Filomat 32:15 (2018), 5441–5451
https://doi.org/10.2298/FIL1815441C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remarks on n-normal Operators

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Abstract. Let T be a bounded linear operator on a complex Hilbert space and $n, m \in \mathbb{N}$. Then T is said to be *n-normal* if $T^*T^n = T^nT^*$ and *(n, m)-normal* if $T^{*m}T^n = T^nT^{*m}$. In this paper, we study several properties of n-normal, (n, m)-normal operators. In particular, we prove that if T is 2-normal with $\sigma(T) \bigcap (-\sigma(T)) \subset \{0\}$, then T is polaroid. Moreover, we study subscalarity of n-normal operators. Also, we prove that if T is (n, m)-normal, then T is decomposable and Weyl's theorem holds for $f(T)$, where f is an analytic function on $\sigma(T)$ which is not constant on each of the components of its domain.

1. Introduction and Motivation

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\, , \,\rangle$ and $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $T = N|_{\mathcal{H}}$, *hyponormal* if $T^*T - TT^* \geq 0$. An operator T is said to be *scalar of order m* if it admits a spectral distribution of order m, i.e., if there is a continuous unital morphism $\Phi : C_0^m(\mathbb{C}) \longrightarrow \mathcal{L}(\mathcal{H})$ such that $\Phi(z) = T$, where z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m $(0 \leq m \leq \infty)$. An operator T is said to be *subscalar of order m* if it is similar to the restriction of a scalar operator of order m to an invariant subspace. It is known that subnormal operators are hyponormal and hyponormal operators are subscalar ([8]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T - \lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \longrightarrow \mathcal{H}$ such that $(T - \lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$.

2010 *Mathematics Subject Classification*. Primary 47B15; Secondary 47A15
Keywords. n-normal; polaroid; subscalar
Received: 04 June 2018; Accepted: 16 August 2018
Communicated by Dragan S. Djordjević
This is partially supported by Grant-in-Aid Scientific Research No.15K04910. The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology(2016R1A2B4007035).
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For an operator $T \in \mathcal{L}(\mathcal{H})$, the quasinilpotent part of $T - \lambda$ is defined as

$$H_0(T-\lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

In general, $\ker((T-\lambda)^m) \subset H_0(T-\lambda)$ and $H_0(T-\lambda)$ is not closed. However, if λ is an isolated point of $\sigma(T)$, then $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda)$ where

$$E_T(\{\lambda\}) = \frac{1}{2\pi i}\int_{\partial D} (z-T)^{-1}\, dz$$

denotes the Riesz idempotent corresponding to λ with D is a closed disk centered at λ which contains no other points of $\sigma(T)$. Hence $H_0(T-\lambda)$ is closed in this case.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known that Property (β) $\Longrightarrow$ Dunford's property (C) $\Longrightarrow$ SVEP, and the converse implications do not hold ([7, Proposision 1.2.19]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{H} = \mathcal{X} + \mathcal{Y}$, $\sigma(T|_{\mathcal{X}}) \subset \overline{U}$ and $\sigma(T|_{\mathcal{Y}}) \subset \overline{V}$. Remark that T is decomposable if and only if T and T^* have the property (β) ([7, Theorem 2.5.19]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if every isolated point of $\sigma(T)$ belongs to the point spectrum of T. Hence, hyponormal operators are isoloid ([6, Theorem 2]). Of course, there are many classes of operators weaker than hyponormal which are isoloid. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be quasinilpotent if $\sigma(T) = \{0\}$.

In [3], S. A. Alzuraiqi and A. B. Patel introduced n-normal operators.

Definition 1.1. *For $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be n-normal if*

$$T^*T^n = T^nT^* \quad . \tag{1}$$

This definition seems natural. S. A. Alzraiqi and A. B. Patel proved characterizations of 2-normal, 3-normal and n-normal operators on $\mathbb{C}^2$. Also, they made several examples of n-normal operators and proved that T is n-normal if and only if T^n is normal. Also, they proved that if T is 2-normal with the following condition

$$\sigma(T) \bigcap (-\sigma(T)) = \emptyset, \tag{2}$$

then T is subscalar. If an operator $T \in \mathcal{L}(\mathcal{H})$ satisfies (2), then T is invertible automatically. Recently, the authors in [4] have studied spectral properties of an n-normal operator T satisfying the following condition (3).

$$\sigma(T) \bigcap (-\sigma(T)) \subset \{0\}. \tag{3}$$

It is a little weaker assumption than this condition (2). We define (n, m)-normality as follows.

Definition 1.2. *For $n, m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (n, m)-normal if*

$$T^{*m}T^n = T^nT^{*m}.$$

In this paper, we study several properties of n-normal or (m, n)-normal operators. In particular, we prove that if T is 2-normal with (3), then T is polarloid. We study subscalarity of n-normal operators. Moreover, we show that if T is (n, m)-normal, then T is decomposable and Weyl's theorem holds for $f(T)$, where f is an analytic function on $\sigma(T)$ which is not constant on each of the components of its domain.

The following proposition is important in this paper.

Proposition 1.3. ([3, Proposition 2.2]) *Let $T \in \mathcal{L}(\mathcal{H})$ and $n \in \mathbb{N}$. Then T is n-normal if and only if T^n is normal.*

Therefore, we have the following result.

