Filomat 32:15 (2018), 5459–5470
https://doi.org/10.2298/FIL1815459M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Measures of Noncompactness in $\left(\bar{N}^q_\Delta\right)$ Summable Difference Sequence Spaces

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Abstract. In this paper we first introduce $\bar{N}^q_\Delta$ summable difference sequence spaces and prove some properties of these spaces. We then obtain the necessary and sufficient conditions for infinite matrices A to map these sequence spaces into the spaces c, c_0, and ℓ_∞. Finally, the Hausdorff measure of noncompactness is then used to obtain the necessary and sufficient conditions for the compactness of the linear operators defined on these spaces.

1. Introduction and Preliminaries

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ and ϕ, c, c_0 and ℓ_∞ for the sets of all finite, convergent sequences, sequences convergent to zero, and bounded sequences respectively. By e we denote the sequence of 1's, $e = (1, 1, 1, \ldots)$ and by $e^{(n)}$ the sequence with 1 as only nonzero term at the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Further by cs and ℓ_1 we denote the convergent and absolutely convergent series respectively. If $x = (x_k)_{k=0}^{\infty} \in w$ then $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$ denotes the $m-$th section of x.

A sequence space X is a linear subspace of ω, such a subspace is called a BK space if it is a Banach space with continuous coordinates
$P_n : X \to \mathbb{C}$ $(n = 0, 1, 2, \ldots)$ where

$$P_n(x) = x_n, \; x = (x_k)_{k=0}^{\infty} \in X.$$

The BK space X is said to have AK if every $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, [16, Definition 1.18]. The spaces c_0, c and ℓ_∞ are BK spaces with respect to the norm

$$\|x\|_\infty = \sup_k \{|x_k| : k \in \mathbb{N}\}.$$

every sequence $(x_k)_{k=0}^{\infty}$ has a unique representation

$$x = \xi e + \sum_{k=0}^{\infty} (x_k - \xi) e^{(k)}, \text{ where } \xi = \lim_{k\to\infty} x_k;$$

2010 *Mathematics Subject Classification*. 40H05, 46A45, 47B07
Keywords. Difference sequence space, BK spaces, matrix transformations, measures of noncompactness
Received: 24 June 2018; Accepted: 16 August 2018
Communicated by Eberhard Malkowsky
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ℓ_∞ has no Schauder basis.

For any two sequences x and y in ω the product xy is given by $xy = (x_k y_k)_{k=0}^{\infty}$.
The $\beta-$dual of a subset X of ω is defined by

$$X^{\beta} = \{a \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}$$

If A is an infinite matrix with complex entries a_{nk} $n, k \in \mathbb{N}$, we write $A_n = (a_{nk})_{k=0}^{\infty}$ for the sequence in the nth row of A. The $A-$transform of any $x = (x_k) \in w$ is given by $Ax = (A_n x)_{n=0}^{\infty}$, where

$$A_n x = \sum_{k=0}^{\infty} a_{nk} x_k \qquad n \in \mathbb{N}$$

provided the series on right must converge for each $n \in \mathbb{N}$.
If X and Y are subsets of ω, we denote by (X, Y), the class of all infinite matrices that map X into Y. So $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$, $n = 0, 1, 2, \ldots$ and $Ax \in Y$ for all $x \in X$. The matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in w : Ax \in X\}$$

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation method has been studied by several authors see [4, 7, 9, 11].

If X and Y are Banach Spaces, then by $\mathcal{B}(X, Y)$ we denote the set of all bounded (continuous) linear operators $L : X \to Y$, which is itself a Banach space with the operator norm $\|L\| = \sup_x \{\|L(x)\|_Y : \|x\| = 1\}$ for all $L \in \mathcal{B}(X, Y)$. The linear operator $L : X \to Y$ is said to be compact if its domain is all of X and for every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a subsequence which converges in Y. The operator $L \in \mathcal{B}(X, Y)$ is said to be of finite rank if $\dim R(L) < \infty$, where $R(L)$ denotes the range space of L. A finite rank operator is clearly compact.[6, Chapter 2]

The concept of difference sequence spaces was first introduced by Kizmaz [13] and later several authors studied new sequence spaces defined by using difference operators like Mursaleen and Noman [19], Mursaleen et al. [18], Jalal [10], Manna et al. [17], Polat et al. [20]. In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of the difference operator, or of the matrices of the classical methods of summability in spaces such as ℓ_p, c_0, c, ℓ_∞ or others. For instance, some matrix domains of the difference operator were studied in [13, 21], of the Riesz matrices in [1].

