such that $p(A)$ is meromorphic.*

Remark 3.9. *It is clear that, a polynomially compact operator more generally a polynomially Riesz operator is a polynomially meromorphic one.*

Now, we are able to characterize the unbounded generalized meromorphic operators densely defined by mean of polynomially meromorphic operators.

Theorem 3.10. *Let $A \in C(X)$ be a densely defined linear operator such that $\rho(A) \neq \emptyset$. Then, A is a polynomially meromorphic operator if and only if A is generalized meromorphic.*

Proof. Let p be a non-zero polynomial, such that $p(A)$ is a meromorphic operator, then from Theorem 3.6, we have $\sigma_{BF}(p(A)) \subset \{0\}$. By using Theorem 3.2, we get $\sigma_{BF}(p(A)) = p(\sigma_{BF}(A)) \subset \{0\}$ which implies that $\sigma_{BF}(A) \subset \{\lambda_1, ..., \lambda_n\}$, where λ_i are the roots of p. Thus, from Theorem 3.6, the operator A is generalized meromorphic. Conversely, if A is a generalized meromorphic operator, then from Theorem 3.6, there exists a finite subset E of $\mathbb{C}$ such that $\sigma_{BF}(A) \subset E$. Set $E = \{\lambda_1, ..., \lambda_n\}$ and $p(z) = \prod_{i=1}^{n}(z - \lambda_i)$. We shaw that $p(A)$ is a meromorphic operator. Let $\lambda \neq 0$, $Q(z) = \lambda - p(z)$ and $\{\mu_1, ..., \mu_n\}$ its roots. For this, let us show that $\mu_i \neq \lambda_j$, for $1 \leq i, j \leq n$. Suppose that there exists $1 \leq i_0, j_0 \leq n$ such that $\mu_{i_0} = \lambda_{j_0}$, then we get $Q(\mu_{i_0}) = Q(\lambda_{j_0}) = \lambda \neq 0$ which contradicts the fact that μ_{i_0} is a zero of Q. Hence, from our hypothesis, we conclude that $\mu_i I - A$ is a B-Fredholm operator, for all $1 \leq i \leq n$ and so by Theorem 3.1, $Q(A) = \lambda - p(A)$ is a B-Fredholm operator, for all $\lambda \neq 0$ which implies that the operator $p(A)$ is meromorphic. ■

Remark 3.11. *The theorem stated as above has been established in [17, p. 93] in the case of bounded linear operators.*

4. Decomposition of Unbounded Generalized Meromorphic Operators

Our main result in this section is to yield a decomposition of unbounded generalized meromorphic operators which generalizes a result of S. R. Caradus in [8] on the structure of unbounded meromorphic operators.
In order to give our main result, we shall need the following Lemma:

Lemma 4.1. *Consider $X = X_1 \oplus X_2 \oplus .... \oplus X_n$ and $A \in C(X)$ such that $A = A_1 \oplus A_2 \oplus .... \oplus A_n$, where X_i is a closed A-invariant subspace of X, A_i is the restriction of A to X_i and $D(A_i) = D(A) \cap X_i$, $\forall\ 1 \leq i \leq n$. Then,*

$$\sigma_D(A) = \bigcup_{i=1}^{n} \sigma_D(A_i)$$

Proof. The fact that $a(A) < \infty$ (resp. $d(A) < \infty$) if and only if $a(A_i) < \infty$ (resp. $d(A_i) < \infty$), $\forall 1 \leq i \leq n$, this conclude the proof. ■

Theorem 4.2. *Let $A \in C(X)$ be an unbounded operator with a non-empty resolvent set and $E = \{\alpha_1, ..., \alpha_n\}$ a finite subset of $\mathbb{C}$. Then, if $A \in GM(E, \infty)$, we can write $A = A_1 + ... + A_n + A_{n+1}$, such that $A_s A_j = A_j A_s = 0$, for all $1 \leq s \neq j \leq n+1$, $A_s - \alpha_s I_s$ is a meromorphic operator, for each $1 \leq s \leq n$ and $A_{n+1} \in GM(\infty)$.*
In this way, we have

$$R_\mu(A) = R_\mu(A_1) + .. + R_\mu(A_n) + R_\mu(A_{n+1}) - \frac{2nI}{\mu} \tag{4.1}$$