Theorem 1.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Then T has the single-valued extension property.*

Proof. Since T^n is normal, it follows that T^n has the single-valued extension property. Hence T has the single-valued extension property by [1, Theorem 2.40]. □

2. 2-normal Operators

In this section, we study some properties of 2-normal operators. Let M be a subspace of $\mathcal{H}$. Then M is said to be a *reducing subspace* for T if $T(M) \subset M$ and $T^*(M) \subset M$, that is, M is an invariant subspace for T and T^*.

Theorem 2.1. ([4]) *Let $T \in \mathcal{L}(\mathcal{H})$ be* 2*-normal and satisfy* (3). *Then the following statements hold.*
(i) *T is isoloid and $\sigma(T) = \sigma_a(T)$.*
(ii) *If z and w are distinct eigen-values of T and $x, y \in \mathcal{H}$ are corresponding eigen-vectors, respectively, then $\langle x, y\rangle = 0$.*
(iii) *If z, w are distinct values of $\sigma_a(T)$ and $\{x_n\}$, $\{y_n\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty} \langle x_n, y_n\rangle = 0$.*
(iv) *If z and w are distinct eigen-values of T, then* $\ker(T - z) \perp \ker(T - w)$.
(v) *If z is a non-zero eigen-value of T, then* $\ker(T - z) = \ker(T^2 - z^2) = \ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$ *and hence* $\ker(T - z)$ *is a reducing subspace for T.*

In 2012, J. T. Yuan and G. X. Ji ([10, Lemma 5.2]) proved the following Lemma.

Lemma 2.2. *Let m be a positive integer, λ be an isolated point of $\sigma(T)$ and $E = E_T(\{\lambda\})$.*
(i) *Then the following assertions are equivalent.*
(a) $E\mathcal{H} = \ker((T - \lambda)^m)$.
(b) $\ker(E) = (T - \lambda)^m\mathcal{H}$.
Hence λ is a pole of the resolvent of T and the order of λ is not greater than m.
(ii) *If λ is a pole of the resolvent of T and the order of λ is m, then the following assertions are equivalent:*
(a) *E is self-adjoint.*
(b) $\ker((T - \lambda)^m) \subset \ker((T - \lambda)^{*m})$.
(c) $\ker((T - \lambda)^m) = \ker((T - \lambda)^{*m})$.

Next we show that if T is 2-normal and satisfies (2), then T is polaroid.

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be* 2*-normal and satisfy* (3)*, and let λ be an isolated point of spectrum of T. Then λ is a pole of the resolvent, that is, T is polaroid and the following statements hold.*
(i) *If $\lambda = 0$, then $H_0(T) = \ker(T^2) = \ker(T^{*2})$, $E_T(\{0\})$ is self-adjoint and the order of* 0 *is not greater than* 2.
(ii) *If $\lambda \neq 0$, then $H_0(T - \lambda) = \ker(T - \lambda) = \ker((T - \lambda)^*)$, $E_T(\{\lambda\})$ is self-adjoint and the order of λ is* 1.

Proof. Let λ be an isolated point of spectrum of T.

(i) Assume that $\lambda = 0$. Since $\sigma(T^2) = \{z^2 : z \in \sigma(T)\}$, it follows that 0 is an isolated point of spectrum of T^2. We want to prove that $H_0(T) = H_0(T^2)$. Let $x \in H_0(T)$. Then $\|T^n x\|^{\frac{1}{n}} \longrightarrow 0$ and hence $\|T^{2n}x\|^{\frac{1}{2n}} = \left(\|T^{2n}x\|^{\frac{1}{n}}\right)^{\frac{1}{2}} \longrightarrow 0$ and $\|T^{2n}x\|^{\frac{1}{n}} \longrightarrow 0$. Hence $x \in H_0(T^2)$. Conversely, let $x \in H_0(T^2)$. Then $\|T^{2n}x\|^{\frac{1}{n}} \longrightarrow 0$ and so $\|T^{2n}x\|^{\frac{1}{2n}} = \left(\|T^{2n}x\|^{\frac{1}{n}}\right)^{\frac{1}{2}} \longrightarrow 0$. Since

$$\|T^{2n+1}x\|^{\frac{1}{2n+1}} \leq \left(\|T\|\|T^{2n}x\|\right)^{\frac{1}{2n+1}} \leq \|T\|^{\frac{1}{2n+1}} \left(\|T^{2n}x\|^{\frac{1}{2n}}\right)^{\frac{2n}{2n+1}} \longrightarrow 0 \ \ (n \to \infty),$$

it follows that $x \in H_0(T)$. Hence $H_0(T) = H_0(T^2)$.

Let $x \in E_T(\{0\}) = H_0(T) = H_0(T^2) = E_{T^2}(\{0\})$. Since T^2 is normal, it follows that $E_{T^2}(\{0\}) = \ker(T^2) = \ker(T^{*2})$. Hence $x \in \ker(T^2)$ and $E_T(\{0\}) \subset \ker(T^2)$. Therefore $E_T(\{0\}) = \ker(T^2) = \ker(T^{*2})$ and 0 is a pole of the resolvent of T and the order of 0 is not greater than 2 by Lemma 2.2.

