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Subsums of Conditionally Convergent Series in Finite Dimensional Spaces

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Abstract. An achievement set of a series is a set of all its subsums. We study the properties of achievement sets of conditionally convergent series in finite dimensional spaces. The purpose of the paper is to answer some of the open problems formulated in [10]. We obtain general result for series with harmonic-like coordinates, that is $A((-1)^{n+1}n^{-\alpha_1}, \dots, (-1)^{n+1}n^{-\alpha_d}) = \mathbb{R}^d$ for pairwise distinct numbers $\alpha_1, \dots, \alpha_d \in (0, 1]$. For $d = 2, \alpha_1 = 1, \alpha_2 = \frac{1}{2}$ this problem was stated in [10].

1. Introduction

For a sequence (x_n) (or a series $\sum_{n=1}^{\infty} x_n$) we call the set $A(x_n) = \{\sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0, 1\}^{\mathbb{N}}\}$ *the set of subsums* or *the achievement set*. This notion was mostly studied for absolutely summable sequences on the real line. Probably the first paper where topological properties of achievement sets were investigated is that of Kakeya [13]. He proved that such sets can be:

- finite sets,
- finite unions of compact intervals (if $|x_k| \le \sum_{n=k+1}^{\infty} |x_n|$ for all but finitely many k and it is a single interval if the inequality holds for all k),
- homeomorphic to the Cantor set (if $|x_k| > \sum_{n=k+1}^{\infty} |x_n|$ for all but finitely many k - so called quickly convergent series $\sum_{n=1}^{\infty} x_n$).

Kakeya conjectured that Cantor-like sets and finite unions of closed intervals are the only possible achievement sets for sequences $(x_n) \in \ell_1 \setminus c_{00}$. The results of Kakeya were rediscovered many times and his conjecture was repeated, even after the first counterexamples were given. In 1970 Renyi in [18] repeated the result of Kakeya Theorem in terms of purely atomic measures and he asked if the Cantor-like sets and finite unions of closed intervals are the only possible sets being the ranges of finite measures. Geometric properties of achievement sets of absolutely summable sequences and ranges of purely atomic finite measures are the same. This follows from the simple observation, that the set of sums of subseries for the series $\sum_{n=1}^{\infty} x_n$ is

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isometric to the analogous set for the series $\sum_{n=1}^{\infty}|x_n|$. Therefore a positive answer for the Renyi's question is equivalent to a positive answer to the Kekeya's conjecture. However the first counterexamples were published by Weinstein and Shapiro [19], Ferens [8] and Guthrie and Nymann [9]. It is worth mentioning that the motivation of Ferens paper [8] came from the measure theory; namely, the author has constructed a purely atomic probabilistic measure whose range is neither a finite union of intervals nor homeomorphic to the Cantor set. Due to Guthrie, Nymann and Saenz [9, 17] we know that the achievement set of an absolutely summable sequence can be a finite set, a finite union of intervals, homeomorphic to the Cantor set or it can be a so-called Cantorval. A Cantorval is a set homeomorphic to the union of the Cantor set and the sets which are removed from the unit segment by the even steps of the construction of the Cantor set. It is known that a Cantorval is such nonempty compact set in $\mathbb{R}$, that it is the closure of its interior and both endpoints of any nontrivial component are accumulation points of its trivial components. Other topological characterizations of Cantorvals can be found in [4] and [14]. All known examples of sequences whose achievement sets are Cantorvals belong to the class of multigeometric sequences or are linear combinations of such sequences, see [1],[2]. This class was deeply investigated in [11], [5] and [3]. In particular, the achievement sets of multigeometric series and similar sets obtained in more general case are the attractors of affine iterated function systems, see [3]. More information on achievement sets can be found in the surveys [4], [15] and [16].

Achievement sets can also be considered for conditionally convergent series. In this case we take only those $(\varepsilon_n) \in \{0,1\}^{\mathbb{N}}$ for which $\sum_{n=1}^{\infty}\varepsilon_n x_n$ converges. By $\mathrm{SR}(x_n) = \{\sum_{n=1}^{\infty} x_{\sigma(n)} : \sigma \in S_\infty\}$ we denote the sum range of a series $\sum_{n=1}^{\infty} x_n$. If $\sum_{n=1}^{\infty} x_n$ is conditionally convergent in $\mathbb{R}^m$, then the classical Levy-Steinitz Theorem states that $\mathrm{SR}(x_n)$ is an affine subset of the underlying space. More precisely, $\mathrm{SR}(x_n) = \sum_{n=1}^{\infty} x_n + \Gamma^\perp$ where $\Gamma^\perp$ is a subspace orthogonal to the set $\Gamma = \{f \in (\mathbb{R}^m)^* : \sum_{n=1}^{\infty}|f(x_n)| < \infty\}$. The theory of rearrangements of conditionally convergent series in Banach spaces, and further in topological vector spaces, has been developed and deeply investigated by many authors; we refer the reader to the monograph [12] for details.