Proof. Let $\sigma_1, ..., \sigma_n, \sigma_{n+1}$ be a pairwise disjoint spectral sets of A such that $\alpha_s \in \sigma_s$, $\infty \notin \sigma_s$, for all $1 \leq s \leq n$ and $\sigma_{n+1} = \widetilde{\sigma}(A) \backslash \sigma_1 \cup ... \cup \sigma_n$, where $\widetilde{\sigma}(A) = \sigma(A) \cup \{\infty\}$.
If $P_1, .., P_n, P_{n+1}$ are, respectively, the associated spectral projections, then it is clear that $P_1 + .. + P_n + P_{n+1} = I$, $P_s P_j = P_j P_s = 0$, for all $1 \leq s \neq j \leq n+1$.
Define $A_s = AP_s$, $1 \leq s \leq n+1$. Then, we have $A = A_1 + ... + A_n + A_{n+1}$ and $A_s A_j = A_j A_s = 0$, for all $1 \leq s \neq j \leq n+1$. Since, for all $1 \leq s \leq n$, σ_s does not contain ∞, then it follows from [15, Theorem 5.7-B], that $R(P_s) \subset D(A)$ and the fact that A_s are closed operators, this implies by the closed-graph theorem, that $A_s \in L(X)$, for each $1 \leq s \leq n$. We use the operational calculus for unbounded operators, as discussed in [15, pp 287-296], to deduce the assertions of this theorem. Let Ω, Ω_s, $1 \leq s \leq n+1$ be Cauchy domains such that $\sigma(A) \subseteq \Omega$, $\sigma_s \subseteq \Omega_s$, $1 \leq s \leq n+1$, $\bigcap_{s=1}^{n+1} \overline{\Omega_s} = \emptyset$ and $\Omega = \bigcup_{s=1}^{n+1} \Omega_s$.
Define the functions $f_s(\lambda)$, for $1 \leq s \leq n+1$ as follows:

$$f_s(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \overline{\Omega_s} \\ 0 & \text{otherwise} \end{cases}$$

Let $\mu \notin \Omega_s$, for each $1 \leq s \leq n$. Denote by $+B(\Omega)$ the positively oriented boundary. Using the above-mentioned operational calculus, we get for each, $1 \leq s \leq n$, that

$$\begin{aligned} R_\mu(A_s) &= \frac{I}{\mu} + \frac{1}{2i\Pi} \oint_{+B(\Omega)} g_s(\lambda) R_\lambda(A) d\lambda \\ &= \frac{I}{\mu} + \frac{1}{2i\Pi} \sum_{s=1}^{n} \oint_{+B(\Omega_s)} \frac{1}{\mu - \lambda} R_\lambda(A) d\lambda + \frac{1}{2i\Pi} \oint_{+B(\Omega_{n+1})} \frac{1}{\mu} R_\lambda(A) d\lambda \\ &= \frac{I + P_{n+1}}{\mu} + \frac{1}{2i\Pi} \sum_{s=1}^{n} \oint_{+B(\Omega)} \frac{f_s(\lambda)}{\mu - \lambda} R_\lambda(A) d\lambda \\ &= \frac{2I - P_s}{\mu} + P_s R_\mu(A) \end{aligned} \tag{4.2}$$

Where, $g_s(\lambda) = \dfrac{1}{\mu - \lambda f_s(\lambda)}$, for $1 \leq s \leq n$. Similarly, we have

$$\begin{aligned} R_\mu(A_{n+1}) &= \frac{1}{2i\Pi} \oint_{+B(\Omega)} g_{n+1}(\lambda) R_\lambda(A) d\lambda \\ &= \frac{1}{2i\Pi} \sum_{s=1}^{n} \oint_{+B(\Omega_s)} \frac{1}{\mu} R_\lambda(A) d\lambda + \frac{1}{2i\Pi} \oint_{+B(\Omega_{n+1})} \frac{1}{\mu - \lambda} R_\lambda(A) d\lambda \\ &= \frac{I - P_{n+1}}{\mu} + \frac{1}{2i\Pi} \oint_{+B(\Omega_{n+1})} \frac{1}{\mu - \lambda} R_\lambda(A) d\lambda \\ &= \frac{I - P_{n+1}}{\mu} + P_{n+1} R_\mu(A) \end{aligned} \tag{4.3}$$