(ii) Next we assume $\lambda \neq 0$. Then λ^2 is an isolated point of $\sigma(T^2)$ by [4, Lemma 2.1]. We will prove $H_0(T-\lambda) = H_0(T^2-\lambda^2)$. Let $x \in H_0(T-\lambda)$. Then $\|(T-\lambda)^n x\|^{\frac{1}{n}} \to 0$. Therefore we have

$$\|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \leq \|(T+\lambda)^n\|^{\frac{1}{n}} \|(T-\lambda)^n x\|^{\frac{1}{n}} \leq \|T+\lambda\| \|(T-\lambda)^n x\|^{\frac{1}{n}} \longrightarrow 0.$$

Hence $H_0(T-\lambda) \subset H_0(T^2-\lambda^2)$. Conversely, let $x \in H_0(T^2-\lambda^2)$. Since $T+\lambda$ is invertible by (3), we have

$$\begin{aligned}\|(T-\lambda)^n x\|^{\frac{1}{n}} &= \|(T+\lambda)^{-n}(T+\lambda)^n(T-\lambda)^n x\|^{\frac{1}{n}} \\ &\leq \| \left\{(T+\lambda)^{-1}\right\}^n \|^{\frac{1}{n}} \|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \leq \|(T+\lambda)^{-1}\| \|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \longrightarrow 0.\end{aligned}$$

Hence $H_0(T-\lambda) \supset H_0(T^2-\lambda^2)$ and $H_0(T-\lambda) = H_0(T^2-\lambda^2)$.

Let $x \in E_T(\{\lambda\})\mathcal{H}$. Since $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda) = H_0(T^2-\lambda^2)$ and T^2 is normal, we have $H_0(T^2-\lambda^2) = E_{T^2}(\{\lambda^2\}) = \ker(T^2-\lambda^2)$. Hence $0 = (T^2-\lambda^2)x = (T+\lambda)(T-\lambda)x$. Since $T+\lambda$ is invertible by (3), we have $(T-\lambda)x = 0$. Hence $E_T(\{\lambda\})\mathcal{H} \subset \ker(T-\lambda)$ and $E_T(\{\lambda\})\mathcal{H} = \ker(T-\lambda)$. Hence λ is a pole of the resolvent of T and the order of λ is 1 by Lemma 2.2. Since $\ker(T-\lambda) = \ker((T-\lambda)^*)$ by [4, Theorem 2.6], it follows that $E_T(\{\lambda\})$ is self-adjoint by Lemma 2.2. □

Let D be a bounded open subset of $\mathbb{C}$ and $L^2(D,\mathcal{H})$ be the Hilbert space of measurable function $f : D \longrightarrow \mathcal{H}$ such that

$$\|f\|_{2,D} = \left(\int_D \|f(z)\|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\mu$ is the planar Lebesgue measure. Let $W^2(D,\mathcal{H})$ be the Sobolev space with respect to $\overline{\partial}$ and of order 2 whose derivatives $\overline{\partial} f$ and $\overline{\partial}^2 f$ in the sense of distributions belong to $L^2(D,\mathcal{H})$. The norm $\|f\|_{W^2}$ is given by

$$\|f\|_{W^2} = \left(\|f\|_{2,D}^2 + \|\overline{\partial} f\|_{2,D}^2 + \|\overline{\partial}^2 f\|_{2,D}^2 \right)^{\frac{1}{2}} \quad \text{for } f \in W^2(D,\mathcal{H}).$$

Then in [3], S. A. Alzuraiqi and A. B. Patel proved the following result.

Proposition 2.4. ([3, Theorem 2.37]) *Let D be an arbitrary bounded disk in $\mathbb{C}$. If $T \in B(\mathcal{H})$ is 2-normal with* (1), *that is, $\sigma(T) \bigcap (-\sigma(T)) = \emptyset$, then the operator*

$$S = z - T : W^2(D,\mathcal{H}) \longrightarrow W^2(D,\mathcal{H})$$

is one to one.

We will revise this result as follows.

Theorem 2.5. *Let D be an arbitrary bounded open disk in $\mathbb{C}$. If $T \in B(\mathcal{H})$ is 2-normal and the planar Lebesgue measure of $\sigma(T) \bigcap (-\sigma(T))$ is 0, then the operator*

$$S = z - T : W^2(D,\mathcal{H}) \longrightarrow W^2(D,\mathcal{H})$$

is one to one.

Proof. Let $f \in W^2(D,\mathcal{H})$ and $Sf = 0$. We show $f = 0$. Then

$$\begin{aligned}\|f\|^2_{W^2} &= \|f\|^2_{2,D} + \|\overline{\partial} f\|^2_{2,D} + \|\overline{\partial}^2 f\|^2_{2,D}\\ &= \int_D \|f(z)\|^2 d\mu(z) + \int_D \|\overline{\partial} f(z)\|^2 d\mu(z) + \int_D \|\overline{\partial}^2 f(z)\|^2 d\mu(z) < \infty,\end{aligned}$$

and

$$\begin{aligned}\|Sf\|^2_{W^2} = \|(z-T)f\|^2_{W^2} &= \|(z-T)f\|^2_{2,D} + \|\overline{\partial}((z-T)f)\|^2_{2,D} + \|\overline{\partial}^2((z-T)f)\|^2_{2,D}\\ &= \|(z-T)f\|^2_{2,D} + \|(z-T)\overline{\partial} f\|^2_{2,D} + \|(z-T)\overline{\partial}^2 f\|^2_{2,D} = 0.\end{aligned}$$

Hence

$$\|(z-T)\,\overline{\partial}^i f\|^2_{2,D} = \int_D \|(z-T)\,\overline{\partial}^i f(z)\|^2 d\mu(z) = 0 \ \ (i = 0,1,2).$$

Let i be $i = 0,1,2$. Since $(z-T)\,\overline{\partial}^i f(z) = 0$ for $z \in D$, if $z \in D \setminus \sigma(T)$, then $\overline{\partial}^i f(z) = 0$ because $z - T$ is invertible. This implies

$$\|(z-T)^*\overline{\partial}^i f\|^2_{2,D\setminus\sigma(T)} = \int_{D\setminus\sigma(T)} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z) = 0.$$

