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Embedding Relations of Besov Classes under MVBV

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Abstract. In this paper, we strengthen some of Leindler's results from [L. Leindler. Embedding relations of Besov classes. Acta Sci. Math. (Szeged), 73(2007) 133–149.] under MVBV condition. First, we discuss embedding relations between two Besov classes. Next, we give an equivalent estimate for the k-order modulus of continuity of $f(x)$ in L^p norm under MVBV condition. Finally, we give the condition to ensure a function $f \in L^p$ having Fourier coefficients of MVBV belongs to the Besov class.

1. Introduction

As we know, many classical results in Fourier analysis have been generalized by weakening the condition imposed on the coefficients of trigonometric series from MS (monotone non-increasing sequences) to RBVS (rest bounded variation sequences [13]), GBVS (group bounded variation sequences [7]), finally to MVBVS (mean value bounded variation sequences [32]) (see [31] for more details). At first we give some definitions.

In [10], Leindler defined a new such class.

Definition 1.1. *Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence $A := \{a_n\}(a_n \to 0)$ of real number satisfying the inequalities*

$$\sum_{i=n}^{\infty} |\Delta a_i| \le K(\mathrm{A})\gamma_n \quad (\Delta a_i := a_i - a_{i+1}), \quad n = 1, 2, \cdots \tag{1}$$

with a positive constant $K(\mathrm{A})$ is said to be a sequence of γ rest bounded variation, in symbol: $A \in \gamma$RBVS.

If $\gamma_n \equiv a_n$ and $a_n > 0$, then γRBVS $\equiv$ RBVS. RBVS was first defined by Leindler in [13]. It is easy to see that if A$\in$ RBVS, then it is also almost monotone, in symbol: A$\in$ AMS, that is for all $n \ge m$, we have

$$a_n \le K(\mathrm{A})a_m.$$

In [8] and [9] , Leindler introduced the class of mean rest bounded variation sequences, where γ is defined by a certain arithmetical mean of the coefficients, e.g., if $\gamma_n = \gamma_n^* := \frac{1}{n}\sum_{i\ge n/2}^{n} a_i$, then γRBVS $\equiv \gamma^*$MRBVS,

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if $\gamma_n = \overline{\gamma}_n := \frac{1}{n}\sum_{i=n}^{2n-1} a_i$, then γRBVS $\equiv \overline{\gamma}$MRBVS. In [19], B. Szal proved that RBVS $\subsetneq$MRBVS . In [21], B. Szal showed that $\overline{\gamma}$MRBVS $\subsetneq \gamma^*$MRBVS. In [20], B. Szal introduced the class of infinity mean rest bounded variation, briefly A $\in$ IMRBVS, if $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$ and $\gamma_n = \sum_{i=n}^{\infty} \frac{a_i}{i}$. Moreover, he showed that $\overline{\gamma}$MRBVS $\neq$ IMRBVS and γ^*MRBVS $\neq$ IMRBVS.

In [13], Leindler defined the following class:

Definition 1.2. *Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence $A := \{a_n\}(a_n \to 0)$ of real number satisfying the inequalities*

$$\sum_{i=n}^{2n-1} |\Delta a_i| \leq K(\mathrm{A})\gamma_n, \quad n = 1, 2, \cdots \tag{2}$$

with a positive constant $K(\mathrm{A})$ is said to be a sequence of γ group bounded variation, in symbol: $A \in \gamma$GBVS.

If $\gamma_n := \max_{n \leq m < n+N} a_n$ (where N is a fixed natural number), then γGBVS $\equiv$ GBVS. This class of sequence was defined and studied by Le and Zhou [7]. They proved that RBVS $\subseteq$ GBVS. In [24], GBVS was also named general monotone sequences (in symbol: GMS). Meanwhile, Tikhonov proved that MRBVS $\nsim$ GBVS. If $\gamma_n = a_n + a_{2n}$, then γGBVS $\equiv$ NBVS (non-onesided bounded variation sequences, [29]). If $\gamma_n = \frac{1}{n}\sum_{k=[n/\lambda]}^{[\lambda n]} a_k$ (where $\lambda \geq 2$), then γGBVS $\equiv$ MVBVS (mean value bounded variation sequences, [32]), MVBVS was also named $\beta-$general monotone sequences(βGMS).