In [6] the authors focused mostly on the case when $\sum_{n=1}^{\infty} x_n$ is conditionally convergent in $\mathbb{R}^2$ and $\mathrm{SR}(x_n)$ is a line. They showed that $\mathrm{A}(x_n)$ can essentially differ from $\mathrm{SR}(x_n)$. In particular when the sum range is one dimensional, affine subspace of $\mathbb{R}^2$ then it is possible to obtain the achievement set, which is one of the following: a graph of function; neither an F_σ nor a G_δ-set or even an open set different from $\mathbb{R}^2$. They made a general observation that $\mathrm{SR}(x_n) = \mathbb{R}^m$ if and only if the closure of $\mathrm{A}(x_n)$ equals $\mathbb{R}^m$ as well. The authors also constructed an example of series on the plane such that $\mathrm{SR}(x_n) = \mathbb{R}^2$ and $\mathrm{A}(x_n)$ is dense and null. The strengthening of this example constructed in [10] shows that it is also possible to obtain $\mathrm{A}(x_n)$ as a graph of a partial function, when $\mathrm{SR}(x_n) = \mathbb{R}^2$.

A partial answer to the question what needs to be assumed on the series $\sum_{n=1}^{\infty} x_n$ with $\mathrm{SR}(x_n) = \mathbb{R}^2$ to obtain $\mathrm{A}(x_n) = \mathbb{R}^2$ is given in [10]. It depends firstly on the number of Levy vectors. A vector $u \in \mathbb{R}^2$, $\|u\| = 1$ is called a Levy vector of the series $\sum_{n=1}^{\infty} v_n$ if for every $\varepsilon > 0$ we have $\sum_{v_n \in S_\varepsilon(u)} \|v_n\| = \infty$, where $S_\varepsilon(u) = \{v : \langle u, v\rangle \geq (1-\varepsilon)\|u\|\|v\|\}$ and $\langle u, v\rangle$ is the scalar product of u and v. The authors showed that if a series has more than two Levy vectors, then $\mathrm{A}(x_n) = \mathrm{SR}(x_n) = \mathbb{R}^2$. They proved even more: for any $a \in \mathbb{R}^2$ there is an increasing sequence (n_k) of indexes such that $\sum_{k=1}^{\infty} x_{n_k}$ is absolutely convergent to a. In symbols: $\mathrm{A}_{\mathrm{abs}}(x_n) = \mathbb{R}^2$, where $\mathrm{A}_{\mathrm{abs}}(x_n) = \{\sum_{n=1}^{\infty}\varepsilon_n x_n : \sum_{n=1}^{\infty}\varepsilon_n\|x_n\| < \infty, \varepsilon_n \in \{0,1\} \text{ for each } n \in \mathbb{N}\}$. The authors also found a sufficient condition (*reduction property*) for $\mathrm{A}(x_n) = \mathbb{R}^2$ for a series with exactly two Levy vectors. They constructed an example of series $\sum_{n=1}^{\infty} x_n$ with two Levy vectors such that $\mathrm{A}_{\mathrm{abs}}(x_n) \neq \mathrm{A}(x_n) = \mathbb{R}^2$. At the end of [10] the authors have formulated some open problems. One of them was to check if the equality $\mathrm{A}(\frac{(-1)^n}{n}, \frac{(-1)^n}{\sqrt{n}}) = \mathbb{R}^2$ holds. The series $\sum_{n=1}^{\infty}(\frac{(-1)^n}{n}, \frac{(-1)^n}{\sqrt{n}})$ is problematic since it has two Levy vectors and it is not known if it satisfies the reduction property. In this paper we give a positive answer to that question, that is $\mathrm{A}(\frac{(-1)^n}{n}, \frac{(-1)^n}{\sqrt{n}}) = \mathbb{R}^2$. We obtain something more general, that is $\mathrm{A}_{\mathrm{abs}}(\frac{(-1)^n}{n}, \frac{(-1)^n}{n^\alpha}) = \mathbb{R}^2$ for each $\alpha \in (0,1)$. We study its generalization in higher dimensions.

2. Main result

In this chapter we often represent a set $A \subset \mathbb{N}$ as $\{a_n;\ n \in \mathbb{N}\}$. In those cases, we assume that this sequence is increasing.

Notation 2.1. *Let $d \in \mathbb{N}$, $A \subset \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (0,1]^d$ and $\alpha, \delta \in (0,1]$. Then we can apply some useful notation.*

- $\mathbb{P} = \{n \in \mathbb{N};\ n \text{ is odd prime number}\}$,
- $< \boldsymbol{x}, \boldsymbol{\alpha} >= (x^1\alpha^1, \dots, x^d\alpha^d)$,
- $\mathbb{E}_\delta = \{\{a_n;\ n \in \mathbb{N}\} \subset \mathbb{N};\ (\forall n \in \mathbb{N} :\ a_{n+1} \le a_n(1+\delta)) \wedge (\exists \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n \ge n_0 :\ a_{n+1} \ge a_n(1+\epsilon))\}$,
- $\mathcal{P} = \{A \subset \mathbb{P};\ (\sum_{n\in A} \frac{1}{n} = \infty) \wedge (\forall \delta \in (0,1] \exists A_\delta \subset A :\ A_\delta \in \mathbb{E}_\delta\}$,
- $\mathcal{O}^d = \{((x^i)_{i=1}^d, (y^i)_{i=1}^d) \in \mathbb{R}^{2d};\ \forall i \in \{1, \dots, d\} : (x^i y^i < 0) \vee (x^i = y^i = 0)\}$,
- $A_{|_d} = A \cap [d+1, +\infty)$.