Since

$$\begin{aligned}\|(z^2-T^2)\overline{\partial}^i f\|^2_{2,D} &= \int_D \|(z^2-T^2)\overline{\partial}^i f(z)\|^2 d\mu(z)\\ &\le \left(\sup_{z\in D}\|z+T\|\right)^2 \int_D \|(z-T)\overline{\partial}^i f(z)\|^2 d\mu(z) = \left(\sup_{z\in D}\|z+T\|\right)^2 \|(z-T)\overline{\partial}^i f\|^2_{2,D} = 0,\end{aligned}$$

we have $(z^2-T^2)\,\overline{\partial}^i f(z) = 0$ for $z \in D$. Moreover, since T^2 is normal, this implies

$$\|(z^2-T^2)^*\overline{\partial}^i f\|^2_{2,D} = \int_D \|(z^2-T^2)^*\overline{\partial}^i f(z)\|^2 d\mu(z) = 0.$$

Hence

$$0 = (z^2-T^2)^*\overline{\partial}^i f(z) = (z+T)^*(z-T)^*\overline{\partial}^i f(z) \ \text{ for } z \in D.$$

If $z \in D \cap (\sigma(T) \setminus (-\sigma(T)))$, then $z + T$ and $(z+T)^*$ are invertible. Hence we obtain $(z-T)^*\overline{\partial}^i f(z) = 0$ for $z \in D\cap(\sigma(T) \setminus (-\sigma(T)))$. Since D is bounded, $\|\overline{\partial}^i f\|^2_{2,D} < \infty$ and the planar Lebesgue measure of $\sigma(T) \bigcap (-\sigma(T))$ is 0, we have

$$\begin{aligned}\|(z-T)^*\overline{\partial}^i f\|^2_{2,D} &= \int_{D\setminus\sigma(T)} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z)\\ &\quad + \int_{D\cap(\sigma(T)\setminus(-\sigma(T)))} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z) + \int_{D\cap\sigma(T)\cap(-\sigma(T))} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z)\\ &\le 0 + 0 + \max_{z\in D}\|(z-T)^*\|^2 \int_{D\cap\sigma(T)\cap(-\sigma(T))} \|\overline{\partial}^i f(z)\|^2 d\mu(z) = 0.\end{aligned}$$

By [8, Proposition 2.1], we obtain $\|(I-P)f\|_{2,D} = 0$. Hence $f(z) = (Pf)(z)$ for $z \in D$. Since $Sf = 0$, we have $(Sf)(z) = (z-T)f(z) = (z-T)(Pf)(z) = 0$ for $z \in D$. Since T has the single-valued extension property by Theorem 1.4 and Pf is analytic, it follows that $0 = (Pf)(z) = f(z)$ for $z \in D$. Hence $f = 0$ and S is one to one. □

3. *n*-normal Operators

Theorem 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Let $\sigma(T)$ be contained in an angle $< 2\pi/n$ with vertex in the origin, i.e., there exists $\theta_1 \in [0, 2\pi)$ such that*

$$\sigma(T) \subset W = \left\{ re^{i\theta} : 0 < r, \theta_1 < \theta < \theta_1 + \frac{2\pi}{n} \right\}.$$

Then T is subscalar of order 2.

Proof. Let D be an open bounded disk such that $\sigma(T) \subset D$. Take an open set U such that $\sigma(T) \subset U \subset \overline{U} \subset D \cap W$. Let $M : W^2(D, \mathcal{H}) \to W^2(D, \mathcal{H})$ be a multiplication operator such that $(Mf)(z) = zf(z)$ for $f \in W^2(D, \mathcal{H})$ and $z \in D$. Then M is scalar of order 2 with a spectral distribution defined by $\Phi(\phi)f = \phi f$ for $\phi \in C_0^2(\mathbb{C})$ and $f \in W^2(D, \mathcal{H})$. Since $(z - T)W^2(D, \mathcal{H})$ is M-invariant, it follows that $S : \mathcal{H}(D) = W^2(D, \mathcal{H})/\overline{(z - T)W^2(D, \mathcal{H})} \to \mathcal{H}(D)$ as

$$S\left(f + \overline{(z - T)W^2(D, \mathcal{H})}\right) \to Mf + \overline{(z - T)W^2(D, \mathcal{H})}$$

for $f \in W^2(D, \mathcal{H})$ is well defined and S is still scalar of order 2 with a spectral distribution

$$\tilde{\Phi}(\phi)\left(f + \overline{(z - T)W^2(D, \mathcal{H})}\right) = \phi f + \overline{(z - T)W^2(D, \mathcal{H})}$$

for $\phi \in C_0^2(\mathbb{C})$ and $f + \overline{(z - T)W^2(D, \mathcal{H})} \in \mathcal{H}(D)$. Let $V : \mathcal{H} \to \mathcal{H}(D)$ be as

$$Vh = 1 \otimes h + \overline{(z - T)W^2(D, \mathcal{H})}$$

for $h \in \mathcal{H}$ where $(1 \otimes h)(z) = h$ for $z \in D$. Then

$$VT = SV.$$

We prove that V is one to one and has dense range. Then $V\mathcal{H}$ is an invariant subspace of S and $T = S|_{V\mathcal{H}}$. Hence T is subscalar of order 2.

Claim. If $Vh_n \to 0$, then $h_n \to 0$.