Where, $g_{n+1}(\lambda) = \dfrac{1}{\mu - \lambda f_{n+1}(\lambda)}$.
Adding and rearranging the equalities (4.2) and (4.3), we obtain (4.1).
Now, we show that for each, $1 \leq s \leq n$, that $A_s - \alpha_s I_s$ is a meromorphic operator and $A_{n+1} \in GM(\infty)$. Indeed, let $\lambda_0 \in \sigma(A_s - \alpha_s I_s)\backslash\{0\}$ an isolated point, for $1 \leq s \leq n$. Since, $\lambda_0 + \alpha_s \neq \alpha_s$, for each, $1 \leq s \leq n$, and the fact that $A \in GM(E, \infty)$, this entails that λ_0 is a pole of the resolvent of $A_s - \alpha_s I$. Moreover, as all the operators $A_s, 1 \leq s \leq n$, are bounded, it is clear that the only possible accumulation point of $\sigma(A_{n+1})$ is ∞. Thus $A_{n+1} \in GM(\infty)$.

Remark 4.3. *The above theorem is also true, if we replace $GM(E, \infty)$, $GM(\infty)$ and meromorphic operators respectively by $GM_f(E, \infty)$, $GM_f(\infty)$ and Riesz operators.*

Theorem 4.4. *Under the hypothesis of Theorem 4.2, if $A \in GM(E, \infty)$, then there exists a closed A-invariant subspaces $X_1, ...X_n, X_{n+1}$ of X such that:*
(a) $A(D(A) \cap X_i) \subseteq X_i$, for all $1 \leq i \leq n + 1$,
(b) $X = X_1 \oplus ...X_n \oplus X_{n+1}$,
(c) $A = A_1 \oplus ...A_n \oplus A_{n+1}$, such that $A_i - \alpha_i I_i$ is a meromorphic operator, for $1 \leq i \leq n$ and $A_{n+1} \in GM(\infty)$, where A_i (resp. I_i) is the reduction of A (resp. of I) on X_i.

Proof. We define $X_i = R(P_i)$, $\forall 1 \leq i \leq n + 1$, where P_i is defined in the proof of Theorem 4.2. So that, from [15, Theorem 5.7-A], we have $X = X_1 \oplus ...X_n \oplus X_{n+1}$, $I = P_1 + ... + P_n + P_{n+1}$, $P_i P_j = 0$; for all $1 \leq i \neq j \leq n + 1$, $P_i(D(A)) \subset D(A)$, $AX_i \subset X_i$, $1 \leq i \leq n + 1$ and $A = A_1 \oplus ...A_n \oplus A_{n+1}$, where $A_i = A_{/X_i}$. Since, $\sigma_1, .., \sigma_n$ does not contain ∞, then again by [15, Theorem 5.7-B(e)], we get $A_i \in L(X_i)$, $1 \leq i \leq n$ and it follows from [15, p. 300], that $\sigma(A_i) = \sigma_i, 1 \leq i \leq n + 1$. Now, we must show that $\sigma_D(A_i) \subseteq \{\alpha_i\}, 1 \leq i \leq n$ and $A_{n+1} \in GM(\infty)$. Since, $A \in GM(E, \infty)$, then by using Theorem 3.6 and Proposition 3.5, we obtain that $\sigma_D(A) \subset E$.
Let $\lambda_0 \in \sigma(A_i)\backslash\{\alpha_1, ..., \alpha_n\}, 1 \leq i \leq n + 1$. The fact that $\sigma_D(A) \subset E$ and the use of Lemma 4.1, this implies that $\lambda_0 \notin \sigma_D(A_i)$, for all $1 \leq i \leq n + 1$. Since, $\sigma_D(A_i) \subset \sigma(A_i)$ and $\sigma(A_i) = \sigma_i$, for each $1 \leq i \leq n$, and the fact that $\alpha_i \in \sigma_i$, this entails that $\sigma_D(A_i) \subseteq \{\alpha_i\}, 1 \leq i \leq n$ and so $A_i - \alpha_i I_i$ is a meromorphic operator, for all $1 \leq i \leq n$. On the other hand, $\alpha_i \notin \sigma_{n+1}$, for each $1 \leq i \leq n$, which shows that α_i cannot be an accumulation points of $\sigma(A_{n+1})$. Hence, we conclude that ∞ is the only point of accumulation of $\sigma(A_{n+1})$ and finally we obtain that $A_{n+1} \in GM(\infty)$. ■

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