Definition 2.2. *For $i \in \mathbb{N}$ we inductively define the collections $\mathcal{W}_i$ and $\mathcal{W}$ of subsets of $\mathbb{N}$. We put*

$$\begin{aligned} \mathcal{W}_1 &= \{A \subset \mathbb{N};\ \exists B \in \mathbb{E}_1 :\ A \subset B\}, \\ \mathcal{W}_{i+1} &= \{A \cdot B \cup C;\ A \in \mathcal{W}_1 \wedge B, C \in \mathcal{W}_i\},\ i \in \mathbb{N} \\ \mathcal{W} &= \bigcup_{i=1}^{\infty} \mathcal{W}_i. \end{aligned}$$

Definition 2.3. *Let $A \subset \mathbb{N}$, $B \subset \mathbb{P}$ and $p \in B$. We say that A is constructed from (B, p) if every element of A is not divisible by any element of $\mathbb{P} \setminus B$ and is divisible by p.*

Lemma 2.4. *Let $\alpha \in (0,1]$.*

(A) Let $A, B \subset \mathbb{N}$. Then $\sum_{n\in A\cdot B} n^{-\alpha} \le \sum_{n\in A} n^{-\alpha} \sum_{n\in B} n^{-\alpha}$ and the equality holds if the mapping $(a,b) \to a \cdot b$ is injective on $A \times B$.

(B) Let $A \in \mathbb{E}_1$. Then $\sum_{n\in A} n^{-\alpha} < +\infty$.

(C) Let $A \in \mathcal{W}$. Then $\sum_{n\in A} n^{-\alpha} < +\infty$.

Proof. The proposition (A) simply follows from the fact that $a^{-\alpha}b^{-\alpha} = (ab)^{-\alpha}$.

Now, we prove the proposition (B). Let $A = \{a_j;\ j \in \mathbb{N}\} \in \mathbb{E}_1$. Then there exist $\epsilon > 0$ and $j_0 \in \mathbb{N}$ such that for every $j \ge j_0$ we have $a_{j+1} \ge a_j(1+\epsilon)$. Clearly,

$$\sum_{n\in A} n^{-\alpha} = \sum_{j=1}^{j_0-1} (a_j)^{-\alpha} + \sum_{j=j_0}^{\infty} (a_j)^{-\alpha} \le \sum_{j=1}^{j_0-1} (a_j)^{-\alpha} + (a_{j_0})^{-\alpha} \sum_{n=0}^{\infty} (1+\epsilon)^{-n\alpha} = \sum_{j=1}^{j_0-1} (a_j)^{-\alpha} + \frac{(a_{j_0})^{-\alpha}}{1-(1+\epsilon)^{-\alpha}} < +\infty.$$

To prove the proposition (C), we need to show that for every $k \in \mathbb{N}$ and every $A \in \mathcal{W}_k$ we have

$$\sum_{n\in A} n^{-\alpha} < +\infty. \tag{1}$$

We will prove this by induction. The case when $k = 1$ immediately follows from (B). Assume that $A \in \mathcal{W}_{k+1}$ and we have already proved (1) for any $C \in \mathcal{W}_k$. Then $A = B \cdot C \cup D$, where $B \in \mathcal{W}_1$ and $C, D \in \mathcal{W}_k$. By proposition (A) we simply obtain

$$\sum_{n\in A} n^{-\alpha} \le \sum_{n\in B} n^{-\alpha} \sum_{n\in C} n^{-\alpha} + \sum_{n\in D} n^{-\alpha} < +\infty,$$

which proves (1).

□

Notation 2.5. *Let* $A \subset \mathbb{P}$ *or* $A \in \mathcal{W}$, $d \in \mathbb{N}$, $\alpha \in (0,1]$ *and* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (0,1]^d$. *We set*

$$\Phi(A,\alpha) = \sum_{n \in A} n^{-\alpha}(-1)^{n+1}$$

and

$$\Psi(A,\boldsymbol{\alpha}) = (\Phi(A,\alpha_1), \dots, \Phi(A,\alpha_d)).$$

By Lemma 2.4 we have that notions $\Phi(A,\alpha)$ and $\Psi(A,\boldsymbol{\alpha})$ are well defined.

Now, we would like to describe the main idea of the proof of our main Theorem 2.10. Let $d \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (0,1]^d$ and $0 < \alpha_1 < \cdots < \alpha_d \leq 1$. We need to find $W \in \mathcal{W}$ such that $\Psi(W,\boldsymbol{\alpha}) = \boldsymbol{x}$. The key tool in our proof is the following property of function $f(x) = x^{\alpha}$: for every $a, b \in \mathbb{R}$ we have $f(ab) = f(a)f(b)$. Using this, we can prove Lemma 2.4(A) and consequently Remark 2.6(iv). Then we use induction and some properties of prime numbers to find disjoint sets $W_1, \dots, W_d \in \mathcal{W}$ such that $\Phi(W_i, \alpha_j) = 0$ for $i \neq j$ and $\Phi(W_i, \alpha_j) = x_i$ for $i = j$ (Lemma 2.8, Lemma 2.9). At the end, we put $W = \bigcup_{i=1}^{d} W_i$ and obtain $\Psi(W,\boldsymbol{\alpha}) = \boldsymbol{x}$.

