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Maximal Ideals in Rings of Real Measurable Functions

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Abstract. Let $M(X)$ be the ring of all real measurable functions on a measurable space $(X, \mathscr{A})$. In this article, we show that every ideal of $M(X)$ is a Z°-ideal. Also, we give several characterizations of maximal ideals of $M(X)$, mostly in terms of certain lattice-theoretic properties of $\mathscr{A}$. The notion of T-measurable space is introduced. Next, we show that for every measurable space $(X, \mathscr{A})$ there exists a T-measurable space $(Y, \mathscr{A}')$ such that $M(X) \cong M(Y)$ as rings. The notion of compact measurable space is introduced. Next, we prove that if $(X, \mathscr{A})$ and $(Y, \mathscr{A}')$ are two compact T-measurable spaces, then $X \cong Y$ as measurable spaces if and only if $M(X) \cong M(Y)$ as rings.

1. Introduction

It is well known that $\mathbb{R}^X$ is the collection of all real-valued functions on X, for every non-empty set X and this with the (pointwise) addition and multiplication is a reduced commutative ring with identity. Let $(X, \mathscr{A})$ be measurable space and $M(X, \mathscr{A})$, abbreviated $M(X)$ be the set of all real measurable functions on X, then $M(X)$ is a subring of $\mathbb{R}^X$. Viertl in [18] shows that if X is a topological space and $\mathscr{A}$ is the set of all Borel sets of X then every maximal ideal of $M(X)$ is real if and only if $\mathscr{A}$ contains only a finite number of elements if and only if every ideal of $M(X)$ is fixed. Hager in [10] shows that if $(X, \mathscr{A})$ is a measurable space, then $M(X)$ is a regular ring in the sense of Von Neumann (i.e., for every given $f \in M(X)$, there is an element g in $M(X)$ with $f^2 g = f$). Azadi et al. in [2] prove that if $(X, \mathscr{A})$ is a measurable space, then $M(X)$ is an $\aleph_0$-self-injective ring. Moreover, if $\mathscr{A}$ contains all singletons, then $M(X)$ has an essential socle. Amini et al. in [1] generalized, simultaneously, the ring of real-valued continuous functions and the ring of real measurable functions. Momtahan in [13] studied essential ideals, socle, and some related ideals of rings of real measurable functions. He also studied the Goldie dimension of rings of measurable functions. In the paper [6] Estaji and Mahmoudi Darghadam investigated rings of real measurable functions vanishing at infinity on a measurable space.

Let X be any topological space and $\mathbb{R}$ be the space of real numbers with its usual topology. $C(X)$ is the set of all real-valued continuous functions with domain X (see [8, 11]). The main progress in the area of rings of real-valued continuous functions defined over a topological space X was provided by three historical subjects as follow:

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(1). Completely regular spaces were first discussed by Tychonoff [17]. We recall from [8, Theorem 3.9] that for every topological space X, there exists a completely regular Hausdorff space Y and a continuous mapping τ of X onto Y, such that the mapping $g \mapsto go\tau$ is an isomorphism of $C(Y)$ onto $C(X)$, which the reduction to completely regular spaces is due to Stone [16, P. 460] and Čech [4, P. 826].
(2). For every $f \in \mathbb{R}^X$, $Z(f) := \{x \in X : f(x) = 0\}$ is called the zero-set of f. An ideal I of $C(X)$ is called fixed if the set $\bigcap_{f\in I} Z(f)$ is non-empty; otherwise, I is called free. The maximal fixed ideals of the ring $C(X)$ are precisely the sets $M_p = \{f \in C(X) : f(p) = 0\}$, for every $p \in X$. The Gelfand-Kolmogoroff theorem generalizes this assertion to the case of arbitrary maximal ideals of $C(X)$ as follows: (Gelfand-Kolmogoroff). A subset M of $C(X)$ is a maximal ideal of $C(X)$ if and only if there is a unique point $p \in \beta X$ such that M coincides with the set $\{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ (see [7, 9])
(3). Again from [8, Theorem 3.9] we recall that two compact spaces X and Y are homeomorphic if and only if rings $C(X)$ and $C(Y)$ are isomorphic. This is due to Gelfand and Kolmogoroff (see [7]).

It is worth noting that measurable spaces have been studied by many authors. We think that it would be of interest to others. The present paper is devoted to placing these results in a measurable space context. We study the three above-mentioned subjects for the ring of all real measurable functions on a measurable space. The paper is organized as follows:

Section 2 of this paper is a prerequisite for the rest of the paper. The definitions and results of this section are taken from [15].

In Section 3, the notion of fixed ideal in $Z_{\mathscr{A}}$-ideal in the ring of all real measurable functions on a measurable space are introduced and we show that for every measurable space $(X, \mathscr{A})$ and every ideal I of $M(X)$, I is a z-ideal à la Mason of $M(X)$ if and only if I is a $Z_{\mathscr{A}}$-ideal of $M(X)$ if and only if I is a Z°-ideal of $M(X)$ (see Proposition 3.12). Also, we prove that every ideal in $M(X)$ is a z-ideal (see Proposition 3.13).

In Section 4, we show that a subset M of $M(X)$ is a maximal ideal if and only if there exists a unique $J \in \Sigma Id(\mathscr{A})$ such that $M = M^J$, where $M^J = \{f \in M(X) : \mathrm{coz}(f) \in J\}$ (see Proposition 4.3). Next, we study the relations between maximal free ideals of $M(X)$ and prime ideals of $\mathscr{A}$ (see Proposition 4.5). Also, we prove that for every subset M of $M(X)$, M is a fixed maximal ideal of $M(X)$ with $\bigcup_{f\in M} \mathrm{coz}(f) \in \mathscr{A}$ if and only if there exists a prime element P of $\mathscr{A}$ such that $M = M_P$, where $M_P = \{f \in M(X) : \mathrm{coz}(f) \subseteq P\}$ (see Proposition 4.6). Finally, we show that a compact measurable $(X, \mathscr{A})$ is determined by fixed maximal ideals of $M(X)$ (see Proposition 4.11).

In Section 5, the notion of T-measurable space is introduced and we show that for every measurable space $(X, \mathscr{A})$, there is a T-measurable space $(Y, \mathscr{A}')$ and an onto function $\theta : X \to Y$ such that $\eta : M(Y) \to M(X)$ given by $g \mapsto g \circ \theta$ is an isomorphism (see Proposition 5.6). In Proposition 5.7, we give an algebraic characterization of T-measurable spaces in terms of maximal ideals and this implies that if $(X, \mathscr{A})$ is a T-measurable space, then for every element P of $\mathscr{A}$, P is a prime element of $\mathscr{A}$ if and only if $|X \setminus P| = 1$ (see Corollary 5.8). Also, we show that a measurable space $(X, \mathscr{A})$ is a T-measurable space if and only if for every prime ideal P in $M(X)$, $|\bigcap_{f\in P} Z(f)| \leq 1$ (see Proposition 5.10).

In Section 6, we prove that for every compact measurable space $(X, \mathscr{A})$ there is a compact T-measurable space $(Y, \mathscr{A}')$ such that $M(X) \cong M(Y)$ as rings (see Corollary 6.2). Also, for every compact T-measurable space $(X, \mathscr{A})$, we show that $X \cong \max(M(X))$ as measurable spaces. If $(X, \mathscr{A})$ and $(Y, \mathscr{A}')$ are two compact T-measurable spaces, then $X \cong Y$ as measurable spaces if and only if $M(X) \cong M(Y)$ as rings (see Proposition 6.7 and Corollary 6.8).

2. Preliminaries

Here, we recall some definitions and results from the literature on measurable spaces and partially ordered sets. For further information see [15] on measurable-theoretic concepts and [5, 14] on lattice-theoretic concepts.