In this paper, we first define a new difference sequence space as the matrix domains X_T of the product T of the triangles $\bar{N}^q$ and Δ and obtain bases for two of them, and determine their β duals. We then find out the necessary and sufficient condition for matrix transformations to map these spaces into c_0, c and ℓ_∞. Finally we characterize the classes of compact matrix operators from these spaces into c_0, c and ℓ_∞.

2. $\bar{N}^q_\Delta$ Summable Difference Sequence Spaces

The difference operator Δ is defined on ω as

$$\Delta_k x = x_k - x_{k-1} \ , k = 0, 1, 2, \ldots \tag{1}$$

where $x_{-1} = 0$, and Δ_k is the kth row of the matrix $\Delta = (e_{nk})_{n,k=0}^{\infty}$ given by

$$e_{nk} = \begin{cases} 1 & k = n \\ -1 & k = n-1 \\ 0 & k > n \end{cases}$$

The inverse of this matrix is $\Sigma = (\sigma_{nk})$ given as

$$\sigma_{nk} = \begin{cases} 1 & 0 \le k \le n \\ 0 & k > n \end{cases}$$

Let $(q_k)_{k=0}^{\infty}$ be a given positive sequences and $(Q_n)_{n=0}^{\infty}$ the sequence with $Q_n = \sum_{i=0}^{n} q_i$.

The $(\bar{N}, q)$ transform of the sequence $(x_k)_{k=0}^{\infty}$ is the sequence $(t_n)_{n=0}^{\infty}$ defined as

$$t_n = \frac{1}{Q_n} \sum_{i=0}^{n} q_i x_i \qquad \text{for } n = 0, 1, \ldots .$$

The matrix $\bar{N}^q$ for this transformation is given by

$$(\bar{N}^q)_{nk} = \begin{cases} \frac{q_k}{Q_n} & 0 \le k \le n \\ 0 & k > n. \end{cases}$$

The inverse of this matrix is [3]

$$(\bar{N}^q)^{-1}_{nk} = \begin{cases} (-1)^{n-k} \frac{Q_k}{q_n} & n-1 \le k \le n \\ 0 & 0 \le k \le n-2, k > n. \end{cases}$$

We define the spaces $(\bar{N}^q_\Delta)_0$, $(\bar{N}^q_\Delta)$ and $(\bar{N}^q_\Delta)_\infty$ that are $\bar{N}^q_\Delta$ summable to zero, summable and bounded respectively as

$$(\bar{N}^q_\Delta)_0 = (c_0, \Delta)_{\bar{N}^q} = \left\{ x \in w : \bar{N}^q \Delta x = \left(\frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta_k x \right)_{n=0}^{\infty} \in c_0 \right\}$$
$$(\bar{N}^q_\Delta) = (c, \Delta)_{\bar{N}^q} = \left\{ x \in w : \bar{N}^q \Delta x = \left(\frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta_k x \right)_{n=0}^{\infty} \in c \right\}$$
$$(\bar{N}^q_\Delta)_\infty = (\ell_\infty, \Delta)_{\bar{N}^q} = \left\{ x \in w : \bar{N}^q \Delta x = \left(\frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta_k x \right)_{n=0}^{\infty} \in \ell_\infty \right\}$$

For any sequence $x = (x_k)_{k=0}^{\infty}$, let $\tau = \tau(x) = (\tau_n(x))_{n=0}^{\infty}$ denote the sequence with nth term given by

$$\tau_n = \left(\bar{N}^q_\Delta x \right)_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta_k x \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

2.1. Basis for the new sequence spaces

First we determine Schauder bases for the spaces $\left(\bar{N}^q_\Delta\right)_0$ and $\left(\bar{N}^q_\Delta\right)$. For the convenience of the reader, we state the following known results:

Proposition 2.1. *[23]*
Every triangle T has a unique inverse $S = (s_{nk})_{n,k=0}^{\infty}$ which is also a triangle, and $x = T(S(x)) = S(T(x))$ for all $x \in w$.

Proposition 2.2. *[12, Theorem 2.3]*
Let T be a triangle and S be its inverse, if $\left(b^{(n)}\right)_{n=0}^{\infty}$ is a basis of the linear metric space (X, d), then $\left(S(b^{(n)})\right)_{n=0}^{\infty}$ is a basis of $Z = X_T$ with the metric d_T defined by $d_T(z, \bar{z}) = d(T(z), T(\bar{z}))$ for all $z, \bar{z} \in Z$.