The following remark demonstrates some simple properties of the above defined notions.

Remark 2.6. *The following assertions hold:*

(i) $\mathbb{P} \in \mathcal{P}$.

(ii) *Let* $A \in \mathcal{P}$ *and* $k \in \mathbb{N}$ *then* $A_{|_k} \in \mathcal{P}$.

(iii) *Let* $A \subset \mathbb{P}$ *then* $(-1)^{n+1} = 1$ *for any* $n \in A$ *and* $\Phi(A,\alpha) = \sum_{n \in A} n^{-\alpha}$.

(iv) *Let* $\alpha \in (0,1]$ *and* $A, B, C \in \mathcal{W}$, $A \subset \mathbb{P}$ *and* $p \in \mathbb{P} \setminus A$ *be such that* $B \cup C$ *is constructed from* $(\mathbb{P} \setminus A, p)$. *Then*

$$\Phi((A \cdot B) \cup C, \alpha) = \Phi(A,\alpha)\Phi(B,\alpha) + \Phi(C,\alpha).$$

(v) *Let* $A \in \mathcal{P}$ *and* $k \in \mathbb{N}$. *Then there exists* $B, C \in \mathcal{P}$ *such that* $B \cup C \subset A_{|_k}$ *and* $B \cap C = \emptyset$.

(vi) *Let* $\alpha \in (0,1]$ *and* $\tilde{B} \in \mathbb{E}_1$. *Then* $\{\Phi(C,\alpha);\ C \subset \tilde{B}\} \supset [0, \Phi(\tilde{B},\alpha)]$.

(vii) *Let* $x > 0$, $\alpha \in (0,1]$ *and* $A \in \mathcal{P}$. *Then there exists* $B \subset A$ *such that* $B \in \mathcal{W}_1$ *and* $\Phi(B,\alpha) = x$.

Proof. It is well known that $\sum_{p \in \mathbb{P}} \frac{1}{p} = +\infty$, which was proved by Euler in [7]. For $x > 0$ we define

$$\begin{aligned} f(x) &= \operatorname{card}(\mathbb{P} \cap [0,x]), \\ g(x) &= \frac{f(x)\log(x)}{x}. \end{aligned}$$

Prime Number Theorem states that

$$\lim_{x \to \infty} g(x) = 1. \tag{2}$$

Now, we prove that for every $0 < \epsilon < \delta \leq 1$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$\mathbb{P} \cap [n(1+\epsilon), n(1+\delta)] \neq \emptyset. \tag{3}$$

Assume on the contrary that there exist $0 < \epsilon < \delta \leq 1$ for which there exists an increasing sequence $\{a_n;\ n \in \mathbb{N}\}$ of integers such that

$$f(a_n(1+\epsilon)) = f(a_n(1+\delta)).$$

By (2) we have

$$1 = \lim_{n \to \infty} \frac{g(a_n(1+\epsilon))}{g(a_n(1+\delta))} = \lim_{n \to \infty} \frac{f(a_n(1+\epsilon))\log(a_n(1+\epsilon))a_n(1+\delta)}{f(a_n(1+\delta))\log(a_n(1+\delta))a_n(1+\epsilon)} = \lim_{n \to \infty} \frac{\log(a_n(1+\epsilon))(1+\delta)}{\log(a_n(1+\delta))(1+\epsilon)} = \frac{1+\delta}{1+\epsilon},$$

which is a contradiction. Let $\delta \in (0,1]$ be arbitrary. By (3) we can find $A_\delta \subset \mathbb{P}$ such that $A_\delta \in \mathbb{E}_\delta$. So, we have proved proposition (i).

Propositions (ii) and (iii) are trivial.

Proposition (iv) simply follows from Lemma 2.4(A). We only need to show that $(A \cdot B) \cap C = \emptyset$ and the mapping $(a,b) \to a \cdot b$ is injective on $A \times B$. This immediately follows from the fact that any element of $B \cup C$ is not divisible by any element of A. We also use the fact that the elements of A are odd. So multiplying by those elements does not change the parity. Thus

$$\begin{aligned}
\Phi((A \cdot B) \cup C, \alpha) &= \Phi(A \cdot B, \alpha) + \Phi(C, \alpha) = \sum_{n \in A \cdot B} (-1)^{n+1} n^{-\alpha} + \Phi(C, \alpha) \\
&= \sum_{n \in A \cdot (B \cap (2\mathbb{N}+1))} (-1)^{n+1} n^{-\alpha} + \sum_{n \in A \cdot (B \cap (2\mathbb{N}))} (-1)^{n+1} n^{-\alpha} + \Phi(C, \alpha) \\
&= \sum_{n \in A \cdot (B \cap (2\mathbb{N}+1))} n^{-\alpha} - \sum_{n \in A \cdot (B \cap (2\mathbb{N}))} n^{-\alpha} + \Phi(C, \alpha) \\
&= \sum_{n \in A} n^{-\alpha} \sum_{n \in B \cap (2\mathbb{N}+1)} n^{-\alpha} - \sum_{n \in A} n^{-\alpha} \sum_{n \in B \cap (2\mathbb{N})} n^{-\alpha} + \Phi(C, \alpha) \\
&= \Phi(A, \alpha)\Phi(B, \alpha) + \Phi(C, \alpha).
\end{aligned}$$