2.1. measurable spaces

Let us recall some general notation from [15]. Let X be a nonempty set. A collection $\mathscr{A}$ of subsets of a set X is said to be a *σ-algebra* in X if $\mathscr{A}$ has the following three properties:

(i) $X \in \mathscr{A}$.

(ii) If $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$, where A^c is the complement of A relative to X.

(iii) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

If $\mathscr{A}$ is a σ-algebra in X, then X or, for clarity $(X, \mathscr{A})$ is called a *measurable space*, and the members of $\mathscr{A}$ are called the measurable sets in X. If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y. If X is a measurable space, then the set of all measurable maps from X into $\mathbb{R}$ is denoted $M(X)$, and the members of $M(X)$ are called the *real measurable functions* on X, where $\mathbb{R}$ denotes the set of all real numbers with the ordinary topology.

We recall from [15, Theorem 1.10] that if $\mathcal{A}$ is any collection of subsets of X, there exists the smallest σ-algebra $\mathscr{A}^*$ in X such that $\mathcal{A} \subseteq \mathscr{A}^*$. This $\mathscr{A}^*$ is called the *σ-algebra generated by* $\mathcal{A}$ and it is denoted by $<\mathcal{A}>$. Since the intersection of any family of σ-algebras in X is a σ-algebra in X, we conclude that $<\mathcal{A}>$ is intersection of the family of all σ-algebras $\mathscr{A}$ in X which contain $\mathcal{A}$. Hence $<\mathcal{A}>=<\mathcal{A}^c>=<\mathcal{A}\cup\mathcal{A}^c>$, where $\mathcal{A}^c := \{A^c : A \in \mathcal{A}\}$. Also, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ with $\mathcal{A} \subseteq \mathcal{B}$, then $<\mathcal{A}>\subseteq<\mathcal{B}>$.

The set $M(X)$ of all real measurable functions on a measurable space $(X, \mathscr{A})$ will be provided with an algebraic structure and an order structure. Since their definitions do not involve measurity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers. Addition, multiplication, joint, and meet in $\mathbb{R}^X$ are defined by the formulas $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $(f\vee g)(x) = \max\{f(x), g(x)\}$, and $(f\wedge g)(x) = \min\{f(x), g(x)\}$. It is obvious that $(\mathbb{R}^X; +, ., \vee, \wedge)$ is an f-ring, this conclusion is the immediate consequence of the corresponding statements about the field $\mathbb{R}$. Also, $(M(X); +, ., \vee, \wedge)$ is an sub-f-ring of $\mathbb{R}^X$.

2.2. *partially ordered sets*

Let us recall some general notation from [5, 14]. A poset L is a *lattice* if and only if for every a and b in L both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist (in L). For subset X of a poset L and $x \in L$ we write:

1. $\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$.
2. $\uparrow X = \{y \in L : y \geq x \text{ for some } x \in X\}$.
3. $\downarrow x = \downarrow\{x\}$.
4. $\uparrow x = \uparrow\{x\}$.

A nonempty subset J of a lattice L is called an *ideal* of L if $x \vee y \in J$ and $\downarrow x \subseteq J$, for all $x, y \in J$. A nonempty subset F of a lattice L is called a *filter* of L if $x \wedge y \in F$ and $\uparrow x \subseteq F$, for all $x, y \in F$. We say an element a of a lattice L is a *top element* (*bottom element*) of L if $x \leq a$ ($a \leq x$), for all $x \in X$. We denote the top element and the bottom element of a lattice L by $\top$ and $\bot$ respectively. A lattice L is said to be *bounded* if there exist the top element and the bottom element in the lattice. An element a of a bounded lattice L is said to be *compact* if $a = \bigvee S$, $S \subseteq L$, implies $a = \bigvee T$ for some finite subset T of S. A bounded lattice L is said to be *compact* whenever its top element $\top$ is compact. A lattice L is said to be distributive lattice if the binary operations $\vee$ and $\wedge$ hold distributive property, i.e.; for any $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. An element p of a bounded distributive lattice L is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ imply that $a \leq p$ or $b \leq p$. An element m of a bounded lattice L is said to be *maximal* (or *dual atom*) if $m < \top$ and $m \leq x \leq \top$ imply that $m = x$ or $x = \top$. As it is well known, every maximal element in a bounded distributive lattice is prime. We write ΣL and $\max(L)$ for the set of all prime elements and maximal elements of L , respectively. A σ-frame is a lattice L with countable joins $\bigvee_n$, finite meets $\wedge$, top $\top$, bottom $\bot$ and satisfying $x \wedge \bigvee_n x_n = \bigvee_n (x \wedge x_n)$, for $n \in J$, a countable index set, $x, x_n \in L$. A *frame* is a complete lattice L in which the distributive law $x \wedge \bigvee S = \bigvee_{s\in S}(x \wedge s)$ holds for all $x \in L$ and $S \subseteq L$. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

If $\mathscr{A}$ is a σ-algebra in X, then the following statements hold.

(1) $(\mathscr{A}, \subseteq)$ is a Boolean algebra.

(2) $(\mathscr{A}, \subseteq)$ is a σ-frame.

3. Filters in σ-algebras

Consider $f \in \mathbb{R}^X$, the set $f^{-1}(0)$ will be called the zero-set of f. We shall find it convenient to denote this set by $Z(f)$, or, for clarity, by $Z_X(f)$:

$$Z(f) = Z_X(f) = \{x \in X : f(x) = 0\}.$$

Any set that is a zero-set of some function in $\mathbb{R}^X$ is called a zero-set in X. For any subset A from $M(X)$, we write $Z_{\mathscr{A}}[A] = \{Z(f) : f \in A\}$ and we put $Z_{\mathscr{A}}[X] = Z_{\mathscr{A}}[M(X)]$.

For every $f, g \in \mathbb{R}^X$, we have

(1) $Z(f) = Z(|f|) = Z(f^n)$, every $n \in \mathbb{N}$.
(2) $Z(fg) = Z(f) \cup Z(g)$.
(3) $Z(f^2 + g^2) = Z(|f| + |g|) = Z(f) \cap Z(g)$.

Remark 3.1. Let $(X, \mathscr{A})$ be a measurable space and $A \in \mathscr{A}$, then the characteristic function $\chi_A : X \to \mathbb{R}$ is a real measurable function on X with $Z(\chi_A) = A^c$.

Proposition 3.2. *If $(X, \mathscr{A})$ is a measurable space, then $Z_{\mathscr{A}}[X] = \mathscr{A}$.*

Proof. Let f be a real measurable function on the measurable space $(X, \mathscr{A})$, then $Z(f) = f^{-1}[0, +\infty) \cap f^{-1}(-\infty, 0] \in \mathscr{A}$. Therefore, $Z_{\mathscr{A}}[X] \subseteq \mathscr{A}$. By Remark 3.1, $\mathscr{A} \subseteq Z_{\mathscr{A}}[X]$ and so $Z_{\mathscr{A}}[X] = \mathscr{A}$. □

Let $(X, \mathscr{A})$ be a measurable space. Then we have $\bigcap_{n\in\mathbb{N}} Z(f_n) \in Z_{\mathscr{A}}[X]$ and $\bigcup_{n\in\mathbb{N}} Z(f_n) \in Z_{\mathscr{A}}[X]$, for every $\{f_n\}_{n\in\mathbb{N}} \subseteq M(X)$. Therefore, $Z_{\mathscr{A}}[X]$ is a σ-frame.