It is obvious that $(c_0, \Delta)_{\bar{N}^q} = (c_0)_{\bar{N}^q \cdot \Delta}$, So the basis for this space is given by $(\bar{N}^q \cdot \Delta)^{-1} \left(e^{(n)}\right) = (\Delta)^{-1} \cdot (\bar{N}^q)^{-1} \left(e^{(n)}\right)$.

Theorem 2.3. *The sequence spaces $\left(\bar{N}^q_\Delta\right)_0$, $\left(\bar{N}^q_\Delta\right)$ and $\left(\bar{N}^q_\Delta\right)_\infty$ are BK-spaces with norm $\|\cdot\|_{\bar{N}^q_\Delta}$ given by*

$$\|x\|_{\bar{N}^q_\Delta} = \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta_k x \right|$$

Let $\tau_k(x) = \left(\left(\bar{N}^q_\Delta\right) x\right)_k$ for all $k \in \mathbb{N}$. Define the sequences $c^{(n)} = \left(c^{(n)}_k\right)_{k=0}^\infty$ for $n = -1, 0, 1, \dots$ by

$$c^{(n)}_k = \begin{cases} 0 & 0 \le k \le n-1 \\ \frac{Q_n}{q_n} & k = n \\ Q_n\left(\frac{1}{q_n} - \frac{1}{q_{n+1}}\right) & k \ge n+1. \end{cases} \quad , \quad c^{(-1)}_k = k+1$$

for every fixed $k \in \mathbb{N}$. Then

i) The sequence $\left(c^{(n)}\right)_{k=0}^\infty$ is a basis for the space $\left(\bar{N}^q_\Delta\right)_0$ and any $x \in \left(\bar{N}^q_\Delta\right)_0$ can be uniquely represented in the form

$$x = \sum_k \tau_k c^{(k)}$$

.

ii) The set $\left\{c^{(-1)}, c^{(n)}\right\}$ is a basis for the spaces $\left(\bar{N}^q_\Delta\right)$ and any $x \in \left(\bar{N}^q_\Delta\right)$ has a unique representation in the form

$$x = lc^{(-1)} + \sum_k (\tau_k - l) c^{(k)}$$

where for all $k \in \mathbb{N}$, $l = \lim_{k\to\infty} \left(\left(\bar{N}^q_\Delta\right) x\right)_k$.

Proof. Since $(X, \Delta)_{\bar{N}^q} = X_{\bar{N}^q \cdot \Delta}$ for all $X = c_0, c$ and the spaces c_0, c, ℓ_∞ are BK spaces with respect to their natural norm [14, pp. 217-218] and the matrix $\bar{N}^q \cdot \Delta$ is a triangle so by [23, Theorem 4.3.12], gives $\left(\bar{N}^q_\Delta\right)_0$, $\left(\bar{N}^q_\Delta\right)$ and $\left(\bar{N}^q_\Delta\right)_\infty$ are BK spaces
The proof of the remaining part of the theorem is a direct consequence of [23, Corollary 2.5 (a) and (c)], since $(e^{(n)})_{n=0}^\infty$ is the standard basis for c_0.
The inverse of the triangle $\bar{N}^q \cdot \Delta$ is the triangle $S = (\bar{N}^q \cdot \Delta)^{-1} = \Sigma \cdot (\bar{N}^q)^{-1} = (s_{nk})_{n,k=0}^\infty$ where

$$s_{nk} = \begin{cases} Q_k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) & 0 \le k < n \\ \frac{Q_n}{q_n} & k = n \\ 0 & k > n \end{cases} \tag{3}$$

Now, if $S_k = (s_{kj})_{j=0}^\infty$ is the kth row of the matrix S then

$$\begin{aligned} S_k e^{(n)} &= \sum_{j=0}^{k} s_{kj} e^{(n)}_j \\ &= \begin{cases} 0 & k \le n-1 \\ s_{kn} & k \ge n \end{cases} \\ &= \begin{cases} 0 & 0 \le k \le n-1 \\ \frac{Q_n}{q_n} & k = n \\ Q_n\left(\frac{1}{q_n} - \frac{1}{q_{n+1}}\right) & k > n. \end{cases} \end{aligned}$$

and

$$\begin{aligned} S_k e &= \sum_{j=0}^{k} s_{kj} \\ &= \sum_{j=0}^{k-1} Q_j \left(\frac{1}{q_j} - \frac{1}{q_{j+1}} \right) + \frac{Q_k}{q_k} \\ &= \left[\sum_{j=0}^{k-1} \frac{Q_j - Q_{j-1}}{q_j} - \frac{Q_{k-1}}{q_k} \right] + \frac{Q_k}{q_k} \\ &= \sum_{j=0}^{k} \frac{Q_j - Q_{j-1}}{q_j} \\ &= \sum_{j=0}^{k} \frac{q_j}{q_j} \\ &= k+1 \end{aligned}$$