In [31], Zhou proved that GBVS $\not\subseteq$ NBVS $\not\subseteq$ MVBVS. Furthermore, the MVBV condition is the weakest possible condition in uniform convergence [32], L^1-convergence [30] and a trigonometric inequality [26] case etc. for trigonometric (Fourier) series, i.e., the constant $K(\mathrm{A})$ in the inequality of the above definition cannot be replaced by any given nonnegative increasing sequence M_n tending to infinity.

MVBV condition has been cited and applied extensively to other classical results about trigonometric series [23], [27], [28] and Fourier integrals [2], [15]. In this paper, we shall obtain three theorems about Besov classes under MVBV condition. The fountainheads of these theorems were proved by [16] and [18] for monotone decreasing coefficients. Later, these theorems were generalized under IMRBV condition [20], γ^*MRBV condition [21], $\overline{\gamma}$MRBV condition [8], and RBV condition [12] respectively.

Remark 1.3. *Without loss of generality, we substitute λ with 2^ν ($\nu = [\log_2 \lambda] + 1$) and set $a_0 = a_1$ in this paper.*

The rest of the paper is organized as follows. In Section 2 we give notions and notations used in the paper. In Section 3 we give our main results. In Section 4 we introduce some lemmas to prove our results. In Section 5 we prove the main results.

2. Notions and Notations

Let $L^p_{[-\pi,\pi]}(1 \leq p \leq \infty)$ be the space of all p-power integrable real functions of period 2π with the norms

$$\|f\|_p := \begin{cases} \left(\int_{-\pi}^{\pi} |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in[-\pi,\pi]} |f(t)|, & p = \infty. \end{cases}$$

The best trigonometric approximation $E_n(f)_p$ and the modulus of smoothness $\omega_k(f;\delta)_p$ are defined as follows:

$$E_n(f)_p = \min\left(\|f - T\|_p : T \in \mathbf{T}_n\right), \ \mathbf{T}_n = \operatorname{span}\{\cos mx, \sin mx : |m| \leq n\}$$

and

$$\omega_k(f;\delta)_p = \sup_{|h|<\delta} \left\|\Delta_h^k f(x)\right\|_p$$
$$\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h f(x))\ \Delta_h f(x) = f(x+h) - f(x),$$

respectively.

A function $\alpha(t)$ is called σ-type if it is measurable on $[0,1]$, integrable on $[\delta,1]$ for every $\delta \in (0,1)$, and there exist positive constants C_1 and C_2 such that

(i) $\alpha(t) \geq C_1$ for all $t \in [0,1]$,

(ii) $\int_0^\delta \alpha(t)t^\sigma \mathrm{d}t \leq C_2\delta^\sigma \int_\delta^{2\delta} \alpha(t)\mathrm{d}t$ for all $\delta \in (0,\delta_0)$, where $0 < \delta_0 \leq \frac{1}{2}$ is given.

A positive function $\alpha(t)$ is said to satisfy the π-condition, $\pi > 0$, if there exists a positive constant C_3 such that

$$\int_{2\delta}^1 \alpha(t)t^\pi \mathrm{d}t \leq C_3\delta^\pi \int_\delta^{2\delta} \alpha(t)\mathrm{d}t, \text{ for all } \delta \in (0,\delta_0).$$

We say that $f \in B(p,\theta,\alpha)$ if

(i) $f \in L^p_{[-\pi,\pi]}$,

(ii) $0 < \theta < \infty$,

(iii) $\alpha(t)$ is σ-type,

(iv) $\int_0^1 \alpha(t)\omega_k^\theta(f;t)_p\,\mathrm{d}t < \infty$, $k \geq \frac{\sigma}{\theta}$.

Later on, we use the notation $L \ll R$ at inequalities if there exists a positive constant K such that $L \leq KR$; and if $L \ll R$ and $R \ll L$ hold simultaneously, then we shall write $L \asymp R$.