Proposition 3.3. *Let f be a real measurable function on a measurable space $(X, \mathscr{A})$. The element f is a unit element of $M(X)$ if and only if $Z(f) = \emptyset$.*

Proof. Necessity. By hypothesis, there is a $g \in M(X)$ such that $fg = \mathbf{1}$, then $Z(f) \cup Z(g) = Z(fg) = Z(\mathbf{1}) = \emptyset$, which implies that $Z(f) = \emptyset$.

Sufficiency. We define $g : X \to \mathbb{R}$ given by $g(x) = \frac{1}{f(x)}$, for every $x \in X$. Since $Z(f) = \emptyset$, we conclude that $g \in \mathbb{R}^X$. We have

$$g^{-1}(r, +\infty) = \Big(\{x \in X : rf(x) < 1\} \cap f^{-1}(0, +\infty)\Big) \cup \Big(\{x \in X : rf(x) > 1\} \cap f^{-1}(-\infty, 0)\Big),$$

for every $r \in \mathbb{R}$. Since $\mathbf{r}f \in M(X)$, we infer that $g^{-1}(r, +\infty) \in \mathscr{A}$, for every $r \in \mathbb{R}$. Therefore, $g \in M(X)$ and $fg = \mathbf{1}$. Hence f is a unit element of $M(X)$. □

Definition 3.4. Let $(X, \mathscr{A})$ be a measurable space. A proper filter of $\mathscr{A}$ is called a $Z_{\mathscr{A}}$*-filter* on X.

In the following proposition, we study relations between proper ideals and $Z_{\mathscr{A}}$-filters.

Proposition 3.5. *Let $(X, \mathscr{A})$ be a measurable space. In $M(X)$, the following statements hold.*

(1) *If I is a proper ideal in $M(X)$, then the family $Z[I] = \{Z(f) \mid f \in I\}$ is a $Z_{\mathscr{A}}$-filter on X.*
(2) *If $\mathcal{F}$ is a $Z_{\mathscr{A}}$-filter on X, then the family $Z^{-1}[\mathcal{F}] = \{f \mid Z(f) \in \mathcal{F}\}$ is a proper ideal in $M(X)$.*

Proof. (1). Consider $f, g \in I$ and $h \in M(X)$. By hypothesis, $f^2 + g^2, fh \in I$, then $Z(f) \cap Z(g) = Z(f^2 + g^2) \in Z[I]$ and if $Z(f) \subseteq Z(h)$, then $Z(h) = Z(f) \cup Z(h) = Z(fh) \in Z[I]$. Also, since I contains no unit, we conclude from Proposition 3.3 that $\emptyset \notin Z[I]$.

(2). Let $J = Z^{-1}[\mathcal{F}]$. By Definition $Z_{\mathscr{A}}$-filter and Proposition 3.3, J contains no unit. Let $f, g \in J$, and let $h \in M(X)$. Then:

$$Z(f - g) = Z(f + (-g)) \geq Z(f) \cap Z(-g)) \geq Z(f) \cap Z(g) \in \mathcal{F}$$

and hence $Z(f - g) \in \mathcal{F}$, by Definition $Z_{\mathscr{A}}$-filter. Therefore, $f - g \in Z^{-1}[\mathcal{F}]$. Moreover,

$$Z(hf) = Z(h) \cup Z(h) \supseteq Z(f) \in \mathcal{F},$$

and hence $Z(fh) \in \mathcal{F}$, by Definition $Z_{\mathscr{A}}$-filter. Therefore $fh \in Z^{-1}[\mathcal{F}]$. This completes the proof that J is a proper ideal in $M(X)$. □

Let $(X, \mathscr{A})$ be a measurable space. A $Z_{\mathscr{A}}$-filter $\mathcal{F}$ on a set X is said to be an *$Z_{\mathscr{A}}$-ultrafilter* if it is maximal (with respect to inclusion) in the family of all $Z_{\mathscr{A}}$-filters on X.

In the following proposition, we study relations between maximal ideals and $Z_{\mathscr{A}}$-ultrafilters.

Proposition 3.6. *Let $(X, \mathscr{A})$ be a measurable space. In $M(X)$, the following statements hold.*

(1) *If M is a maximal ideal in $M(X)$, then $Z[M]$ is a $Z_{\mathscr{A}}$-ultrafilter on X.*
(2) *If $\mathcal{F}$ is a $Z_{\mathscr{A}}$-ultrafilter on X, then $Z^{-1}[\mathcal{F}]$ is a maximal ideal in $M(X)$.*
The mapping Z is one-one from the set of all maximal ideals in $M(X)$ onto the set of all $Z_{\mathscr{A}}$-ultrafilters on X.

Proof. Since $Z_{\mathscr{A}}$ and $Z^{-1}_{\mathscr{A}}$ preserve inclusion, the result follows at once from Proposition 3.5. □

Let $A \in \mathscr{A}$ and $\mathcal{F} \subseteq \mathscr{A}$, we say A meets $\mathcal{F}$ if and only if $A \cap B \neq \emptyset$, for all $B \in \mathcal{F}$. It is evident that

1. A $Z_{\mathscr{A}}$-filter $\mathcal{F}$ on X is a $Z_{\mathscr{A}}$-ultrafilter if and only if A meets $\mathcal{F}$, it implies that $A \in \mathcal{F}$, for every $A \in \mathscr{A}$.
2. If $\mathcal{F}$ and $\mathcal{G}$ are disjoint $Z_{\mathscr{A}}$-ultrafilter on X, then there are elements $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $A \cap B = \emptyset$.
3. If $\{\mathcal{F}_i\}_{i\in I}$ is a nonempty collection of $Z_{\mathscr{A}}$-filters on X, then $\bigcap_{i\in I} \mathcal{F}_i$ is a $Z_{\mathscr{A}}$-filter on X.
4. Every $Z_{\mathscr{A}}$-filter on X is contained in a $Z_{\mathscr{A}}$-ultrafilter on X .

Proposition 3.7. *Let M be a maximal ideal in $M(X)$. If $Z(f)$ meets every member of $Z[M]$, then $f \in M$.*

Proof. The set $Z[M]$ is a $Z_{\mathscr{A}}$-ultrafilter on X, by Proposition 3.6, and so, if $Z(f)$ meets every member of $Z[M]$, then $Z(f) \in Z[M]$. Therefore $f \in Z^{-1}[Z[M]]$; moreover, $M \subseteq Z^{-1}[Z[M]]$, and M is a maximal ideal, so that $f \in M = Z^{-1}[Z[M]]$. □

Proposition 3.8. *Let $(X, \mathscr{A})$ be a measurable space. For every $p \in X$,*

$$M_p := \{f \in M(X) : f(p) = 0\}$$

is a maximal ideal in $M(X)$.

Proof. By Proposition 3.3, it is clear that M_p is a proper ideal in $M(X)$. Consider $f \in M(X) \setminus M_p$ with $f(p) = r \neq 0$. From $\mathbf{1} - \frac{1}{\mathbf{r}} f \in M(X)$ and $(\mathbf{1} - \frac{1}{\mathbf{r}} f)(p) = 0$, we infer that $\mathbf{1} - \frac{1}{\mathbf{r}} f \in M_p$, which implies that M_p is a maximal ideal in $M(X)$. □

Recall the notion of a z-ideal of a ring R as was introduced by Mason in [12]. In lattice theory this notion is known as "z-ideals à la Mason". Denote by $\max(R)$ the set of all maximal ideals of the ring R. For $a \in R$, let

$$\mathfrak{M}(a) = \{M \in \max(R) \mid a \in M\}.$$

An ideal I of a ring R is called a ***z*-ideal à la Mason** if whenever $\mathfrak{M}(a) \subseteq \mathfrak{M}(b)$ and $a \in I$, then $b \in I$.