Now, we prove proposition (v). Let $A_{|_k} = \{a_n;\ n \in \mathbb{N}\}$. Put $B = \{a_{2n-1};\ n \in \mathbb{N}\}$ and $C = \{a_{2n};\ n \in \mathbb{N}\}$. Clearly

$$\sum_{n \in B} \frac{1}{n} = \sum_{n \in C} \frac{1}{n} = +\infty.$$

Let $\delta \in (0,1]$. We need to find $B_\delta \subset B$ such that $B_\delta \in \mathbb{E}_\delta$. By the definition of $\mathcal{P}$ there exists an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $\{a_{n_k};\ k \in \mathbb{N}\} \in \mathbb{E}_{\frac{\delta}{7}}$. Thus there exists $\epsilon > 0$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have $a_{n_{k+1}} \geq (1+\epsilon)a_{n_k}$ and for every $k \in \mathbb{N}$ we have $a_{n_{k+1}} \leq (1+\frac{\delta}{7})a_{n_k}$. We define a set $B_\delta = \{x_i;\ i \in \mathbb{N}\}$ by

$$x_i = \begin{cases} a_{n_{2i}} : & n_{2i} \text{ is odd}, \\ a_{n_{2i}-1} : & n_{2i} \text{ is even}. \end{cases}$$

Clearly, $B_\delta \subset B$. So, it remains to be shown that $B_\delta \in \mathbb{E}_\delta$. Let $i \geq k_0$. Then

$$x_{i+1} \geq a_{n_{2i+2}-1} \geq a_{n_{2i+1}} \geq (1+\epsilon)a_{n_{2i}} \geq (1+\epsilon)x_i.$$

Let now $i \in \mathbb{N}$ be arbitrary. Then

$$x_{i+1} \leq a_{n_{2i+2}} \leq \left(1 + \frac{\delta}{7}\right)^3 a_{n_{2i-1}} \leq (1+\delta)a_{n_{2i-1}} \leq (1+\delta)a_{n_{2i}-1} \leq (1+\delta)x_i.$$

So, $B_\delta \in \mathbb{E}_\delta$. Similarly, we can find $C_\delta \subset C$ such that $C_\delta \in \mathbb{E}_\delta$.

Assume that $\tilde{B} = \{a_n;\ n \in \mathbb{N}\}$. The proposition (vi) follows from the fact that the terms $(a_n)^{-\alpha}$ tend to 0 and $(a_{n+1})^\alpha \leq 2(a_n)^\alpha$.

Finally, we prove proposition (vii). Since $A \in \mathcal{P}$, we can find $C \subset A$ such that $C \in \mathbb{E}_1$. By Lemma 2.4(B), there exists $y > 0$ such that $\Phi(C, \alpha) = y$. Since $A \in \mathcal{P}$, we have $\Phi(A, 1) = +\infty$. Thus

$$\Phi(A \setminus C, \alpha) = \Phi(A, \alpha) - \Phi(C, \alpha) \geq \Phi(A, 1) - \Phi(C, \alpha) = +\infty - y = +\infty.$$

Since sequence $\{n^{-\alpha}\}_{n=1}^{\infty}$ tends to 0, we can find a finite set $D \subset (A \setminus C)$ such that $\Phi(D, \alpha) \in [x - y, x)$. Put $\tilde{B} = C \cup D$. Clearly $\tilde{B} \subset A$, $\tilde{B} \in \mathbb{E}_1$ and $\Phi(\tilde{B}, \alpha) = \Phi(C, \alpha) + \Phi(D, \alpha) \geq x$. We use proposition (vi) to find $B \subset \tilde{B}$ such that $\Phi(B, \alpha) = x$. $\square$

In the following lemma, we prove a stronger version of Remark 2.6(vii).

Lemma 2.7. *Let $x, y, z > 0$, $\tilde{A} \in \mathcal{P}$, $0 < \alpha < \beta < \gamma$ and $\beta \leq 1$. Then there exists $A \subset \tilde{A}$ such that*

$$\begin{aligned} A &\in \mathcal{W}_1, \\ \Phi(A, \alpha) &> z, \\ \Phi(A, \beta) &= x, \\ \Phi(A, \gamma) &< y. \end{aligned}$$

Proof. Let $k \in \mathbb{N}$ be arbitrary. By Remark 2.6(ii) we have $\tilde{A}_{|_k} \in \mathcal{P}$. By Remark 2.6(vii), there exists $C_k \subset \tilde{A}_{|_k}$ such that $C_k \in \mathcal{W}_1$ and $\Phi(C_k, \beta) = x$. Clearly,

$$\Phi(C_k, \alpha) = \sum_{n \in C_k} n^{-\alpha} = \sum_{n \in C_k} n^{-\beta} n^{-\alpha+\beta} \geq \sum_{n \in C_k} n^{-\beta} k^{-\alpha+\beta} = \Phi(C_k, \beta) k^{-\alpha+\beta} = x k^{-\alpha+\beta},$$

$$\Phi(C_k, \gamma) = \sum_{n \in C_k} n^{-\gamma} = \sum_{n \in C_k} n^{-\beta} n^{-\gamma+\beta} \leq \sum_{n \in C_k} n^{-\beta} k^{-\gamma+\beta} = \Phi(C_k, \beta) k^{-\gamma+\beta} = x k^{-\gamma+\beta}.$$