3. Main Results

We formulate our results as follows:

Theorem 3.1. *If $1 < p < q \leq \infty$, the function $\alpha(t)$ satisfies π-condition with*

$$\pi = \left(\frac{1}{p} - \frac{1}{q}\right)\theta,\ 0 < \theta < \infty,\ \alpha^*(t) := \alpha(t)t^\pi,$$

$A := \{a_n\}_{n=1}^\infty \in$ MVBVS, and f has the Fourier expansion

$$f(x) \sim \sum_{n=1}^\infty a_n \cos nx, \tag{3}$$

then the Besov classes $B(p,\theta,\alpha)$ and $B(p,\theta,\alpha^*)$ coincide. Furthermore, for any

$$k_1 \geq \frac{\sigma}{\theta}, k_2 \geq \frac{\sigma^*}{\theta}, k_3 \geq \frac{\sigma^*}{\theta}, \sigma^* = \sigma - \pi,$$

we have

$$\int_0^1 \alpha^*(t)\omega_{k_2}^\theta(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha(t)\omega_{k_1}^\theta(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha^*(t)\omega_{k_3}^\theta(f;t)_q \mathrm{d}t. \tag{4}$$

Theorem 3.2. *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, f has the Fourier expansion* (3) *with $A := \{a_n\} \in$ MVBVS, then*

$$S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n), \tag{5}$$

where

$$S(A,q,k,n) := \begin{cases} a_n, & if\ q=1, \\ n^{-k}\left(\sum\limits_{i=1}^{n} a_i^q i^{(k+1)q-2}\right)^{1/q} + \left(\sum\limits_{i=n+1}^{\infty} a_i^q i^{q-2}\right)^{1/q}, & if\ 1<q<\infty, \\ n^{-k}\sum\limits_{i=1}^{n} a_i i^k + \sum\limits_{i=n+1}^{\infty} a_i, & if\ q=\infty. \end{cases}$$

Theorem 3.3. *If* $f \in L^p_{[-\pi,\pi]}$, $1<p<\infty$, f *has the Fourier expansion* (3) *with* $A := \{a_n\} \in$ MVBVS, $\alpha(t) = t^{-r\theta-1}$ *and* $k>r$. *If* $\theta \geq 1$, *then* $f \in B(p,\theta,\alpha)$ *if and only if*

$$J_1 := \sum_{n=1}^{\infty} a_n^{\theta} n^{r\theta+\theta-\frac{\theta}{p}-1} < \infty. \tag{6}$$

If $0<\theta\leq 1$, *then a sufficient condition for* $f \in B(p,\theta,\alpha)$ *is*

$$J_2 := \sum_{n=1}^{\infty} a_n^{\theta} n^{r\theta-\theta/p} < \infty \tag{7}$$

and a necessary condition is

$$J_1 := \sum_{n=1}^{\infty} a_n^{\theta} n^{r\theta+\theta-\frac{\theta}{p}-1} < \infty. \tag{8}$$

4. Lemmas

In order to verify our theorems we need several lemmas: most of them are the analogues of the lemmas used in the proofs of the theorems with monotone coefficients or other conditions.

Lemma 4.1. ([28], pp. 218) *If* A $:= \{a_n\}_{n=1}^{\infty} \in$ MVBVS, *we have*

$$a_n \ll \sum_{i=[n/(2\lambda)]}^{[\lambda n]} \frac{a_i}{i+1} \ll \sum_{i=[2^{-\nu-1}n]}^{2^{\nu}n} \frac{a_i}{i+1} \tag{9}$$

for all $n = 1, 2, \cdots$.