Lemma 3.9. *Let $(X, \mathscr{A})$ be a measurable space. In $M(X)$, the following statements are equivalent, for every $f, g \in M(X)$.*

(1) $\mathfrak{M}(f) \subseteq \mathfrak{M}(g)$.
(2) $Z(f) \subseteq Z(g)$.
(3) $\mathrm{Ann}(f) \subseteq \mathrm{Ann}(g)$.

Proof. (1)⇒(2). We have

$$p \in Z(f) \Rightarrow f \in M_p \in \mathfrak{M}(f) \subseteq \mathfrak{M}(g) \Rightarrow g \in M_p \Rightarrow p \in Z(g).$$

(2)⇒(3). We have

$$h \in \mathrm{Ann}(f) \Rightarrow X = Z(hf) = Z(h) \cup Z(f) \subseteq Z(h) \cup Z(g) = Z(hg) \Rightarrow h \in \mathrm{Ann}(g).$$

(3)⇒(1). Consider $M \in \mathfrak{M}(f)$. Since

$$Z(\chi_{Z(f)} f) = Z(\chi_{Z(f)}) \cup Z(f) = (X \setminus Z(f)) \cup Z(f) = X,$$

we conclude that $\chi_{Z(f)} \in \mathrm{Ann}(f) \subseteq \mathrm{Ann}(g)$, which implies that $\chi_{Z(f)} g = \mathbf{0} \in M$. Since $Z(\chi^2_{Z(f)} + f^2) = \emptyset$, we conclude from Proposition 3.3 that $g \in M$, we infer that $M \in \mathfrak{M}(g)$. Therefore $\mathfrak{M}(f) \subseteq \mathfrak{M}(g)$. □

For each element a of a ring R, let P_a be the intersection of all minimal prime ideals containing a and by convention, we put the intersection of an empty set of ideals equal to R. We recall from [3] that a proper ideal I of a ring R is called a Z°-ideal if for each $a \in I$ we have $P_a \in I$. Also,

Proposition 3.10. [3] *Let R be a reduced ring and I be a proper ideal in R, then the following are equivalent.*

(1) *I is a Z°-ideal in R.*
(2) *$P_a = P_b$ and $a \in I$, imply that $b \in I$.*
(3) $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ *and $a \in I$, imply that $b \in I$.*
(4) *$a \in I$ implies that* $\mathrm{Ann}(\mathrm{Ann}(a)) \subseteq I$.

Definition 3.11. An ideal I of the ring $M(X)$ is called a *$Z_{\mathscr{A}}$-ideal* if whenever $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in M(X)$, then $g \in I$.

Proposition 3.12. *Let $(X, \mathscr{A})$ be a measurable space. If I is an ideal in $M(X)$, then the following statements are equivalent.*

(1) *I is a z-ideal à la Mason of $M(X)$.*
(2) *I is a $Z_{\mathscr{A}}$-ideal of $M(X)$.*
(3) *I is a Z°-ideal of $M(X)$.*

Proof. By Lemma 3.9 and Proposition 3.10, it is evident. □

Proposition 3.13. *Let $(X, \mathscr{A})$ be a measurable space. Every ideal in $M(X)$ is a z-ideal.*

Proof. Let I be an ideal of $M(X)$. Suppose that $f, g \in M(X)$ with $Z(f) \subseteq Z(g)$ and $f \in I$. We define $h : X \to \mathbb{R}$ by $h(x) = \frac{1}{f(x)}$, if $x \in \mathrm{coz}(f)$ and 0 otherwise. Then $h \in M(X)$ and $g = ghf \in I$. This completes the proof, by Proposition 3.12. □

The following proposition shows that the primeity of a ideal in $M(X)$ coincides with its semiprimeity.

Proposition 3.14. *Let $(X, \mathscr{A})$ be a measurable space. Let I be a proper ideal in $M(X)$. Then the following statements are equivalent.*

(1) *I is a prime ideal.*
(2) *I contains a prime ideal.*
(3) *For every $f, g \in M(X)$, if $fg = \mathbf{0}$, then $f \in I$ or $g \in I$.*
(4) *For every $f \in M(X)$, there is a zero set belonging to $Z[I]$ on which f does not change sign.*

Proof. (1)⇒(2) and (2)⇒(3) are trivial.

(3)⇒(4). We observe that for every $f \in M(X)$,

$$(f \vee \mathbf{0})(f \wedge \mathbf{0}) = f^+(-f^-) = \mathbf{0}.$$

Then, by hypothesis, either $f \vee \mathbf{0}$ or $f \wedge \mathbf{0}$ is in I , and hence $Z(f \vee \mathbf{0})$ or $Z(f \wedge \mathbf{0})$ is in $Z[I]$. However, f does not change sign on one of them, since

$$Z(f \wedge \mathbf{0}) \cap f^{-1}(-\infty, 0) = f^{-1}[0, +\infty) \cap f^{-1}(-\infty, 0) = \emptyset$$

and

$$Z(f \vee \mathbf{0}) \cap f^{-1}(0, +\infty) = f^{-1}(-\infty, 0] \cap f^{-1}(0, +\infty) = \emptyset.$$

(4)⇒(1). Given $gh \in I$, consider the function $|g| - |h|$. By hypothesis, there is a zero set Z of $Z[I]$ on which $|g| - |h|$ is non-negative, say, $Z \cap (|g| - |h|)^{-1}(-\infty, 0) = \emptyset$. Then there is an element f in I such that $Z = Z(f)$, it implies that $Z(f) \cap Z(g) \subseteq Z(h)$. From

$$Z((hg)^2 + f^2) = Z(hg) \cap Z(f) = [Z(h) \cap Z(f)] \cup [Z(g) \cap Z(f)] \subseteq Z(h)$$

and $(hg)^2 + f^2 \in I$, we conclude that $h \in I$, since, by Propositions 3.12 and 3.13, I is the $Z_{\mathscr{A}}$-ideal in $M(X)$. Thus, I is prime. □

4. Maximal ideals in $M(X)$

Let $(X, \mathscr{A})$ be a measurable space. For each $I \in Id(\mathscr{A})$, the ideal M^I is defined by

$$M^I = \{f \in M(X) : \mathrm{coz}(f) \in I\}.$$

Lemma 4.1. *Let $(X, \mathscr{A})$ be a measurable space. For each $I, J \in Id(\mathscr{A})$, $M^I = M^J$ if and only if $I = J$.*

Proof. Necessity. If $a \in I$ then $\chi_a \in M^I$, which implies that $a = \mathrm{coz}(\chi_a) \in J$. Therefore, $I = J$.
Sufficiency. It is clear. □

Remark 4.2. Let $(X, \mathscr{A})$ be a measurable space. Consider $I \in \Sigma Id(\mathscr{A})$ and $I \subsetneq J \in Id(\mathscr{A})$. Then there exists an element a in $J \setminus I$ and since $a \wedge a' = \bot \in I \in \Sigma Id(\mathscr{A})$, we conclude that $a' \in I \subseteq J$, which implies that $\top = a \vee a' \in J$ and so, $J = \mathscr{A}$. Therefore,

$$\Sigma Id(\mathscr{A}) = \max(Id(\mathscr{A})).$$

In the following proposition, we investigate the relations between maximal ideals of $M(X)$ and maximal ideals of $\mathscr{A}$.