Hence, $\left(c^{(n)}\right)_{n=0}^{\infty}$ is a basis for the space $(c_0, \Delta)_{\bar{N}^q} = \left(\bar{N}_{\Delta}^q\right)_0$ and $\left\{c^{(-1)}, \left(c^{(n)}\right)_{n=0}^{\infty}\right\}$ is a basis for the space $(c, \Delta)_{\bar{N}^q} = \left(\bar{N}_{\Delta}^q\right)$.
The representations in Parts (i) and (ii) now are immediate from [12, (2.1) and (2.3)]. □

2.2. β dual of the new spaces

To obtain the β dual we need the following results:

Lemma 2.4. *[13, 22] If $A = (a_{nk})_{n,k=0}^{\infty}$, then $A \in (c_0, c)$ if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \tag{4}$$

and

$$\alpha_k = \lim_{n\to\infty} a_{nk} \qquad \text{exists for each } k. \tag{5}$$

Lemma 2.5. *[13, 22] If $A = (a_{nk})_{n,k=0}^{\infty}$, then $A \in (c, c)$ if and only if condition* (4) *and* (5) *hold and*

$$\alpha = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} \text{ exists.} \tag{6}$$

Lemma 2.6. *[5] If $A = (a_{nk})_{n,k=0}^{\infty}$, then $A \in (\ell_\infty, c)$ if and only if condition* (5) *holds and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n\to\infty} a_{nk} \right| \tag{7}$$

Theorem 2.7. *Let $(q_k)_{k=0}^{\infty}$ be a given positive sequences, $Q_n = \sum_{i=0}^{n} q_i$ and $a = (a_k) \in w$ we define a matrix $C = (c_{nk})_{n,k=0}^{\infty}$ as*

$$c_{nk} = \begin{cases} Q_k \left[\left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{j=k+1}^{n} a_j + \frac{a_k}{q_k} \right] & 0 \le k \le n \\ 0 & k > n \end{cases}$$

and consider the sets

$$c_1 = \left\{ a \in w : \sup_n \sum_k |c_{nk}| < \infty \right\} \quad ; c_2 = \left\{ a \in w : \lim_{n\to\infty} c_{nk} \text{ exists for each } k \in \mathbb{N} \right\}$$

$$c_3 = \left\{ a \in w : \lim_{n\to\infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n\to\infty} c_{nk} \right| \right\} \quad ; c_4 = \left\{ a \in w : \lim_{n\to\infty} \sum_k c_{nk} \text{ exists} \right\}$$

Then $\left[\left(\bar{N}^q_\Delta\right)_0\right]^\beta = c_1 \cap c_2$, $\left[\left(\bar{N}^q_\Delta\right)\right]^\beta = c_1 \cap c_2 \cap c_4$ *and* $\left[\left(\bar{N}^q_\Delta\right)_\infty\right]^\beta = c_2 \cap c_3$.

Proof. We prove the result for $\left[\left(\bar{N}^q_\Delta\right)_0\right]^\beta$. Let $x \in \left(\bar{N}^q_\Delta\right)_0$ then there exists a y such that $y = \bar{N}^q_\Delta x$.
Writing S for the inverse of the matrix $\bar{N}^q_\Delta$ we obtain

$$\begin{aligned}
\sum_{k=0}^n a_k x_k &= \sum_{k=0}^n a_k S_k y \\
&= \sum_{k=0}^n a_k \left[\sum_{j=0}^{k-1} Q_j \left(\frac{1}{q_j} - \frac{1}{q_{j+1}} \right) y_j + \frac{Q_k}{q_k} y_k \right] \\
&= \sum_{k=0}^n Q_k \left[\left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{j=k+1}^n a_j + \frac{a_k}{q_k} \right] y_k \\
&= \sum_{k=0}^n c_{nk} y_k \\
&= C_n y
\end{aligned} \tag{8}$$

So $ax = (a_n x_n) \in cs$ whenever $x \in \left(\bar{N}^q_\Delta\right)_0$ if and only if $Cy \in cs$ whenever $y \in c_0$.
Using Lemma 2.4 we get $\left[\left(\bar{N}^q_\Delta\right)_0\right]^\beta = c_1 \cap c_2$.
The other two results can be shown in similar way using Lemma 2.5, Lemma 2.6. □

Let $X \subset \omega$ be a normed space and $a \in \omega$. Then we write

$$\|a\|^* = \sup \left\{ \left| \sum_{k=0}^\infty a_k x_k \right| : \|x\| = 1 \right\}$$

provided the term on the right side exists and is finite, which is the case whenever X is a BK space and $a \in X^\beta$ [23, Theorem 7.2.9].