To finish the proof we only need to find $k \in \mathbb{N}$ such that $xk^{-\alpha+\beta} > z$ and $xk^{-\gamma+\beta} < y$ and set $A = C_k$. □

The following lemma helps us to do an inductive step in the proof of Lemma 2.9.

Lemma 2.8. *Let $d \in \mathbb{N}$, $l \in \{1, \ldots, d\}$, $\boldsymbol{x} = (x^1, \ldots, x^d)$, $\boldsymbol{y} = (y^1, \ldots, y^d)$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{O}^d$, $0 < \alpha_1 < \cdots < \alpha_d \leq 1$ and $M \in \mathcal{P}$. Then there exist $A, B, C \subset \mathbb{N}$ and $\boldsymbol{z}_1 = (z_1^1, \ldots, z_1^d)$, $\boldsymbol{z}_2 = (z_2^1, \ldots, z_2^d) \in \mathbb{R}^d$ such that*

(a) $C \in \mathcal{P}$,

(b) $A \cup B \cup C \subset M$,

(c) A, B, C *are pairwise disjoint,*

(d) $A, B \in \mathcal{W}_1$,

(e) $(\boldsymbol{z}_1, \boldsymbol{z}_2) \in \mathcal{O}^d$,

(f) $z_1^i = 0$ *if and only if* ($x^i = 0$ *or* $i = l$),

(g)

$$\boldsymbol{z}_1 = < \Psi(A, (\alpha_1, \ldots, \alpha_d)), \boldsymbol{x} > + \boldsymbol{y},$$
$$\boldsymbol{z}_2 = < \Psi(B, (\alpha_1, \ldots, \alpha_d)), \boldsymbol{y} > + \boldsymbol{x}.$$

Proof. If $x^l = 0$, then we can put $A = B = \emptyset$, $C = M$ satisfying (a)-(d). We define $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ by (g). Thus, we simply obtain (e), (f).

Assume $x^l \neq 0$. Since $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{O}^d$ we have $-\frac{y^l}{x^l} > 0$. By Remark 2.6(v), we can find some pairwise disjoint sets $\tilde{A}, \tilde{B}, C \subset M$ such that $\tilde{A}, \tilde{B}, C \in \mathcal{P}$. Thus ($a$) is satisfied. Now, we use Lemma 2.7 to find the set A. We put constants in Lemma 2.7 in the following way. For $1 < l < d$ let us put $x = -\frac{y^l}{x^l}$, $z = \frac{\max\{|y^i|;\ i \in \{1, \ldots, l-1\} \wedge y^i \neq 0\}}{\min\{|x^i|;\ i \in \{1, \ldots, l-1\} \wedge x^i \neq 0\}}$, $y = \frac{\min\{|y^i|;\ i \in \{l+1, \ldots, d\} \wedge y^i \neq 0\}}{\max\{|x^i|;\ i \in \{l+1, \ldots, d\} \wedge x^i \neq 0\}}$, $\alpha = \alpha_{l-1}$ $\beta = \alpha_l$, $\gamma = \alpha_{l+1}$. We put $\max\{\emptyset\} = \min\{\emptyset\} = 1$. For $l \in \{1, d\}$ we put them analogously but in case $l = 1$ we put $z = 1$ and $\alpha = \frac{\alpha_1}{2}$ and in case $l = d$ we put $y = 1$ and $\gamma = 2$. Analogously, we find $B \subset \tilde{B}$, we only interchange $\boldsymbol{x}$ and $\boldsymbol{y}$.

Since $A, B \in \mathcal{W}_1$ and Lemma 2.4(C) we can define $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ by (g).

Since $A \subset \tilde{A}$ and $B \subset \tilde{B}$ we have (b) and (c). By Lemma 2.7 we have (d).

Let $j \in \{1, \ldots, d\}$ be arbitrary.

If $x^j = 0$ then clearly $z_1^j = z_2^j = 0$.

Assume $j = l$. Then

$$z_1^l = \Phi(A, \alpha_l) x_l + y_l = -\frac{y_l}{x_l} x_l + y_l = 0.$$

Similarly $z_2^l = 0$.

Now assume that $1 \leq j < l$ and $x^j \neq 0$. Thus

$$\Phi(A, \alpha_j) \geq \Phi(A, \alpha_{l-1}) > \frac{\max\{|y^i|;\ i \in \{1, \ldots, l-1\} \wedge y^i \neq 0\}}{\min\{|x^i|;\ i \in \{1, \ldots, l-1\} \wedge x^i \neq 0\}}.$$

So

$$\Phi(A, \alpha_j)|x^j| > |y^j|.$$

Obviously, $z_1^j = \Phi(A, \alpha_j)x^j + y^j$ has the same sign as x^j. Similarly z_2^j has the same sign as y^j. Since $x^j y^j < 0$ we have $z_1^j z_2^j < 0$.