Proposition 4.3. *Let $(X, \mathscr{A})$ be a measurable space. A subset M of $M(X)$ is a maximal ideal if and only if there exists a unique $J \in \Sigma Id(\mathscr{A})$ such that $M = M^J$.*

Proof. Necessity. Let M be a maximal ideal of $M(X)$. We set $I = \{a \in \mathscr{A} : a \leq \mathrm{coz}(f)$ for some $f \in M\}$. By Proposition 3.3, $\mathscr{A} \neq I \in Id(\mathscr{A})$ and since $Id(\mathscr{A})$ is a compact frame, we conclude that $J \in \Sigma Id(\mathscr{A})$ such that $I \subseteq J$. From $\top \notin J$, we infer that $M \subseteq M^J \neq M(X)$, and in view of the maximality of M we must have $M = M^J$. By Lemma 4.1, there exists a unique $J \in \Sigma Id(\mathscr{A})$ such that $M = M^J$.
Sufficiency. Consider $J \in \Sigma Id(\mathscr{A})$ and $Q \in Id(M(X))$ with $M^J \subsetneq Q$. Then there exists an element f in $Q \setminus M^J$. From $\mathrm{coz}(f) \notin J$, we infer from Remark 4.2 that there exists an element a in J such that

$$\mathrm{coz}(\chi_a + f^2) = \mathrm{coz}(\chi_a) \vee \mathrm{coz}(f) = a \vee \mathrm{coz}(f) = \top,$$

which implies that $\chi_a + f^2 \in Q$ is a unit element of $M(X)$ and so, $Q = M(X)$. Therefore, M^J is a maximal ideal of $M(X)$. □

Definition 4.4. Let I be an ideal in $M(X)$. If $\bigcap_{f \in I} Z(f)$ is nonempty, we call I a *fixed ideal*; if $\bigcap_{f \in I} Z(f) = \emptyset$, then I is a *free ideal*. Also, if $\mathcal{K} \subseteq \mathscr{A}$ with $\bigcap \mathcal{K}$ is nonempty, we call $\mathcal{K}$ a *fixed subset* of $\mathscr{A}$; if $\bigcap \mathcal{K} = \emptyset$, then $\mathcal{K}$ is a *free subset* of $\mathscr{A}$.

In the following proposition, we investigate the relations between fixed maximal ideals of $M(X)$ and prime ideals of $\mathscr{A}$.

Proposition 4.5. *Let $(X, \mathscr{A})$ be a measurable space. For every subset M of $M(X)$, M is a fixed maximal ideal of $M(X)$ if and only if there exists a prime ideal P of $\mathscr{A}$ such that $\bigcup P \subsetneq X$ and $M = M^P$.*

Proof. Necessity. Let M be a fixed maximal ideal of $M(X)$. Then, by Proposition 4.3, $M = M^P$ for some P of $\Sigma Id(\mathscr{A})$. Since for every $A \in P$, $\chi_A \in M^P$, we infer that $\bigcup P = \bigcup_{f \in M^P} \mathrm{coz}(f) \subsetneq X$.
Sufficiency. Consider $P \in \Sigma Id(\mathscr{A})$ with $\bigcup P \subsetneq X$. Then, by Proposition 4.3, M^P is a maximal ideal in $M(X)$. Since for every $A \in P$, $\chi_A \in M^P$, we conclude that $\bigcup_{f \in M^P} \mathrm{coz}(f) = \bigcup P \subsetneq X$, which implies that M^P is a fixed maximal ideal of $M(X)$. □

Let $(X, \mathscr{A})$ be a measurable space. For each $A \in \mathscr{A}$ with $A \subsetneq X$, define the subset M_A of $M(X)$ by

$$M_A := \{f \in M(X) : \mathrm{coz}(f) \subseteq A\}$$

In the following proposition, we investigate the relations between fixed maximal ideals of $M(X)$ and prime elements of $\mathscr{A}$.

Proposition 4.6. *Let $(X, \mathscr{A})$ be a measurable space. For every subset M of $M(X)$, M is a fixed maximal ideal of $M(X)$ with $\bigcup_{f\in M} \operatorname{coz}(f) \in \mathscr{A}$ if and only if there exists a prime element P of $\mathscr{A}$ such that $M = M_P$.*

Proof. Necessity. We claim that $P := \bigcup_{f\in M} \operatorname{coz}(f)$ is a prime element of $\mathscr{A}$. If not, there exist $V, W \in \mathscr{A}$ such that $V \cap W \subseteq P$, $V \nsubseteq P$ and $V \nsubseteq P$, then $\chi_{X\setminus V}, \chi_{X\setminus W} \in M$ and this implies $X = (X \setminus V) \cup (X \setminus W) \cup (V \cap W) = P$, but this is a contradiction to the fact that M is a fixed maximal ideal of $M(X)$, which proves the claim. By the maximality of M, we have $M = M_P$, since $M \subseteq M_P \subsetneq M(X)$.

Sufficiency. Consider $P \in \Sigma\mathscr{A}$. From $\downarrow P \in \Sigma Id(\mathscr{A})$, we conclude from Proposition 4.3 that $M_P = M^{\downarrow P}$ is a maximal ideal in $M(X)$. Since $\chi_P \in M^{\downarrow P}$, we conclude that $\bigcup_{f\in M^{\downarrow P}} \operatorname{coz}(f) = P \subsetneq X$, which implies that M_P is a fixed maximal ideal in $M(X)$, by Proposition 4.5. □

As an immediate consequence we now have the following corollary.

Corollary 4.7. *Let $(X, \mathscr{A})$ be a measurable space. For every subset M of $M(X)$ with $\bigcup_{f\in M} \operatorname{coz}(f) \in \mathscr{A}$, M is a fixed prime ideal of $M(X)$ if and only if M is a fixed maximal ideal of $M(X)$.*

A measurable space $(X, \mathscr{A})$ is called a *compact measurable space* if $\mathscr{A}$ is a compact lattice.

Definition 4.8. Let $\mathcal{K}$ be a nonempty family of sets. $\mathcal{K}$ is said to have the *finite intersection property* provided that the intersection of any finite number of members of $\mathcal{K}$ is nonempty.

Remark 4.9. Let $(X, \mathscr{A})$ be a measurable space. It is evident that every subfamily of $\mathscr{A}$ with the finite intersection property is contained in some $Z_{\mathscr{A}}$-ultrafilter.

Proposition 4.10. *Let $(X, \mathscr{A})$ be a measurable space. Then $(X, \mathscr{A})$ is a compact measurable space if and only if every family of measurable subsets of X with the finite intersection property has nonempty intersection.*

Proof. Necessity. Let $\mathcal{K}$ be any family of measurable subsets of X with the finite intersection property. If $\bigcap \mathcal{K} = \emptyset$, then $\bigcup_{A\in\mathcal{K}} A^c = X$ and from $\{A^c : A \in \mathcal{K}\} \subseteq \mathscr{A}$, we conclude that there exist $A_1, \dots, A_n \in \mathcal{K}$ such that $\bigcup_{i=1}^n A_i^c = X$, by hypothesis. Then $\bigcap_{i=1}^n A_i = \emptyset$, and this is a contradiction to the fact that $\mathcal{K}$ has the finite intersection property.

Sufficiency. Let $\mathcal{B} \subseteq \mathscr{A}$ such that $X = \bigcup \mathcal{B}$. Suppose that for every finite subset $\mathcal{A}$ of $\mathcal{B}$, $X \neq \bigcup \mathcal{A}$, then $\mathcal{K} := \{B^c : B \in \mathcal{B}\} \subseteq \mathscr{A}$ have the finite intersection property, but $\bigcap \mathcal{K} = \emptyset$, and this is a contradiction. Hence there exists a finite subset $\mathcal{A}$ of $\mathcal{B}$ such that $X = \bigcup \mathcal{A}$. Therefore, $(X, \mathscr{A})$ is a compact measurable space. □

Our next result shows that compact measurable spaces admit a simple characterization in terms of fixed ideals and fixed $Z_{\mathscr{A}}$-filters.