Theorem 2.8. *For* $\left[\left(\bar{N}^q_\Delta\right)_0\right]^\beta$, $\left[\left(\bar{N}^q_\Delta\right)\right]^\beta$ *and* $\left[\left(\bar{N}^q_\Delta\right)_\infty\right]^\beta$ *the norm* $\|\cdot\|^*$ *is given by*

$$\|a\|^* = \sup_n \left[\sum_{k=0}^n Q_k \left| \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{j=k+1}^n a_j + \frac{a_k}{q_k} \right| \right].$$

Proof. We write $\mathcal{N}$ for any of the spaces $\left(\bar{N}^q_\Delta\right)_0$, $\left(\bar{N}^q_\Delta\right)$ or $\left(\bar{N}^q_\Delta\right)_\infty$. Let $a \in [\mathcal{N}]^\beta$ and $n \in \mathbb{N}$ be given. We write $C_n = (c_{nk})_{k=0}^\infty$ for the sequence in the nth row of the matrix C of Theorem 2.7 and

$\|C\| = \sup_n \|C_n\|_1 = \sup_n \sum_{k=0}^{n} |c_{nk}|$, and note that $\|C\| < \infty$, since $a \in [\mathcal{N}]^\beta$. Then we obtain as in (8) for all $x \in \mathcal{N}$ with y_k replaced by $\tau_k = \tau_k(x)$

$$\begin{aligned}\left|\sum_{k=0}^{n} a_k x_k\right| &\leq \sum_{k=0}^{n} |c_{nk}\tau_k| \\ &\leq \sup_k |\tau_k| \sup_n \|C_n\|_1 \\ &\leq \|C\| \cdot \|x\|_{\left(\bar{N}_\Delta^q\right)_\infty}\end{aligned}$$

Since n was arbitrary, we obtain

$$\left|\sum_{k=0}^{\infty} a_k x_k\right| \leq \|C\| \cdot \|x\|_{\left(\bar{N}_\Delta^q\right)_\infty}$$

Therefore,

$$\|a\|^* \leq \|C\| \qquad (2.2a).$$

Note that since $a \in [\mathcal{N}]^\beta$ and $x \in \mathcal{N}$ so $\sum_{k=0}^{\infty} a_k x_k$ converges.
Now for the converse consider an arbitrary integer n and let $x^{(n)}$ be a sequence such that

$$\tau_k\left(x^{(n)}\right) = \operatorname{sign}(c_{nk}) \qquad (k = 0, 1, \ldots)$$

where τ is defined as in (2).
Then

$$\tau_k\left(x^{(n)}\right) = 0 \qquad \text{for } k > n, \text{ i.e } x^{(n)} \in \left(\bar{N}_\Delta^q\right)_0, \quad \|x^{(n)}\|_{\left(\bar{N}_\Delta^q\right)_\infty} = \|\tau\left(x^{(n)}\right)\|_\infty \leq 1$$

and

$$\left|\sum_{k=0}^{\infty} a_k x_k^{(n)}\right| = \left|\sum_{k=0}^{n} c_{nk} x_k^{(n)}\right| = \sum_{k=0}^{n} |c_{nk}| \leq \|a\|^*$$

Since n was chosen arbitrarily, we obtain

$$\|C\| \leq \|a\|^* \qquad (2.2b)$$

We conclude by (2.2a) and (2.2b). □

Some well known results that are required for proving the compactness are:

Proposition 2.9. *[15, Theorem 7]*
Let X and Y be BK spaces, then $(X, Y) \subset \mathcal{B}(X, Y)$ that is every matrix A from X into Y defines an element L_A of $\mathcal{B}(X, Y)$ where

$$L_A(x) = Ax \qquad \forall\, x \in X$$

Also $A \in (X, \ell_\infty)$ if and only if

$$\|A\|^* = \sup_n \|A_n\|^* = \|L_A\| < \infty$$

If $\left(b^{(k)}\right)_{k=0}^{\infty}$ is a basis of X, Y and Y_1 are FK spaces with Y_1 a closed subspace of Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A\left(b^{(k)}\right) \in Y_1$ for all $k = 0, 1, 2, \ldots$.