Let $Vh_n \to 0$. Then there exists $f_n \in W^2(D, \mathcal{H})$ such that

$$\|(z - T)f_n + 1 \otimes h_n\|_{W^2}^2 = \|(z - T)f_n + 1 \otimes h_n\|_{2,D}^2 + \|(z - T)\overline{\partial} f\|_{2,D}^2 + \|(z - T)\overline{\partial}^2 f\|_{2,D}^2 \to 0.$$

Let $\zeta = \exp(2\pi i/n)$. Then

$$\begin{aligned}\|(z^n - T^n)\overline{\partial}^i f\|_{2,D}^2 &= \int_D \|(z^n - T^n)\overline{\partial}^i f_n(z)\|^2 d\mu(z) = \int_D \left\| \prod_{k=1}^{n} (\zeta^k z - T)\overline{\partial}^i f_n(z) \right\|^2 d\mu(z) \\ &\leq \sup_{z \in D} \left\| \prod_{k=1}^{n-1} (\zeta^k z - T) \right\|^2 \int_D \|(z - T)\overline{\partial}^i f_n(z)\|^2 d\mu(z) \to 0.\end{aligned}$$

Since T^n is normal, we have

$$\|(z^n - T^n)^* \overline{\partial}^i f\|_{2,D}^2 \to 0.$$

If $z \in \overline{U}$, then $\zeta^k z \notin \sigma(T)$ for $k = 1, 2, \cdots, n-1$ by the assumption. Hence

$$\begin{aligned}
\|(z-T)^*\overline{\partial}^i f_n\|^2_{2,\overline{U}} &= \int_{\overline{U}} \|(z-T)^*\overline{\partial}^i f_n(z)\|^2 d\mu(z) \\
&= \int_{\overline{U}} \prod_{k=1}^{n-1} \left((\zeta^k z - T)^{-1}\right)^* \left(\prod_{k=1}^{n-1} (\zeta^k z - T)^*\right)(z-T)^*\overline{\partial}^i f_n(z) d\mu(z) \\
&\le \prod_{k=1}^{n-1} \sup_{z \in \overline{U}} \left\|\left((\zeta^k z - T)^{-1}\right)^*\right\|^2 \left\|(z^n - T^n)^*\overline{\partial}^i f_n\right\|^2_{2,D} \to 0.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{D\setminus\overline{U}} \|\overline{\partial}^i f_n(z)\|^2 d\mu(z) &= \int_{D\setminus\overline{U}} \|(z-T)^{-1}(z-T)\overline{\partial}^i f_n(z)\|^2 d\mu(z) \\
&\le \sup_{z\in D\setminus\overline{U}} \|(z-T)^{-1}\|^2 \int_{D\setminus\overline{U}} \|(z-T)\overline{\partial}^i f_n(z)\|^2 d\mu(z) \\
&\le \sup_{z\in D\setminus\overline{U}} \|(z-T)^{-1}\|^2 \|(z-T)\overline{\partial}^i f_n\|^2_{2,D} \to 0,
\end{aligned}$$

we have

$$\begin{aligned}
\|(z-T)^*\overline{\partial}^i f_n\|^2_{2,D} &= \int_{D\setminus\overline{U}} \|(z-T)^*\overline{\partial}^i f_n(z)\|^2 d\mu(z) + \int_{\overline{U}} \|(z-T)^*\overline{\partial}^i f_n(z)\|^2 d\mu(z) \\
&\le \sup_{z\in D\setminus\overline{U}} \|(z-T)^*\|^2 \int_{D\setminus\overline{U}} \|\overline{\partial}^i f_n(z)\|^2 d\mu(z) + \|(z-T)^*\overline{\partial}^i f_n\|^2_{2,\overline{U}} \to 0.
\end{aligned}$$

Let P be the orthogonal projection of $L^2(D, \mathcal{H})$ onto $A^2(D, \mathcal{H})$. Then there exists a constant $0 < C_D$ such that

$$\|(1-P)f_n\|_{2,D} \le C_D \left(\|(z-T)^*\overline{\partial} f_n\|_{2,D} + \|(z-T)^*\overline{\partial}^2 f_n\|_{2,D}\right) \to 0$$

by Proposition 2.1 of [8]. Hence

$$\begin{aligned}
\|(z-T)Pf_n + 1 \otimes h_n\|_{2,D} &\le \|(z-T)f_n + 1 \otimes h_n\|_{2,D} + \|(z-T)(1-P)f_n\|_{2,D} \\
&\le \|(z-T)f_n + 1 \otimes h_n\|_{2,D} + \sup_{z\in D} \|z-T\| \|(1-P)f_n\|_{2,D} \to 0.
\end{aligned}$$

Hence

$$\|(z-T)Pf_n + 1 \otimes h_n\|_{\infty,U} = \sup_{z\in U} \|(z-T)Pf_n(z) + h_n\| \to 0$$

by [8, Lemma 1.1]. Define $\Psi : A^2(U, \mathcal{H}) \to \mathcal{H}$ as

$$\Psi(g) = \frac{1}{2\pi i} \int_{\partial G} (z-T)^{-1} g(z) dz$$

for $g \in A^2(U, \mathcal{H})$ where G is an open set such that $\sigma(T) \subset G \subset \overline{G} \subset U$ and ∂G is a Jordan curve. Since

$$\|\Psi(g)\| \le \frac{1}{2\pi} \max_{z\in\partial G} \|(z-T)^{-1}\| \|g\|_{\infty,U} \ell(\partial G)$$

for $g \in A^2(U, \mathcal{H})$ where $\ell(\partial G)$ denotes the length of ∂G and

$$(z-T)Pf_n + 1 \otimes h_n \in A^2(U, \mathcal{H}),$$

we have

$$\Psi((z-T)Pf_n + 1\otimes h_n) = \frac{1}{2\pi i}\int_{\partial G}(z-T)^{-1}\left((z-T)Pf_n(z)+h_n\right)dz$$
$$= \frac{1}{2\pi i}\int_{\partial G}\left(Pf_n(z)+(z-T)^{-1}h_n\right)dz = 0 + h_n \to 0.$$