Proposition 4.11. *Let $(X, \mathscr{A})$ be a measurable space. The following statements are equivalent.*

(1) *$(X, \mathscr{A})$ is compact.*
(2) *Every proper ideal in $M(X)$ is fixed.*
(3) *Every maximal ideal in $M(X)$ is fixed.*
(4) *Every $Z_{\mathscr{A}}$-filter in $\mathscr{A}$ is fixed.*
(5) *Every $Z_{\mathscr{A}}$-ultrafilter in $\mathscr{A}$ is fixed.*

Proof. The equivalence of (2) with (4) and (3) with (5) are evident, by Propositions 3.5 and 3.6.

(1)⇒(2). Let I be a proper ideal in $M(X)$. Then, by Proposition 3.5, $Z[I]$ is $Z_{\mathscr{A}}$-filter in $\mathscr{A}$. Since $Z[I]$ has the finite intersection property, we conclude from Proposition 4.9 that $\bigcap Z[I] \neq \emptyset$, which implies that I is a fixed ideal.

(2)⇒(3). Evident.

(3)⇒(1). Suppose that $\mathcal{K} \subseteq \mathscr{A}$ has the finite intersection property. Then, by Remark 4.9, there exists a $Z_{\mathscr{A}}$-ultrafilter $\mathcal{F}$ such that $\mathcal{K} \subseteq \mathcal{F}$. Therefore, by Proposition 3.6, $\emptyset \neq \bigcap \mathcal{F} \subseteq \bigcap \mathcal{K}$ and we conclude from Proposition 4.10 that $(X, \mathscr{A})$ is compact. □

5. *T*-measurable spaces

In this section, we show that in the study of rings of measurable functions on a measurable space there is no need to deal with measurable spaces that are not T-measurable.

Definition 5.1. Let X be an abstract set, and consider an arbitrary subfamily C of $\mathbb{R}^X$. The *weak measurable space induced by* C on X is defined to be the smallest σ-algebra in X such that all functions in C are measurable.

Let $C \subseteq \mathbb{R}^X$ and $\mathcal{A} := \{f^{-1}(O) : f \in C, O \in \mathfrak{O}\mathbb{R}\}$. Then $(X, <\mathcal{A}>)$ is the weak measurable space induced by C on X.

Lemma 5.2. *Let $(X, \mathscr{A})$ be a measurable space, $Y \neq \emptyset$ and $\mathcal{A} \subseteq \mathcal{P}(Y)$. Suppose that $f : X \to Y$ is a function and $<\mathcal{A}>$ is the σ-algebra generated by $\mathcal{A}$ on Y. Then*

$$\{f^{-1}(A) : A \in \mathcal{A}\} \subseteq \mathscr{A}$$

if and only if

$$\{f^{-1}(B) : B \in <\mathcal{A}>\} \subseteq \mathscr{A}.$$

Proof. It is evident, because $f^{-1}(\bigcup_{\lambda\in\Lambda} B_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(B_\lambda)$, $f^{-1}(\bigcap_{\lambda\in\Lambda} B_\lambda) = \bigcap_{\lambda\in\Lambda} f^{-1}(B_\lambda)$ and $f^{-1}(B^c) = (f^{-1}(B))^c$, for every $\{B_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{P}(Y)$ and every $B \in \mathcal{P}(Y)$. □

Proposition 5.3. *Let $(X, \mathscr{A})$ be a measurable space, $Y \neq \emptyset$ and $C \subseteq \mathbb{R}^Y$. Let $\mathscr{A}'$ be the weak measurable space induced by C on Y and $f : X \to Y$ be a function. Then for every $g \in C$, $g \circ f \in M(X)$ if and only if for every $A \in \mathscr{A}'$, $f^{-1}(A) \in \mathscr{A}$.*

Proof. Necessity. Let $O \in \mathfrak{O}\mathbb{R}$, then $f^{-1}(g^{-1}(O)) \in \mathscr{A}$. Therefore, By Definition 5.1 and Lemma 5.2, $f^{-1}(A) \in \mathscr{A}$ for every $A \in \mathscr{A}'$.

Sufficiency. Let $O \in \mathfrak{O}\mathbb{R}$, then $g^{-1}(O) \in \mathscr{A}'$, which implies that $f^{-1}(g^{-1}(O)) \in \mathscr{A}$. Therefore, $g \circ f \in M(X)$ for every $g \in C$. □

Definition 5.4. A measurable space $(X, \mathscr{A})$ is said to be *T-measurable* if whenever x and y are distinct points in X, there is a measurable set containing one and not the other (see [19]).

Lemma 5.5. *Let $(X, \mathscr{A})$ be a measurable space. Define $x \sim x'$ in X to mean that $f(x) = f(x')$ for every $f \in M(X)$. Then the following statements hold.*

(1) *The relation $\sim$ is an equivalence relation.*
(2) *If $A \in \mathscr{A}$ then $A = \bigcup_{x\in A}[x]_\sim$.*
(3) *For each $f \in M(X)$, associate a function $h_f \in \mathbb{R}^{X/\sim}$ given by $h_f([x]_\sim) = f(x)$. If $\mathscr{A}_{X/\sim}$ is the weak measurable space induced by $\{h_f : f \in M(X)\}$ on $X/\sim$, then $(X/\sim, \mathscr{A}_{X/\sim})$ is a T-measurable space.*

Proof. (1). It is evident.

(2). Consider $x \in A \in \mathscr{A}$, then, by Proposition 3.2, there exists an element f in $M(X)$ such that $z(f) = A$. If $y \in [x]$, then $g(x) = g(y)$ for all $g \in M(X)$, which implies that $f(y) = f(x) = 0$, hence $y \in z(f) = A$.

(3). By the statement (2), $(X/\sim, \mathscr{A}_{X/\sim})$ is a measurable space. Consider $[x], [x'] \in X/\sim$ with $[x] \neq [x']$. Then there exists an element f in $M(X)$ such that $f(x) \neq f(x')$, which implies that $h_f([x]) \neq h_f([x'])$. Consider

$$r := \frac{|h_f([x]) - h_f([x'])|}{3}.$$

Thus, by Definition 5.1,

$$[x] \in h_f^{-1}(h_f([x]) - r, h_f([x]) + r) \in \mathscr{A}_{X/\sim},$$

$$[x'] \in h_f^{-1}(h_f([x']) - r, h_f([x']) + r) \in \mathscr{A}_{X/\sim}$$

and

$$h_f^{-1}(h_f([x]) - r, h_f([x]) + r) \cap h_f^{-1}(h_f([x']) - r, h_f([x']) + r) = \emptyset.$$

Therefore, $(X/\sim, \mathscr{A}_{X/\sim})$ is a T-measurable space. □

The next proposition eliminates any reason for considering rings of real measurable functions on other than T-measurable spaces.

Proposition 5.6. *For every measurable space $(X, \mathscr{A})$ there is a T-measurable space $(Y, \mathscr{A}')$ and an onto function $\theta : X \to Y$ such that $\eta : M(Y) \to M(X)$ given by $g \mapsto g \circ \theta$ is an isomorphism and the following statements hold.*

(1) *For every $A \in \mathscr{A}$, $\theta(A) \in \mathscr{A}'$.*
(2) *For every $B \in \mathscr{A}'$, $\theta^{-1}(B) \in \mathscr{A}$.*

Proof. Suppose $\sim$ is the same equivalence relation Lemma 5.5. We put $Y := X/\sim$ and $\mathscr{A}' := \mathscr{A}_{X/\sim}$. Then, by Lemma 5.5, $(Y, \mathscr{A}')$ is a T–measurable space. Define $\theta : X \to Y$ by $\theta(x) = [x]_\sim$. Hence θ is an onto function and $h_f \circ \theta = f$ for any $f \in M(X)$, where h_f is the same function Lemma 5.5. Then $\eta : M(Y) \to M(X)$ given by $g \mapsto g \circ \theta$ is an onto function and we have

$$\eta(g_1 + g_2)(x) = ((g_1 + g_2) \circ \theta)(x) = (g_1 \circ \theta)(x) + (g_2 \circ \theta)(x) = (\eta(g_1) + \eta(g_2))(x)$$

and

$$\eta(g_1 g_2)(x) = ((g_1 g_2) \circ \theta)(x) = (g_1 \circ \theta)(x)(g_2 \circ \theta)(x) = (\eta(g_1)\eta(g_2))(x),$$

for every $g_1, g_2 \in M(Y)$ and every $x \in X$. Since θ is onto, we infer that

$$ker(\eta) = \{g \in M(Y) : \forall x \in X, g(\theta(x)) = \{0\}\} = \{g \in M(X) : g(Y) = \{0\}\} = \{\mathbf{0}\},$$

which implies that η is an isomorphism, i.e., $M(X) \cong M(Y)$ as rings.