Finally, let $l < j \leq d$ and $x^j \neq 0$. Thus

$$\Phi(A, \alpha_j) \leq \Phi(A, \alpha_{l+1}) < \frac{\min\{|y^i|;\ i \in \{l+1, \ldots, d\} \wedge y^i \neq 0\}}{\max\{|x^i|;\ i \in \{l+1, \ldots, d\} \wedge x^i \neq 0\}}.$$

So

$$\Phi(A, \alpha_j)|x^j| < |y^j|.$$

Clearly $z_1^j = \Phi(A, \alpha_j)x^j + y^j$ has the same sign as y^j. Similarly z_2^j has the same sign as x^j. Since $x^j y^j < 0$ we have $z_1^j z_2^j < 0$. Thus we have proved $(e), (f)$. $\square$

Lemma 2.9. *Let $d \in \mathbb{N}$, $k \in \{1, \ldots, d\}$, $0 < \alpha_1 < \cdots < \alpha_d \leq 1$, $V \in \mathcal{P}$, $p \in V$ and $x \in \mathbb{R}$. Then there exists $W \in \mathcal{W}$ such that W is constructed from (V, p) and*

$$\Phi(W, \alpha_i) = \begin{cases} 0;\ i \neq k, \\ x;\ i = k. \end{cases}$$

Proof. If $x = 0$ then put $W = \emptyset$.

If $d = 1$ then this lemma simply follows from Remark 2.6(vii). If $x > 0$, then we find $\tilde{W} \subset V$ such that $\tilde{W} \in \mathcal{W}$ and $\Phi(W, \alpha_1) = xp^{\alpha_1}$. Then we set $W = p \cdot \tilde{W}$. If $x < 0$, then we find $\tilde{W} \subset V$ such that $\tilde{W} \in \mathcal{W}$ and $\Phi(\tilde{W}, \alpha_1) = |x|(2p)^{\alpha_1}$. Then we set $W = (2p) \cdot \tilde{W}$.

Assume that $x \neq 0$ and $d \geq 2$. For $i \in \{1, \ldots, d\}$ we inductively construct $\boldsymbol{x}_i = (x_i^1, \ldots, x_i^d)$, $\boldsymbol{y}_i = (y_i^1, \ldots, y_i^d) \in \mathbb{R}^d$, $l_i \in \mathbb{N}$, $M_i \in \mathcal{P}$ and $W_i^+, W_i^- \in \mathcal{W}_i$ satisfying

(1) $(\boldsymbol{x}_i, \boldsymbol{y}_i) \in \mathcal{O}^d$, $i \in \{1, \ldots, d\}$,

(2) $x_i^j = 0$ if and only if $j \in \{l_1, \ldots, l_{i-1}\}$, $i \in \{2, \ldots, d\}$,

(3) $l_i \in \{1, \ldots, d\} \setminus (\{k\} \cup \bigcup_{1 \leq j < i} \{l_j\})$, $i \in \{1, \ldots, d-1\}$,

(4) $\Phi(W_i^+, \boldsymbol{\alpha}) = \boldsymbol{x}_i$ and $\Phi(W_i^-, \boldsymbol{\alpha}) = \boldsymbol{y}_i$, $i \in \{1, \ldots, d\}$,

(5) $W_i^+ \cup W_i^-$ is constructed from $(V \setminus M_i, p)$, $i \in \{1, \ldots, d\}$,

(6) $M_i \subset V \setminus \{p\}$, $i \in \{1, \ldots, d\}$.

For $i = 1$ we put $W_1^+ = \{p\}$, $W_1^- = \{2p\}$ and $M_1 = V \setminus \{p\}$. Clearly, $W_1^+, W_1^- \in \mathcal{W}_1$, $M_1 \in \mathcal{P}$ and the conditions (5) and (6) are satisfied. Then we define $\boldsymbol{x}_1$, $\boldsymbol{y}_1$ by (4). Thus $x_1^j > 0$ and $y_1^j < 0$ for every $j \in \{1, \ldots, d\}$. So, (1) is also satisfied. Finally put $l_1 \in \{1, \ldots, d\} \setminus \{k\}$ arbitrarily and (3) is also satisfied.

Assume $i < d$ and $\boldsymbol{x}_i, \boldsymbol{y}_i \in \mathbb{R}^d$, $l_i \in \mathbb{N}$, $M_i \in \mathcal{P}$ and $W_i^+, W_i^- \in \mathcal{W}_i$ have already been constructed and satisfy (1)-(6). We use l_i, $\boldsymbol{x}_i$, $\boldsymbol{y}_i$ and M_i in Lemma 2.8 and obtain $A, B, C \subset \mathbb{N}$ and $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^d$ such that (a)-(g) are satisfied. We put $M_{i+1} = C$. Thus $M_{i+1} \in \mathcal{P}$ and $M_{i+1} \subset M_i \subset V \setminus \{p\}$. So, (6) is satisfied. We set $W_{i+1}^+ = (A \cdot W_i^+) \cup W_i^-$ and $W_{i+1}^- = (B \cdot W_i^-) \cup W_i^+$. Thus $W_{i+1}^+, W_{i+1}^- \in \mathcal{W}_{i+1}$. Since every element of $W_i^+ \cup W_i^-$