Lemma 4.2. ([28], Theorem 1) *If* $1<p<\infty$, f *has the Fourier expansion* (3) *with* A $: = \{a_n\}_{n=1}^{\infty} \in$ MVBVS, *then* $x^{-\gamma} f(x) \in L^p_{[-\pi,\pi]}$, $1/p - 1 < \gamma < 1/p$, *if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2} a_n^p < \infty. \tag{10}$$

or more precisely,

$$\|x^{-\gamma} f(x)\|_p^p \asymp \sum_{n=1}^{\infty} n^{p+p\gamma-2} a_n^p. \tag{11}$$

Lemma 4.3. *Assume that f has the Fourier expansion* (3) *with* $\mathrm{A} := \{a_n\}_{n=1}^{\infty} \in$ MVBVS. *If* $1 < p < \infty$ *and* (10) *holds with* $\gamma = 0$, *then*

$$\begin{aligned} E_n(f)_p &\ll (n+1)^{1-1/p} \sum_{i=n+1}^{[2\lambda(n+1)]} |\Delta a_i| + \left(\sum_{i=n+1}^{\infty} a_i^p i^{p-2} \right)^{1/p} \\ &\ll (n+1)^{1-1/p} \sum_{i=n+1}^{2^{\nu+1}(n+1)} |\Delta a_i| + \left(\sum_{i=n+1}^{\infty} a_i^p i^{p-2} \right)^{1/p} . \end{aligned} \tag{12}$$

We omit the proof, since the proof of this lemma follows the same line as that of Theorem 3.1. in [4], where (12) for the Fourier sine expansion is proved.

Lemma 4.4. ([*11*], Corollary 1) *If* $\beta_n > 0$ *and* $a_n \geq 0$, *then*

$$\sum_{n=1}^{\infty} \beta_n \left(\sum_{i=1}^{n} a_i \right)^p \leq p^p \sum_{n=1}^{\infty} \beta_n^{1-p} a_n^p \left(\sum_{i=n}^{\infty} \beta_i \right)^p \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n \left(\sum_{i=n}^{\infty} a_i \right)^p \leq p^p \sum_{n=1}^{\infty} \beta_n^{1-p} a_n^p \left(\sum_{i=1}^{n} \beta_i \right)^p \tag{14}$$

hold for any $p \geq 1$; *while if* $0 < p < 1$, *then the inequality in* (13) *and* (14) *hold with opposite direction.*

Lemma 4.5. ([*5*], Theorem 19) *If* $a_n \geq 0$ *and* $0 < p_1 < p_2 < \infty$, *then*

$$\left(\sum_{n=1}^{\infty} a_n^{p_2} \right)^{\frac{1}{p_2}} \leq \left(\sum_{n=1}^{\infty} a_n^{p_1} \right)^{\frac{1}{p_1}} . \tag{15}$$

Lemma 4.6. ([*1*], p. 293) *If* $f \in L^{\infty}_{[-\pi,\pi]} \equiv C_{[-\pi,\pi]}$ *and* $a_n \geq 0$,

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx, x \in [-\pi, \pi],$$

then

$$\sum_{i=2n}^{\infty} a_i \leq 4E_n(f)_C.$$

Lemma 4.7. ([*6*], Theorem 5) *If* $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, *and f has the Fourier expansion* (3) *with* $a_n \geq 0$, *then for* $\eta > \frac{1}{p}$

$$\sum_{i=n}^{\infty} \frac{a_i}{i^{\eta}} \leq n^{-\eta+\frac{1}{p}} E_n(f)_p.$$

Lemma 4.8. *If* $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, *and f has the Fourier expansion* (3) *with* $\mathrm{A} :=\{a_n\} \in$ MVBVS, *then*

$$E_n(f)_p^p \gg \sum_{i=2^{\nu+1}n}^{\infty} a_i^p i^{p-2}.$$

Proof. *We want to apply Lemma 4.2 with $\gamma = 0$ to the following function:*

$$f_0(x) := f(x) - \sum_{i=1}^{2^{\nu+1}n-1} a_i \cos ix + b_n \sum_{i=1}^{2^{\nu+1}n-1} \cos ix$$

where

$$b_n := \frac{1}{n} \sum_{i=n}^{2^{\nu+1}n-1} a_i.$$

We easily know that $a_i^0 = b_n$ when $i \le 2^{\nu+1}n - 1$ and $a_i^0 = a_i$ when $i \ge 2^{\nu+1}n$. Now, we show that the $\mathrm{A}^0 :=\{a_n^0\}$ *of coefficients of f_0 belongs to* MVBVS*. That is to say that there exists $\lambda_0 = 2^{\nu+1}$ such that for all m*

$$\sum_{i=m}^{2m-1} |\Delta a_i^0| \ll \frac{1}{m} \sum_{i=[m/\lambda_0]}^{\lambda_0 m} a_i^0.$$