If $A \in \mathscr{A}$ then, by Proposition 3.2, there is an element f in $M(X)$ such that $A = z(f)$, which implies that

$$z(h_f) = \{[x] : h_f([x]) = 0\} = \{[x] : f(x) = 0\} = \{[x] : x \in z(f)\} = \theta(z(f)).$$

Therefore, by Proposition 3.2, $\theta(A) \in \mathscr{A}'$. Thus, the statement (1) holds.

Since $\mathscr{A}'$ is the weak measurable space induced by $\{h_f : f \in M(X)\}$ on Y and for every $f \in M(X)$, $h_f \circ \theta = f$, we conclude from Lemma 5.3 that for every $B \in \mathscr{A}'$, $\theta^{-1}(B) \in \mathscr{A}$. Thus, the statement (2) holds. □

As a consequence of the foregoing theorem, algebraic or lattice properties that hold for all $M(X)$, with T-measurable space X, hold just as well for all $M(X)$, with arbitrary measurable space X.

Now, we give an algebraic characterization of T-measurable spaces in terms of maximal ideals.

Proposition 5.7. *Let $(X, \mathscr{A})$ be a measurable space. Then the following statements are equivalent.*

(1) *The measurable space $(X, \mathscr{A})$ is a T-measurable space.*
(2) *If x and y are distinct points in X, then $M_x \neq M_y$.*
(3) *For every maximal ideal M in $M(X)$, $|\bigcap_{f \in M} Z(f)| \leq 1$.*

Proof. (1)⇒(2). Assume that x and y are distinct points in X. By hypothesis, there exists a measurable set A in X such that $x \in A$ and $y \notin A$. By Remark 3.1, $\chi_A \in M_y \setminus M_x$.

(2)⇒(3). Let M be a maximal ideal in $M(X)$ with $|\bigcap_{f \in M} Z(f)| \geq 2$. If x and y are distinct points in $\bigcap_{f \in M} Z(f)$, then $M \subseteq M_x$ and $M \subseteq M_y$. Since M is maximal, we conclude from Proposition 3.8 that $M_y = M = M_x$, and this is a contradiction.

(3)⇒(1). Assume that x and y are distinct points in X. Then, by Proposition 3.8, $\bigcap_{f \in M_x} Z(f) = \{x\}$ and $\bigcap_{f \in M_y} Z(f) = \{y\}$, which imply that there exists an element f in $M_x \setminus M_y$. Since $Z(f) \in \mathscr{A}$, $x \in Z(f)$ and $y \notin Z(f)$, we infer that $(X, \mathscr{A})$ is a T-measurable space. □

Corollary 5.8. *Let $(X, \mathscr{A})$ be a T-measurable space. For every element P of $\mathscr{A}$, P is a prime element of $\mathscr{A}$ if and only if $|X \setminus P| = 1$.*

Proof. Necessity. By Propositions 4.6 and 5.7, $|X \setminus P| = |\bigcap_{f \in M_P} z(f)| = 1$.

Sufficiency. It is clear. □

We recall that a ring R is called a Gelfand ring or a PM-ring if each of its proper prime ideals is contained in a unique maximal ideal. The following proposition shows that $M(X)$ is a Gelfand ring.

Proposition 5.9. *Let $(X, \mathscr{A})$ be a measurable space. Every prime ideal in $M(X)$ is contained in a unique maximal ideal.*

Proof. Let P be a prime ideal. We know that every ideal is contained in at least one maximal ideal. If M and M' are distinct maximal ideals such that $P \subseteq M \cap M'$, then, by Proposition 3.14, $M \cap M'$ is a prime ideal, since M and M' are $Z_{\mathscr{A}}$-ideals. This is a contradiction to the fact that $M \cap M'$ is not prime. □

The following proposition shows that T-measurable spaces have a nice characterization in terms of prime ideals.

Proposition 5.10. *Let $(X, \mathscr{A})$ be a measurable space. Then the measurable space $(X, \mathscr{A})$ is a T-measurable space if and only if for every prime ideal P in $M(X)$, $|\bigcap_{f \in P} Z(f)| \leq 1$.*

Proof. Necessity. Let P be a prime ideal in $M(X)$ such that $x, y \in \bigcap_{f \in P} Z(f)$ with $x \neq y$. Then, by hypothesis and Proposition 5.7, M_x and M_y are distinct maximal ideals such that $P \subseteq M_x \cap M_y$. This is a contradiction to the fact that every prime ideal in $M(X)$ is contained in a unique maximal ideal.

Sufficiency. Let M be a maximal ideal in $M(X)$, then M is a prime ideal in $M(X)$, which implies that $|\bigcap_{f \in M} Z(f)| \leq 1$, by hypothesis. Therefore, by Proposition 5.7, $(X, \mathscr{A})$ is a T-measurable space. □

Definition 5.11. [5] Let L be a distributive lattice and F be a filter of L. The filter F is called *prime filter* if $I \neq L$ and $x \vee y \in F$ implies $x \in F$ or $y \in F$.

In the following proposition, we study relations between prime ideals and prime $Z_{\mathscr{A}}$-filters.

Proposition 5.12. *Let $(X, \mathscr{A})$ be a measurable space. In $M(X)$, the following statements hold.*

(1) *If P is a prime ideal in $M(X)$, then $Z[P]$ is a prime $Z_{\mathscr{A}}$-filter on X.*
(2) *If $\mathcal{F}$ is a prime $Z_{\mathscr{A}}$-filter on X, then $Z^{-1}[\mathcal{F}]$ is a prime ideal in $M(X)$.*
The mapping Z is one-one from the set of all prime ideals in $M(X)$ onto the set of all prime $Z_{\mathscr{A}}$-filters on X.

Proof. (1). Consider $f, g \in M(X)$ with $Z(fg) = Z(f) \cup Z(g) \in Z[P]$. Since $P_z := Z^{-1}[Z[P]]$ is a $Z_{\mathscr{A}}$-ideal in $M(X)$ and $Z(fg) \in Z[P] = Z[P_z]$, we conclude that $fg \in P_z$. From $P \subseteq P_z$ and Proposition 3.14, we infer that P_z is a prime ideal in $M(X)$, which implies that $f \in P_z$ or $g \in P_z$. Hence $Z(f) \in Z[P_z] = Z[P]$ or $Z(g) \in Z[P_z] = Z[P]$ and, by Proposition 3.5, the proof is now complete.

(2). Consider $f, g \in M(X)$ with $fg \in Z^{-1}[\mathcal{F}]$, then $Z(f) \cup Z(g) = Z(fg) \in F$, which implies that $Z(f) \in F$ or $Z(g) \in F$, by hypothesis. Hence $f \in Z^{-1}[\mathcal{F}]$ or $g \in Z^{-1}[\mathcal{F}]$ and, by Proposition 3.5, the proof is now complete. □

6. On compact measurable spaces

In this section, we study the third subject. We begin with the following lemma.