is divisible by p, we obtain that every element of $W_{i+1}^+ \cup W_{i+1}^-$ is also divisible by p. Since every element of $W_i^+ \cup W_i^-$ is not divisible by any element of M_i, $M_{i+1} \subset M_i$ and every element of $A \cup B$ is not divisible by any element of M_{i+1} we have that every element of $W_{i+1}^+ \cup W_{i+1}^-$ is not divisible by any element of M_{i+1}. Thus we have (5). Put $\boldsymbol{x}_{i+1} = \boldsymbol{z}_1$ and $\boldsymbol{y}_{i+1} = \boldsymbol{z}_2$. Since $W_i^+ \cup W_i^-$ is constructed from $(V \setminus M_i, p)$, Remark 2.6(iv) and Lemma 2.8(b),(g) we obtain (4). Conditions (1),(2) immediately follow from Lemma 2.8(e),(f). If $i + 1 < d$ we choose some l_{i+1}, which satisfies (3). Otherwise, we put $l_d = k$. So the construction is finished.

Thus, we have constructed $\boldsymbol{x}_d, \boldsymbol{y}_d \in \mathbb{R}^d$, $M_d \in \mathcal{P}$ and $W_d^+, W_d^- \in \mathcal{W}_i$ satisfying (1),(2),(4),(5),(6). Without loss of generality, we can assume that the sign of x is the same as the sign of x_d^k. By Remark 2.6(vii), we can find the set $K \subset M_d$ such that $\Phi(K, \alpha_k) = \frac{x}{x_d^k}$ and $K \in \mathcal{W}_1$. Let us put $W = K \cdot W_d^+$. By (5) and (6) we obtain that W is constructed from (V, p). Clearly $W \in \mathcal{W}_{d+1} \subset \mathcal{W}$. By (2),(4), (5) and Remark 2.6(iv) we obtain

$$\Phi(W, \alpha_i) = \begin{cases} 0; \ i \neq k, \\ x; \ i = k, \end{cases}$$

and the proof is finished. □

Theorem 2.10. *Let $d \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_d \in (0, 1]$ and $a_1, \ldots, a_d \in \mathbb{R} \setminus \{0\}$. Then*

$$\mathrm{A_{abs}}(a_1(-1)^{n+1} n^{-\alpha_1}, \ldots, a_d(-1)^{n+1} n^{-\alpha_d}) = \mathbb{R}^d$$

if and only if $\alpha_i,\ i = 1, \ldots, d$ are pairwise distinct numbers.

Proof. If there exist $i \neq j$ such that $\alpha_i = \alpha_j$, then clearly $\mathrm{A_{abs}}((-1)^{n+1} n^{-\alpha_1}, \ldots, (-1)^{n+1} n^{-\alpha_d}) \neq \mathbb{R}^d$.

Assume that $\alpha_i,\ i = 1, \ldots, d$ are pairwise distinct numbers. Without loss of generality, we can assume that $0 < \alpha_1 < \cdots < \alpha_d \leq 1$ and $a_1 = \cdots = a_d = 1$. Let $(x^1, \ldots, x^d) \in \mathbb{R}^d$ be arbitrary. By Remark 2.6(v) we can find pairwise disjoint sets $V^k \in \mathcal{P}$, $p_k \in V_k$, $k \in \{1, \ldots, d\}$. Using Lemma 2.9, we can find $W^k \in \mathcal{W}$ such that W^k is constructed from (V^k, p_k) and

$$\Phi(W^k, \alpha_i) = \begin{cases} 0; \ i \neq k, \\ x^k; \ i = k. \end{cases}$$

Put $W = \bigcup_{k=1}^d W^k$. Assume that $i \neq j$, $i, j \in \{1, \ldots, d\}$. Then element of W^i is not divisible by p_j and element of W^j is divisible by p_j. Thus, the sets W^k, $k \in \{1, \ldots, d\}$ are pairwise disjoint. Hence

$$\Psi(W, (\alpha_1, \ldots, \alpha_d)) = \sum_{k=1}^d \Psi(W^k, (\alpha_1, \ldots, \alpha_d)) = (x^1, \ldots, x^d)$$

and we are done. □

3. Open problems

Problem 3.1. *Decide, whether the series $\sum(\frac{(-1)^n}{n}, \frac{(-1)^n}{\sqrt{n}})$ satisfies the reduction property.*

Problem 3.2. *Characterize the family of all functions f such that $\mathrm{A_{abs}}((\frac{(-1)^n}{n}, \frac{(-1)^n}{f(n)}) = \mathbb{R}^2$.*

Problem 3.3. *Characterize the family of all functions f such that $\mathrm{A}((\frac{(-1)^n}{n}, \frac{(-1)^n}{f(n)}) = \mathbb{R}^2$.*

Clearly the family defined in the second problem is smaller than the one defined in the third problem. In the paper we have shown that it contains any function $f(n) = n^\alpha$ for $\alpha \in (0, 1)$.

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