□

Corollary 3.2. *Under the same hypothesis as in Theorem* 3.1, *if* $\sigma(T)$ *has nonempty interior, then* T *has a nontrivial invariant subspace.*

Proof. By the hypothesis, T is subscalar of order 2 from Theorem 3.1. Since $\sigma(T)$ has nonempty interior, we get this result from [5, Theorem 2.1]. □

In [9], C.R. Putnam proved that if T is hyponormal, then

$$\pi\|T^*T - TT^*\| \le m(\sigma(T))$$

where m is the Lebesque measure in the complex plane. This is well known as Putnam's inequality.

Lemma 3.3. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be n-normal and let* $\mathcal{M} \subset \mathcal{H}$ *be an invariant subspace for* T. *Then the following assertions hold.*

(i) $(T|_{\mathcal{M}})^n$ *is subnormal.*

(ii) *Let* $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$. *Then* $T|_{\mathcal{M}} = \lambda$ *if* $\lambda \neq 0$ *and* $(T|_{\mathcal{M}})^n = 0$ *if* $\lambda = 0$.

Proof. (i) Since $(T|_{\mathcal{M}})^n = T^n|_{\mathcal{M}}$ and T^n is normal, $(T|_{\mathcal{M}})^n$ is subnormal.

(ii) Suppose $\lambda = 0$. Then $(T|_{\mathcal{M}})^n$ is subnormal by (1) and

$$\sigma\left((T|_{\mathcal{M}})^n\right) = \{z^n | z \in \sigma(T|_{\mathcal{M}})\} = \{0\}.$$

It follows that $(T|_{\mathcal{M}})^n = 0$ by Putnam's inequality.

Suppose $\lambda \neq 0$. Then $(T|_{\mathcal{M}})^n$ is subnormal and $\sigma\left((T|_{\mathcal{M}})^n\right) = \{\lambda^n\}$. It follows that $(T|_{\mathcal{M}})^n = \lambda^n$ by Putnam's inequality. Since $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$ and

$$0 = (T|_{\mathcal{M}})^n - \lambda^n = \left(\prod_{k=1}^{n-1}(T|_{\mathcal{M}} - \lambda\zeta^k)\right)(T|_{\mathcal{M}} - \lambda),$$

we have

$$T|_{\mathcal{M}} - \lambda = \left(\prod_{k=1}^{n-1}(T|_{\mathcal{M}} - \lambda\zeta^k)\right)^{-1} \cdot 0 = 0,$$

where $\zeta = \exp(2\pi i/n)$. □

Definition 3.4. *Let* $\lambda \in \sigma(T)$ *be arbitrary,* $n \in \mathbb{N}$ *and* $\zeta := \exp(2\pi i/n)$. *We say that* T *has property* (n) *at* λ *if*

$$\lambda\zeta^k \notin \sigma(T) \text{ for } k = 1, \cdots, n-1.$$

Remark. We do not need the assumption that λ is an isolated point of $\sigma(T)$ in the following theorem.

Theorem 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Then the following assertions hold.*

(i) $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$.

(ii-1) *If* $\lambda \neq 0$, *then* $H_0(T-\lambda) = \ker(T-\lambda)$.

(ii-2) *If* $\lambda \neq 0$ *and T has property* (n) *at* λ, *then* $H_0(T-\lambda) = \ker(T-\lambda) = \ker((T-\lambda)^*)$.

Proof. (i) Since T^n is normal, we have $H_0(T) \subset H_0(T^n) = \ker(T^n) = \ker(T^{*n})$. It is known that $\ker(T^n) \subset H_0(T)$. Hence $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$.

(ii-1) We claim $H_0(T-\lambda) \subset H_0(T^n - \lambda^n)$.

Let $x \in H_0(T-\lambda)$ and $\zeta = \exp(2\pi i/n)$. Then

$$\begin{aligned}\|(T^n-\lambda^n)^m x\|^{\frac{1}{m}} &= \|(T-\lambda\zeta)^m(T-\lambda\zeta^2)^m\cdots(T-\lambda\zeta^{m-1})^m(T-\lambda)^m x\|^{\frac{1}{m}}\\ &\le \|T-\lambda\zeta\|\|T-\lambda\zeta^2\|\cdots\|T-\lambda\zeta^{m-1}\|\|(T-\lambda)^m x\|^{\frac{1}{m}} \longrightarrow 0\ (m\to\infty).\end{aligned}$$

Hence $x \in H_0(T^n-\lambda^n)$.