We consider three cases in term of m:
Case (i): If $m \ge 2^{2\nu+1}n$,

$$\begin{aligned}\sum_{i=m}^{2m-1} |\Delta a_i^0| &= \sum_{i=m}^{2m-1} |\Delta a_i| \\ &\ll \frac{1}{m} \sum_{i=[m/2^\nu]}^{m2^\nu} a_i = \frac{1}{m} \sum_{i=[m/2^\nu]}^{m2^\nu} a_i^0 \ll \frac{1}{m} \sum_{i=[m/\lambda_0]}^{\lambda_0 m} a_i^0.\end{aligned}$$

Case (ii): If $m < 2^\nu n$,

$$\sum_{i=m}^{2m-1} |\Delta a_i^0| = 0 \ll \frac{1}{m} \sum_{i=[m/2^\nu]}^{m2^\nu} a_i^0 \ll \frac{1}{m} \sum_{i=[m/\lambda_0]}^{\lambda_0 m} a_i^0.$$

Case (iii): If $2^\nu n \le m < 2^{2\nu+1}n$, by (9)*,*

$$\begin{aligned}\sum_{i=m}^{2m-1} \left|\Delta a_i^0\right| &\ll \sum_{i=m}^{2^{\nu+1}n-2} \left|\Delta a_i^0\right| + \left|\Delta a_{2^{\nu+1}n-1}^0\right| + \sum_{i=2^{\nu+1}n}^{2m-1} \left|\Delta a_i^0\right| \\ &\ll a_{2^{\nu+1}n-1}^0 + a_{2^{\nu+1}n}^0 + \sum_{i=2^{\nu+1}n}^{2m-1} |\Delta a_i| \ll b_n + a_{2^{\nu+1}n} + \frac{1}{n} \sum_{i=2n}^{2^\nu m} a_i \\ &\ll b_n + \frac{1}{n} \sum_{i=n}^{2^{2\nu+1}n} a_i + \frac{1}{n} \sum_{i=2n}^{2^\nu m} a_i \\ &\ll b_n + \frac{1}{n} \sum_{i=n}^{2^{\nu+1}n-1} a_i + \frac{1}{n} \sum_{i=2^{\nu+1}n}^{2^{2\nu+1}n} a_i + \frac{1}{n} \sum_{i=2n}^{2^{\nu+1}n-1} a_i + \frac{1}{n} \sum_{i=2^{\nu+1}n}^{2^\nu m} a_i \\ &\ll b_n + \frac{1}{n} \sum_{i=2^{\nu+1}n}^{2^{2\nu+1}n} a_i^0 + \frac{1}{n} \sum_{i=2^{\nu+1}n}^{2^\nu m} a_i^0 \ll \frac{1}{n} \sum_{i=2^\nu n}^{2^{\nu+1}n-1} a_i^0 + \frac{1}{n} \sum_{i=2^{\nu+1}n}^{2^{2\nu+1}n} a_i^0 + \frac{1}{n} \sum_{i=2^{\nu+1}n}^{2^\nu m} a_i^0 \\ &\ll \frac{1}{m} \sum_{i=[m/(2^{\nu+1})]}^{2^{\nu+1}m} a_i^0 \ll \frac{1}{m} \sum_{i=[m/\lambda_0]}^{\lambda_0 m} a_i^0.\end{aligned}$$

That means $A^0 \in MVBVS$*, we can apply Lemma 4.2 with* $\gamma = 0$ *to* f_0*, thus we obtain*

$$\left\|f - S_{2^{\nu+1}n-1}(f)\right\|_p^p + b_n^p \left\|\sum_{i=1}^{2^{\nu+1}n-1} \cos ix\right\|_p^p \gg \|f_0\|_p^p \gg \sum_{i=2^{\nu+1}n}^{\infty} a_i^p i^{p-2}.$$