By Proposition 2.9 and Theorem 2.8 we obtain the following corollary.

Corollary 2.10. *Let* $(q_k)_{k=0}^{\infty}$ *be a positive sequence,* $Q_n = \sum_{k=0}^{n} q_k$ *and* Δ *be the difference operator as defined in* (1), *then*

i) $A \in (\mathcal{N}, \ell_\infty)$ *if and only if*

$$\sup_{m,n}\left[\sum_{k=0}^{m} Q_k \left|\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right)\sum_{j=k+1}^{m} a_{nj} + \frac{a_{nk}}{q_k}\right|\right] < \infty \tag{9}$$

and

$$A_n \in [\mathcal{N}]^\beta \quad \forall\, n = 0, 1, \ldots \tag{10}$$

where $\mathcal{N}$ *is any of the spaces* $\left(\bar{N}_\Delta^q\right)_0$, $\left(\bar{N}_\Delta^q\right)$ *and* $\left(\bar{N}_\Delta^q\right)_\infty$.

ii) $A \in \left(\left(\bar{N}_\Delta^q\right)_0, c_0\right)$ *if and only if condition* (9) *holds and*

$$\lim_{n\to\infty} A_n c^{(k)} = 0 \quad \textit{for all } k = 0, 1, 2\ldots \tag{11}$$

where $c^{(k)}$ *is as given in Theorem 2.3.*

iii) $A \in \left(\left(\bar{N}_\Delta^q\right)_0, c\right)$ *if and only if condition* (9) *holds and*

$$\lim_{n\to\infty} A_n c^{(k)} = \alpha_k \quad \textit{for all } k = 0, 1, 2\ldots \tag{12}$$

where $c^{(k)}$ *is as given in Theorem 2.3.*

iv) $A \in \left(\left(\bar{N}_\Delta^q\right), c_0\right)$ *if and only if conditions* (9) *and* (11) *holds,*
$A_n \in \left[\bar{N}_\Delta^q\right]^\beta \quad \forall\, n = 0, 1, \ldots$ *and*

$$\lim_{n\to\infty} A_n c^{(k)} = 0 \quad \textit{for all } k = -1, 0, 1, 2\ldots \tag{13}$$

where $c^{(k)}$ *is as given in Theorem 2.3.*

v) $A \in \left(\left(\bar{N}_\Delta^q\right), c\right)$ *if and only if conditions* (9) *and* (12) *holds,*
$A_n \in \left[\bar{N}_\Delta^q\right]^\beta \quad \forall\, n = 0, 1, \ldots$ *and*

$$\lim_{n\to\infty} A_n c^{(k)} = \alpha_k \quad \textit{for all } k = -1, 0, 1, 2\ldots \tag{14}$$

where $c^{(k)}$ *is as given in Theorem 2.3.*

Proof. First we assume $A \in (\mathcal{N}, \ell_\infty)$.
Then it follows that $A_n \in [\mathcal{N}]^\beta$ for all n, hence $\|A_n\|^*$ is given by the formula in Theorem 2.8 and consequently (9) holds by Proposition 2.9.
Conversely, we assume that $A_n \in [\mathcal{N}]^\beta$ for all n and (9) is satisfied. Then $\|A_n\|^*$ for each n is given by the formula in Theorem 2.8, and so $A \in (\mathcal{N}, \ell_\infty)$ by Proposition 2.9.
The proof of the other parts is a direct consequence of second part of Proposition 2.9, where

$$\begin{aligned} A_n c^{(k)} &= \sum_{j=0}^{\infty} a_{nj} c_j^{(k)} \\ &= \sum_{j=k}^{\infty} a_{nj} c_j^{(k)} \\ &= a_{nk}\frac{Q_k}{q_k} + Q_k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right)\sum_{j=k+1}^{\infty} a_{nj} \qquad \text{for } n = 0, 1, 2, \ldots \end{aligned}$$

and

$$\begin{aligned} A_n c^{(-1)} &= \sum_{j=0}^{\infty} a_{nk} c_j^{(-1)} \\ &= \sum_{j=0}^{\infty} (j+1) a_{nj} \end{aligned}$$