Since T^n is normal, $H_0(T^n-\lambda^n) = \ker(T^n-\lambda^n)$. Put $\mathcal{M} = \ker(T^n-\lambda^n)$. Then $\mathcal{M}$ is an invariant subspace of T and $\sigma((T|_{\mathcal{M}})^n) = \sigma(T^n|_{\mathcal{M}}) = \{\lambda^n\}$. Hence $\sigma(T|_{\mathcal{M}}) \subset \{\lambda, \lambda\zeta, \cdots, \lambda\zeta^{n-1}\}$. Put $\sigma(T|_{\mathcal{M}}) = \{\mu_1, \cdots, \mu_r\}$ with $\mu_i \neq \mu_j$ $(i \neq j)$ and $\mu_i^n = \lambda^n$ for $i = 1, 2, \cdots, r$. Let F_i be the Riesz idempotent corresponding to $\mu_i \in \sigma(T|_{\mathcal{M}})$. Then $F_iF_j = 0$ $(i \neq j)$, $F_1 + \cdots + F_r = I_{\mathcal{M}}$, $\sigma((T|_{\mathcal{M}})|_{F_i\mathcal{M}}) = \sigma(T|_{F_i\mathcal{M}}) = \{\mu_i\}$ and $\sigma\left(T|_{(I_{\mathcal{M}}-F_i)\mathcal{M}}\right) = \sigma(T|_{\mathcal{M}}) \setminus \{\mu_i\}$ for $i = 1, 2, \cdots, r$. This shows that $T|_{F_i\mathcal{M}} = \mu_i$ for $i = 1, 2, \cdots, r$ by Lemma 3.3. Put $C = (\|F_1\| + \|F_2\| + \cdots + \|F_r\|)^{-1} > 0$. Since

$$\begin{aligned}\|x\| = \|F_1x + F_2x + \cdots + F_rx\| &\le \|F_1x\| + \|F_2x\| + \cdots + \|F_rx\|\\ &\le (\|F_1\| + \|F_2\| + \cdots + \|F_r\|)\,\|x\|,\end{aligned}$$

we have

$$\|x\| \ge C(\|F_1x\| + \|F_2x\| + \cdots + \|F_rx\|) \text{ for all } x \in \mathcal{M}.$$

Let $0 \neq x \in H_0(T-\lambda) \subset \mathcal{M}$. Then

$$\begin{aligned}\|(T-\lambda)^n x\|^{\frac{1}{n}} = \|(T|_{\mathcal{M}}-\lambda)^n x\|^{\frac{1}{n}} &\ge \left(C\sum_{k=1}^{r}\|F_k(T|_{\mathcal{M}}-\lambda)^n x\|\right)^{\frac{1}{n}}\\ &= \left(C\sum_{k=1}^{r}\|(T|_{\mathcal{M}}-\lambda)^n F_kx\|\right)^{\frac{1}{n}} = \left(C\sum_{k=1}^{r}\|(T|_{F_k\mathcal{M}}-\lambda)^n F_kx\|\right)^{\frac{1}{n}}\\ &= C^{\frac{1}{n}}\left(\sum_{k=1}^{r}|\mu_k-\lambda|^n\|F_kx\|\right)^{\frac{1}{n}} \ge |\mu_k-\lambda|C^{\frac{1}{n}}\|F_kx\|^{\frac{1}{n}}.\end{aligned}$$

By letting $n \to \infty$, it follows that $F_kx = 0$ for all k such as $\mu_k \neq \lambda$. Hence if there does not exist k such that $\mu_k = \lambda$, then $x = F_1x + F_2x + \cdots + F_rx = 0$ which is a contradiction. Hence there exists a unique number $k' \in \{1, \cdots, r\}$ such that $\mu_{k'} = \lambda$ and $F_{k'}x = x$. Hence $x \in F_{k'}\mathcal{M} = \ker(T|_{F_{k'}\mathcal{M}} - \lambda) \subset \ker(T-\lambda)$. Hence $H_0(T-\lambda) \subset \ker(T-\lambda)$. Since the converse inclusion is clear, we have $H_0(T-\lambda) = \ker(T-\lambda)$.

(ii-2) Let T have property (n) at λ. Since T^n is normal, we have

$$H_0(T-\lambda) = \ker(T-\lambda) \subset H_0(T^n-\lambda^n) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*).$$

Conversely, let $y \in H_0(T^n-\lambda^n) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*)$. Then $(T^n-\lambda^n)y = 0$ and $(T^n-\lambda^n)^*y = 0$. Since $\lambda\zeta^k \notin \sigma(T)$ for $k = 1, \cdots, n-1$, it follows that

$$(T-\lambda)y = \left(\prod_{k=1}^{n-1}(T|_{\mathcal{M}}-\lambda\zeta^k)\right)^{-1}(T^n-\lambda^n)y = 0$$

and

$$(T-\lambda)^* y = \left(\prod_{k=1}^{n-1} (T|_{\mathcal{M}} - \lambda\zeta^k)^* \right)^{-1} (T^n - \lambda^n)^* y = 0.$$

Hence $H_0(T-\lambda) = \ker(T-\lambda) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*) \subset \ker((T-\lambda)^*)$. Since $\ker((T-\lambda)^*) \subset \ker((T^n-\lambda^n)^*)$ is clear, we have

$$H_0(T-\lambda) = \ker(T-\lambda) = \ker((T-\lambda)^*) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*).$$

□

Theorem 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Then T is isoloid and polaroid.*
Moreover, let λ be an isolated point of the spectrum of T. Then λ is a pole of the resolvent and following statements hold.
(i) *If $\lambda = 0$, then $E_T(\{0\})\mathcal{H} = H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$, $E_T(\{0\})$ is self-adjoint and the order of* 0 *is not greater than n.*
(ii) *If $\lambda \neq 0$, then $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda) = \ker(T-\lambda)$ and the order of λ is* 1.