Since

$$\begin{aligned}\left\|\sum_{i=1}^{2^{\nu+1}n-1} \cos ix\right\|_p^p &= 2\int_0^{\pi} \left|\sum_{i=1}^{2^{\nu+1}n-1} \cos ix\right|^p \mathrm{d}x \\ &= 2\left(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right) \left|\frac{\cos 2^{\nu} nx \sin \frac{2^{\nu+1}n-1}{2}x}{\sin \frac{x}{2}}\right|^p \mathrm{d}x \\ &\ll n^p \int_0^{\frac{\pi}{n}} \mathrm{d}x + \int_{\frac{\pi}{n}}^{\pi} \frac{1}{x^p} \mathrm{d}x \ll n^{p-1},\end{aligned}$$

by a theorem of M. Riesz ([14], Theorem 3, p. 221), we obtain

$$\sum_{i=2^{\nu+1}n}^{\infty} a_i^p i^{p-2} \ll E^p_{2^{\nu+1}n-1}(f)_p + b_n^p n^{p-1} < E_n^p(f)_p + b_n^p n^{p-1}. \tag{16}$$

Applying Lemma 4.7 with $\eta = 1$ *and* (9)*, we obtain*

$$b_n^p n^{p-1} \le n^{p-1} \left(\sum_{i=n}^{2^{\nu+1}n-1} \frac{a_i}{i}\right)^p \ll n^{p-1} \left(\sum_{i=n}^{\infty} \frac{a_i}{i}\right)^p \ll E_n^p(f)_p. \tag{17}$$

The inequalities (16) *and* (17) *imply the assertion.* □

Lemma 4.9. *If* $f \in L^p_{[-\pi,\pi]}$, $1 < p < q \le \infty$, *and* f *has the Fourier expansion* (3) *with* $A := \{a_n\}_{n=1}^{\infty} \in$ MVBVS. *If* $q < \infty$*, then*

$$S_1 := \sum_{i=2^{2\nu+1}n}^{\infty} i^{\frac{q}{p}-2} E_i^q(f)_p \ll E_n^q(f)_q;$$

while if $q = \infty$*, then*

$$S_2 := \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{\frac{1}{p}-1} E_i(f)_p \ll E_n(f)_q.$$

Proof. *By Lemma 4.3, we have*

$$\begin{aligned} S_1 &\ll \sum_{i=2^{2\nu+1}n}^{\infty} i^{\frac{q}{p}-2} (i+1)^{q-q/p} \left(\sum_{j=i+1}^{2^{\nu+1}(i+1)} \left|\Delta a_j\right|\right)^q + \sum_{i=2^{2\nu+1}n}^{\infty} i^{\frac{q}{p}-2} \left(\sum_{j=i+1}^{\infty} a_j^p j^{p-2}\right)^{q/p} \\ &:= S_{11} + S_{12}.\end{aligned}$$

Using the inequality of Lemma 4.4 and Lemma 4.8, we obtain

$$\begin{aligned} S_{11} &\ll \sum_{i=2^{2\nu+1}n}^{\infty} i^{q-2}\left(\sum_{j=i+1}^{2^{\nu+1}(i+1)}\left|\Delta a_j\right|\right)^q \ll \sum_{i=2^{2\nu+1}n}^{\infty} i^{q-2}\left(\sum_{j=[i/2^\nu]}^{2^{2\nu+1}(i+1)}\frac{a_j}{j}\right)^q \\ &\ll \sum_{i=2^{2\nu+1}n}^{\infty} i^{q-2}\left(\sum_{j=[i/2^\nu]}^{\infty}\frac{a_j}{j}\right)^q \ll \sum_{i=2^{\nu+1}n}^{\infty} i^{q-2}\left(\sum_{j=i}^{\infty}\frac{a_j}{j}\right)^q \\ &\ll \sum_{i=2^{\nu+1}n}^{\infty} a_i^q i^{q-2} \ll E_n^q(f)_q. \end{aligned}$$