Which completes the proof. □

3. Hausdorff Measure of Noncompactness

Let S and M be the subsets of a metric space (X, d) and $\epsilon > 0$. Then S is called an ϵ−net of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x, s) < \epsilon$. Further, if the set S is finite, then the ϵ−net S of M is called *finite ϵ−net* of M. A subset of a metric space is said to be *totally bounded* if it has a finite ϵ−net for every $\epsilon > 0$.
If $\mathcal{M}_X$ denotes the collection of all bounded subsets of metric space (X, d). If $Q \in \mathcal{M}_X$ then the *Hausdorff Measure of Noncompactness* of the set Q is defined by

$$\chi(Q) = \inf \{\epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called *Hausdorff Measure of Noncompactness* [2]
The basic properties of *Hausdorff Measure of Noncompactness* can be found in ([3], [16], [2]). Some of those properties are:
If Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then

$$\begin{aligned} \chi(Q) = 0 &\Leftrightarrow Q \text{ is totally bounded set}, \\ \chi(Q) &= \chi(\bar{Q}), \\ Q_1 \subset Q_2 &\Rightarrow \chi(Q_1) \le \chi(Q_2), \\ \chi(Q_1 \cup Q_2) &= \max \{\chi(Q_1), \chi(Q_2)\}, \\ \chi(Q_1 \cap Q_2) &= \min \{\chi(Q_1), \chi(Q_2)\}. \end{aligned}$$

Further if X is a normed space then χ has the additional properties connected with the linear structure.

$$\begin{aligned} \chi(Q_1 + Q_2) &\le \chi(Q_1) + \chi(Q_2), \\ \chi(\eta Q) &= |\eta| \chi(Q) \qquad \forall\, \eta \in \mathbb{C} \end{aligned}$$

The most effective way of characterizing compact operators between Banach spaces is by applying Hausdorff Measure of Noncompactness. If X and Y are Banach spaces, and $L \in \mathcal{B}(X, Y)$, then the Hausdorff Measure of Noncompactness of L, denoted by $\|L\|_\chi$ is given by

$$\|L\|_\chi = \chi\left(L(B_X)\right),$$

where $B_X = \{x \in X : \|x\| \le 1\}$ is the unit ball in X [16, Theorem 2.25].
From [16, Corollary 2.26 (2.58)] we know that

$$L \text{ is compact if and only if } \|L\|_\chi = 0.$$

4. Compact Operators on the Spaces $\left(\bar{N}_\Delta^q\right)_0$, $\left(\bar{N}_\Delta^q\right)$ and $\left(\bar{N}_\Delta^q\right)_\infty$

Let R be the transpose of the inverse matrix S defined in (3) then by [8, Lemma 2.5], if X is a BK space with AK or $X = \ell_\infty$ and Y be an arbitarry subset of ω. Then $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and

$W^{(A_n)} \in (X, c_0)$ for all $n = 0, 1, \ldots$ where $\hat{A}$ is the matrix with rows $\hat{A}_n = RA_n$ for $n = 0, 1, \ldots$ and the triangles $W^{(A_n)}$ $(n = 0, 1, \ldots)$ are defined as

$$w_{mk}^{(A_n)} = \begin{cases} \sum_{j=m}^{\infty} a_{nj} s_{jk} & (0 \leq k \leq m), \\ 0 & (k > m) \end{cases} \qquad (m = 0, 1, \ldots)$$

Hence using (3) we have

$$\hat{a}_{nk} = R_k A_n = Q_k \left(\frac{a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{j=k+1}^{\infty} a_{nj} \right) \qquad \text{for all } n, k = 0, 1, \ldots \tag{15}$$

$$\gamma_n = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(A_n)} = \lim_{m\to\infty} \left(\sum_{k=0}^{m} Q_k \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{j=k+1}^{\infty} a_{nj} + a_{nm} \frac{Q_m}{q_m} \right) \tag{16}$$

Let $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}_{nk}$, $\hat{\alpha} = (\hat{\alpha}_k)_{k=0}^{\infty}$, $\beta = \lim_{n\to\infty}(\sum_{k=0}^{\infty} \hat{a}_{nk} - \gamma_n)$ and $\hat{B}$ be the matrix with $\hat{B}_n = \hat{A}_n - \hat{\alpha}$ for all n.