Lemma 6.1. *Let $(X, \mathscr{A})$ be a compact measurable space. Suppose θ and $(Y, \mathscr{A}')$ are the same symptoms of Proposition 5.6. Then the following statements hold.*

(1) *If A is a compact element of $\mathscr{A}$, then $\theta(A)$ is a compact element of $\mathscr{A}'$.*
(2) *If $B \in \mathscr{A}'$ is a compact element of $\mathscr{A}'$, then $\theta^{-1}(B)$ is a compact element of $\mathscr{A}$.*

Proof. Let A be a compact element of $\mathscr{A}$ and $\{B_\lambda : \lambda \in \Lambda\} \subseteq \mathscr{A}'$ such that $\theta(A) \subseteq \bigcup_{\lambda \in \Lambda} B_\lambda$. From $A \subseteq \theta^{-1}(\bigcup_{\lambda \in \Lambda} B_\lambda) = \bigcup_{\lambda \in \Lambda} \theta^{-1}(B_\lambda)$, we conclude that there exists a finite subset Λ' of Λ such that $A \subseteq \bigcup_{\lambda \in \Lambda'} \theta^{-1}(B_\lambda)$. Hence $\theta(A) \subseteq \theta(\bigcup_{\lambda \in \Lambda'} \theta^{-1}(B_\lambda)) = \bigcup_{\lambda \in \Lambda'} B_\lambda$. Therefore, $\theta(A)$ is a compact element of $\mathscr{A}'$. Thus, the statement (1) holds.

Let $B \in \mathscr{A}'$ be a compact element of $\mathscr{A}'$ and $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathscr{A}$ such that $\theta^{-1}(B) \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda$. Since θ is the onto function, we infer that

$$B = \theta(\theta^{-1}(B)) \subseteq \theta(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcup_{\lambda\in\Lambda} \theta(A_\lambda),$$

which implies that there is a finite subset Λ' of Λ such that $B \subseteq \bigcup_{\lambda\in\Lambda'} \theta(A_\lambda)$, in other words, $\theta^{-1}(B) \subseteq \bigcup_{\lambda\in\Lambda'} A_\lambda$. Therefore, $\theta^{-1}(B)$ is a compact element of $\mathscr{A}$. Thus, the statement (2) holds. □

Corollary 6.2. *For every compact measurable space $(X, \mathscr{A})$ there is a compact T-measurable space $(Y, \mathscr{A}')$ such that $M(X) \cong M(Y)$, as rings.*

Proof. It is evident, by Proposition 5.6 and Lemma 6.1. □

Remark 6.3. Let $(X, \mathscr{A})$ be a compact measurable space. Consider $A \in \mathscr{A}$ and $\{B_\lambda\}_{\lambda\in\Lambda} \subseteq \mathscr{A}$ with $A \subseteq \bigcup_{\lambda\in\Lambda} B_\lambda$. From $A^c \in \mathscr{A}$ and $X = A^c \cup \bigcup_{\lambda\in\Lambda} B_\lambda$, we infer that there exists a finite subset Λ_0 of Λ such that $X = A^c \cup \bigcup_{\lambda\in\Lambda_0} B_\lambda$, which implies that $A \subseteq \bigcup_{\lambda\in\Lambda_0} B_\lambda$. Therefore, for every $A \in \mathscr{A}$, A is a compact element of $\mathscr{A}$.

Corollary 6.4. *Let $(X, \mathscr{A})$ be a measurable space and $A, B \in \mathscr{A}$. The following statements hold.*

(1) *If A or B is a compact element of $\mathscr{A}$, then $A \cap B$ is a compact element of $\mathscr{A}$.*
(2) *If A and B are compact elements of $\mathscr{A}$, then $A \cup B$ is a compact element of $\mathscr{A}$.*
(3) *A is a compact element of $\mathscr{A}$ if and only if $\uparrow A^c$ is a compact lattice.*
(4) *If A is a compact element of $\mathscr{A}$ then B is a compact element of $\mathscr{A}$, for every $B \in \downarrow A$.*

Proof. It is evident. □

Throughout this paper, we put

$$\mathcal{F}(f) := \{M \in \max(M(X)) : f \in M\},$$

for every $f \in M(X)$. Hence $\mathcal{F}(\mathbf{1}) = \emptyset$ and $\mathcal{F}(\mathbf{0}) = \max(M(X))$. Also, by Propositions 4.11 and 5.7, if $(X, \mathscr{A})$ is a compact T-measurable space, then

$$\mathcal{F}(f) = \{M_x : x \in z(f)\},$$

for every $f \in M(X)$.

Proposition 6.5. *For every compact T-measurable space $(X, \mathscr{A})$,*

$$\Big(\max(M(X)), \{\mathcal{F}(f) : f \in M(X)\}\Big)$$

is a T-measurable space.

Proof. Consider $f \in M(X)$ and $\{f_n\}_{n\in\mathbb{N}} \subseteq M(X)$. Therefore,

$$\mathcal{F}(f)^c = \{M_x : x \in coz(f)\} = \{M_x : x \in z(\chi_{z(f)})\} = \mathcal{F}(\chi_{z(f)})$$

and

$$\bigcup_{n\in\mathbb{N}} \mathcal{F}(f_n) = \bigcup_{n\in\mathbb{N}} \{M_x : x \in z(f_n)\} = \{M_x : x \in \bigcup_{n\in\mathbb{N}} z(f_n)\} = \mathcal{F}(\chi_{(\bigcap_{n\in\mathbb{N}} \mathrm{coz}(f_n))}).$$

Also, if M_1 and M_2 are distinct points in $\max(M(X))$, then there exists an element $f \in M(X)$ such that $f \in M_1 \setminus M_2$, which implies that $M_1 \in \mathcal{F}(f)$ and $M_2 \notin \mathcal{F}(f)$. Hence $\Big(\max(M(X)), \{\mathcal{F}(f) : f \in M(X)\}\Big)$ is a T-measurable space. □

Definition 6.6. Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be two measurable spaces. We say that $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ are *homeomorphic*, provided that there exists a one to one and onto function $f : X_1 \to X_2$ such that

$$A \in \mathscr{A}_1 \Leftrightarrow f(A) \in \mathscr{A}_2,$$

for every $A \in \mathscr{A}_1$.

If we denote "$(X_1, \mathscr{A}_1)$ is homeomorphic with $(X_2, \mathscr{A}_2)$" by $X \cong Y$, then the relationship $\cong$ is an equivalence relation on any set of measurable spaces.

Proposition 6.7. *For every compact T-measurable space $(X, \mathscr{A})$, $X \cong \max(M(X))$ as measurable spaces.*

Proof. We define

$$\begin{array}{rcl} \varphi : X & \longrightarrow & \max(M(X)) \\ x & \longmapsto & M_x. \end{array}$$

By Propositions 4.11 and 5.7, φ is a one-one correspondence and also, for every $f \in M(X)$, we have

$$\varphi[z(f)] = \{M_x : x \in z(f)\} = \mathcal{F}(f)$$

and

$$\varphi^{-1}(\mathcal{F}(f)) = \{x \in X : x \in z(f)\} = z(f).$$

Therefore, by Proposition 3.2, $X \cong \max(M(X))$ as measurable spaces. □

The measure defined on $\max(M(X))$ in Proposition 6.7, is called the Stone measure on $\max(M(X))$.

Corollary 6.8. *If $(X, \mathscr{A})$ and $(Y, \mathscr{A}')$ are two compact T-measurable spaces, then $X \cong Y$ as measurable spaces if and only if $M(X) \cong M(Y)$ as rings.*

Proof. By Proposition 6.7, it is obvious. □

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