Similarly,

$$S_{12} \ll \sum_{i=2^{2\nu+1}n}^{\infty} i^{\frac{q}{p}-2}\left(\sum_{j=i}^{\infty} a_j^p j^{p-2}\right)^{q/p} \ll \sum_{i=2^{2\nu+1}n}^{\infty} a_i^q i^{q-2} \ll \sum_{i=2^{\nu+1}n}^{\infty} a_i^q i^{q-2} \ll E_n^q(f)_q.$$

To estimate S_2, we apply again Lemma 4.3. Thus

$$\begin{aligned} S_2 &\ll \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{1/p-1}(i+1)^{1-1/p}\sum_{j=i+1}^{2^{\nu+1}(i+1)}\left|\Delta a_j\right| + \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{1/p-1}\left(\sum_{j=i+1}^{\infty} a_j^p j^{p-2}\right)^{1/p} \\ &:= S_{21} + S_{22}. \end{aligned}$$

First, by Lemma 4.6, we have

$$\begin{aligned} S_{21} &\ll \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} \frac{1}{i}\sum_{j=[(i+1)/2^\nu]}^{2^{2\nu+1}(i+1)} a_j \\ &\ll \sum_{j=2^{\nu+1}n}^{\infty} a_j \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n}\frac{1}{i} \\ &\ll \sum_{j=2n}^{\infty} a_j \ll E_n(f)_q. \end{aligned}$$

Applying Lemma 4.1 and Lemma 4.6, we obtain

$$\begin{aligned} S_{22} &\ll \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{1/p-1}\left(\sum_{j=i}^{\infty}\left(\sum_{l=[j/(2^{\nu+1})]}^{2^\nu j}\frac{a_l}{l}\right)^p j^{p-2}\right)^{1/p} \\ &\ll \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{1/p-1}\left(\sum_{j=i}^{\infty}\left(\sum_{l=2^\nu n}^{\infty} a_l\right)^p j^{-2}\right)^{1/p} \\ &\ll \sum_{l=2^\nu n}^{\infty} a_l \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{1/p-1}\left(\sum_{j=i}^{\infty} j^{-2}\right)^{1/p} \\ &\ll \sum_{l=2^\nu n}^{\infty} a_l \sum_{i=2^{2\nu+1}n}^{2^{2\nu+2}n} i^{-1} \ll E_n(f)_q. \end{aligned}$$

Collecting our estimates, we obtain that $S_2 \ll E_n(f)_q$, herewith the proof lemma is complete. □

Lemma 4.10. [25] *If* $f \in L^p_{[-\pi,\pi]}$, $1 < p \le 2$, *then*

$$\omega_k\left(f;\frac{1}{n}\right)_p \ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1}E_i^p(f)_p\right)^{\frac{1}{p}};$$

while if $p > 2$, *then the reverse inequality holds.*

Lemma 4.11. ([18], Theorem 1) *If* $f \in L^p_{[-\pi,\pi]}$, $1 \le p \le \infty$, *f has the Fourier expansion* (3), *and* $P_1 := \min\{2,p\}$, $P_2 := \max\{2,p\}$, *then*

$$S(A,P_1,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,P_2,k,n).$$

Lemma 4.12. ([17], pp. 847 – 848) *If* $f \in L^p_{[-\pi,\pi]}$, $1 \le p \le \infty$, $0 < \theta < \infty$, *α is a σ-type function and* $k \ge \frac{\sigma}{\theta}$, *then*

$$E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{i=1}^{\infty}\mu(i)E_{2^i}^\theta(f)_p \asymp \int_0^1 \alpha(t)\omega_k^\theta(f;t)_p\mathrm{d}t,$$

where

$$\mu(n) := \int_{2^{-n}}^{2^{-n+1}} \alpha(t)\mathrm{d}t, n \ge 1 \text{ and } \mu(0) = 1.$$

Lemma 4.13. ([21], Lemma 6) *If α is a σ-type function, then*

$$\mu(n+1) \ll \mu(n) \tag{18}$$

holds for all n.