Theorem 4.1. *Let $X = \left(\bar{N}_\Delta^q\right)_0$ or $X = \left(\bar{N}_\Delta^q\right)_\infty$.*

(a) If $A \in (X, c_0)$ then we have

$$\|L_A\|_\chi = \lim_{m\to\infty} \|\hat{A}^{[m]}\| = \lim_{m\to\infty} \left(\sup_{n\geq m} \left(\sum_{k=0}^{\infty} |\hat{a}_{nk}| \right) \right) \tag{17}$$

(b) If $A \in (X, c)$ then we have

$$\frac{1}{2} \cdot \lim_{m\to\infty} \|\hat{B}^{[m]}\| = \lim_{m\to\infty} \left(\sup_{n\geq m} \left(\sum_{k=0}^{\infty} |\hat{a}_{nk} - \hat{\alpha}_k| \right) \right) \leq \|L_A\|_\chi \leq \lim_{m\to\infty} \|\hat{B}^{[m]}\|. \tag{18}$$

(c) If $A \in (X, \ell_\infty)$ then

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty} \|\hat{A}^{[m]}\|. \tag{19}$$

Proof. Using [8, Corollary 3.6 (a), (c)], the statements in (17) and (18) can be easily shown.
(c) Define $P_m : \ell_\infty \to \ell_\infty$ by $P_m(x) = x^{[m]}$ for all $x \in \ell_\infty$ and $m = 0, 1, \ldots$. Then using the properties of χ and [8, Lemma 3.1, (3.1), (3.2)] we get

$$\begin{aligned} 0 \leq \chi\left(L_A(\hat{B}_{\ell_\infty})\right) &\leq \chi\left(P_m\left(L_A(\hat{B}_{\ell_\infty})\right)\right) + \chi\left((I - P_m)\left(L_A(\hat{B}_{\ell_\infty})\right)\right) \\ &= \chi\left((I - P_m)\left(L_A(\hat{B}_{\ell_\infty})\right)\right) \\ &\leq \sup_{x\in\hat{B}} \|(I - P_m)(L_A(x))\| \\ &= \|\hat{A}^{[m]}\| \qquad \text{for all } m. \end{aligned}$$

Which implies (19). □

Theorem 4.2. *(a) If $A \in \left(\left(\bar{N}_\Delta^q\right), c_0\right)$ then we have*

$$\|L_A\|_\chi = \lim_{m\to\infty} \left(\sup_{n\geq m} \left(\sum_{k=0}^{\infty} |\hat{a}_{nk}| + |\gamma_n| \right) \right). \tag{20}$$

(b) If $A \in \left(\left(\bar{N}^q_\Delta\right), c\right)$ then we have

$$\begin{aligned}\frac{1}{2}\cdot\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}-\hat{\alpha}_k|+\left|\sum_{k=0}^{\infty}\hat{\alpha}_k-\beta-\gamma_n\right|\right)\right)&\leq\|L_A\|_\chi\\ \leq\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}-\hat{\alpha}_k|+\left|\sum_{k=0}^{\infty}\hat{\alpha}_k-\beta-\gamma_n\right|\right)\right).\end{aligned} \tag{21}$$

(c) If $A \in \left(\left(\bar{N}^q_\Delta\right), \ell_\infty\right)$ then we have

$$0\leq\|L_A\|_\chi\leq\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}|+|\gamma_n|\right)\right). \tag{22}$$

Proof. Using [8, Theorem 3.7 (a), (c)], the statements in (20) and (21) can be easily shown.
The proof of (c) can be obtained using similar arguments as in the proof of Theorem 4.1(c). □

Using the notations of Theorem 4.1 and Theorem 4.2, we obtain the following corollaries for the compactness of an operator on the above spaces.

Corollary 4.3. *Let $X = \left(\bar{N}^q_\Delta\right)_0$ or $X = \left(\bar{N}^q_\Delta\right)_\infty$.*

(a) If $A \in (X, c_0)$ then L_A is compact if and only if

$$\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}|\right)\right)=0. \tag{23}$$

(b) If $A \in (X, c)$ then L_A is compact if and only if

$$\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}-\hat{\alpha}_k|\right)\right)=0. \tag{24}$$

(c) If $A \in (X, \ell_\infty)$ then L_A is compact if and only if (23) is satisfied.

Corollary 4.4. *(a) If $A \in \left(\left(\bar{N}^q_\Delta\right), c_0\right)$ then L_A is compact if and only if*

$$\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}|+|\gamma_n|\right)\right)=0. \tag{25}$$

(b) If $A \in \left(\left(\bar{N}^q_\Delta\right), c\right)$ then L_A is compact if and only if

$$\lim_{m\to\infty}\left(\sup_{n\geq m}\left(\sum_{k=0}^{\infty}|\hat{a}_{nk}-\hat{\alpha}_k|+\left|\sum_{k=0}^{\infty}\hat{\alpha}_k-\beta-\gamma_n\right|\right)\right)=0. \tag{26}$$

(c) If $A \in \left(\left(\bar{N}^q_\Delta\right), \ell_\infty\right)$ then L_A is compact if and only if (25) holds.

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