Proof. (i) Assume that 0 is an isolated point of $\sigma(T)$. Since $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$ by Theorem 3.5, we have $E_T(\{0\})\mathcal{H} = H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$. Hence 0 is a pole of the resolvent of T, $E_T(\{0\})$ is self-adjoint and the order of pole is not greater than n by Lemma 2.2.
(ii) Next we assume λ is a nonzero isolated point of $\sigma(T)$. Since $H_0(T-\lambda) = \ker(T-\lambda)$ by Theorem 3.5, we have $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda) = \ker(T-\lambda)$. Hence λ is a pole of the resolvent of T and the order of pole is 1 by Lemma 2.2. □

4. (n, m)-normal Operators

Definition 4.1. *For $n, m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (n, m)-normal if*

$$T^{*m}T^n = T^nT^{*m}.$$

From the definition, it is clear that T is (n, m)-normal if and only if T^* is (m, n)-normal. Moreover, if T^n is normal, then T is (n, m)-normal for every m. Indeed, since T^n is normal and $T^m \cdot T^n = T^n \cdot T^m$, it follows from Fuglede theorem that $T^{*m} \cdot T^n = T^n \cdot T^{*m}$. Hence T is (n, m)-normal. From [4], we restate the properties of (m, n)-normal operators.

Lemma 4.2. Let $T \in \mathcal{L}(\mathcal{H})$ be (n, m)-normal. Then the following statements hold.
(i) T^* is (m, n)-normal.
(ii) If T^{-1} exists, then T^{-1} is (n, m)-normal.
(iii) If $S \in \mathcal{L}(\mathcal{H})$ is unitary equivalent to T, then S is (n, m)-normal.
(iv) If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces T, then $T|_{\mathcal{M}}$ is (n, m)-normal on $\mathcal{M}$.
(v) If T is (n, m)-normal, then T^k is normal where k is the least common multiple of n and m.
(vi) If T is quasi-nilpotent, then T is nilpotent.

Proof. The proofs of the statements of (i), (ii), (iii), and (iv) are clearly holds by the definition.
(v) Let $k := n \cdot j$ and $k := m \cdot \ell$. Since T is (n, m)-normal, it follows that

$$T^{*k}T^k = \overbrace{T^{*m} \cdots T^{*m}}^{\ell} \cdot \overbrace{T^n \cdots T^n}^{j} = T^n \cdots T^n \cdot T^{*m} \cdots T^{*m} = T^kT^{*k},$$

which means that T^k is normal.
(vi) If T is quasi-nilpotent, i.e., $\sigma(T) = \{0\}$, then $\sigma(T^k) = \{0\}$ for every $k \in \mathbb{N}$. Let k_0 be the least common multiple of n and m. Then T^{k_0} is normal by Lemma 4.2 (v). Hence $T^{k_0} = 0$. □

Corollary 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be (n, m)-normal. Then T is isoloid and polaroid.*
Moreover, let λ be an isolated point of the spectrum of T. Then λ is a pole of the resolvent and following statements hold.
(i) *If $\lambda = 0$, then $H_0(T) = E_T(\{0\})\mathcal{H} = \ker(T^{nm}) = \ker(T^{*nm})$, $E_T(\{0\})$ is self-adjoint and the order of* 0 *is not greater than n.*
(ii) *If $\lambda \neq 0$, then $H_0(T - \lambda) = E_T(\{\lambda\})\mathcal{H} = \ker(T - \lambda)$ and the order of λ is* 1.

Proof. Since T^{nm} is normal by Lemma 4.2, we have these results from Theorem 3.6. □

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$, $\pi_{00}(T) = \{\lambda \in \operatorname{iso}(\sigma(T)) : 0 < \dim \ker(T - \lambda) < \infty\}$, and $\operatorname{iso}(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$.

Theorem 4.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be (n, m)-normal. Then the following statements hold.*
(i) *T is decomposable.*
(ii) *If f is an analytic function on $\sigma(T)$ which is not constant on each of the components of its domain, then Weyl's theorem holds for $f(T)$.*

Proof. (i) Since T^{nm}, T^{*nm} are normal by Lemma 4.2, it follows T^{nm} is decomposable. Hence T is decomposable by [7, Theorem 3.3.9].

(ii) Since T is polaroid by Theorem 3.6 or Corollary 4.3 and T has the single-valued extension property by Theorem 1.4, it follows that Weyl's theorem holds for $f(T)$ by [2, Theorem 3.14]. □

References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Publishers, Dordrecht, 2004.
[2] P. Aiena, E. Aponte and E. Balzan E, *Weyl type theorems for left and right polaroid operators*, Integ. Eq. Op. Th. **66**, (2010), 1–20.
[3] S. A. Alzuraiqi and A. B. Patel, *On n-normal operators*, General Math. Notes **1**, (2010), 61–73.
[4] M. Chō and B. Načevska Nastovska, *Spectral properties of n-normal operators*, to appear in Filomat.
[5] J. Eschmeier, *Invariant subspaces for operators with Bishop's property (β) and thick spectrum*, J. Funct. Anal. **94**, (1990), 196–222.
[6] J. G. Stampfli, *Hyponormal operators*, Pacific J. Math. **12**, (1962), 1453–1458.
[7] K. Laursen and M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.
[8] M. Putinar, *Hyponormal operators are subscalar*, J. Op. Th. **12**, (1984), 385–395.
[9] C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. **116**, (1970), 323–330.
[10] J. T. Yuan and G. X. Ji, *On (n, k)-quasiparanormal operators*, Studia Math. **209**, (2012), 289–301.