Lemma 4.14. ([16], Theorem 1) *If* $f \in B(p,\theta,\alpha)$, $1 < p < q \le \infty$ *and α satisfies π-condition with* $\pi = \left(\frac{1}{p}-\frac{1}{q}\right)\theta$, *then* $f \in B(q,\theta,\alpha^*)$, *where*

$$\alpha^*(t) := \alpha(t)t^\pi, \text{ that is, } B(p,\theta,\alpha) \subset B(q,\theta,\alpha^*);$$

furthermore,

$$\int_0^1 \alpha^*(t)\omega_{k_2}^\theta(f;t)_q\mathrm{d}t \ll \int_0^1 \alpha(t)\omega_{k_1}^\theta(f;t)_p\mathrm{d}t$$

for any

$$k_1 \ge \frac{\sigma}{\theta}, k_2 \ge \frac{\sigma^*}{\theta} \text{ and } \sigma^* := \sigma - \left(\frac{1}{p}-\frac{1}{q}\right) + \varepsilon, \varepsilon > 0.$$

5. Proofs of the Theorems

5.1. Proof of Theorem 3.1

By Lemma 4.14 the first inequality in (4) is proved, whence

$$B(p,\theta,\alpha) \subset B(q,\theta,\alpha^*) \tag{19}$$

also holds. To prove the second inequality of (4), we apply Lemma 4.12, which yields, by $f \in B(q,\theta,\alpha^*)$, that

$$I_q := E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty}\mu^*(n)E_{2^n}^\theta(f)_q \ll \int_0^1 \alpha^*(t)\omega_{k_3}^\theta(f;t)_q\mathrm{d}t < \infty,$$

where $k_3 \geq \frac{\sigma^*}{\theta}$ and

$$\mu^*(n) := \int_{2^{-n}}^{2^{1-n}} \alpha^*(t)\mathrm{d}t, n > 1 \text{ and } \mu^*(0) = 1.$$

Since $1 < p < q$,

$$\mu(n) \ll \mu^*(n) 2^{n(1/p-1/q)\theta}.$$

Applying Lemma 4.13, we get easily that

$$\begin{aligned} I_p : &= E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{\infty} \mu(n) E_{2^n}^\theta(f)_p \\ &\ll E_0^\theta(f)_p + E_1^\theta(f)_p + \sum_{n=1}^{2\nu+2} \mu(n) E_{2^n}^\theta(f)_p + \sum_{n=1}^{\infty} \mu(n+2\nu+2) E_{2^{n+2\nu+2}}^\theta(f)_p \\ &\ll E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu(n) E_{2^{n+2\nu+2}}^\theta(f)_p \\ &\ll E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu^*(n) \left(2^{n(1/p-1/q)} E_{2^{n+2\nu+2}}(f)_p\right)^\theta \\ &\ll E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu^*(n) \left(\sum_{i=2^{n+2\mu+1}}^{2^{n+2\mu+2}} i^{(1/p-1/q)-1} E_i(f)_p\right)^\theta \end{aligned}$$

Hence, if $q = \infty$, by Lemma 4.9, we obtain

$$I_p \ll E_0^\theta(f)_q + E_1^\theta(f)_q + \sum_{n=1}^{\infty} \mu^*(n) E_{2^n}^\theta(f)_q$$

and immediately $I_p \ll I_q$. If $1 < q < \infty$, first applying Hölder's inequality, we obtain that

$$\begin{aligned} \sum_{i=2^{n+2\mu+1}}^{2^{n+2\mu+2}} i^{(1/p-1/q)-1} E_i(f)_p &= \sum_{i=2^{n+2\mu+1}}^{2^{n+2\mu+2}} i^{1/p-2/q} E_i(f)_p i^{1/q-1} \\ &\leq \left(\sum_{i=2^{n+2\mu+1}}^{2^{n+2\mu+2}} i^{q/p-2} E_i^q(f)_p\right)^{1/q} \left(\sum_{i=2^{n+2\mu+1}}^{2^{n+2\mu+2}} \left(i^{1/q-1}\right)^{q/(q-1)}\right)^{1-1/q} \\ &\ll \left(\sum_{i=2^{n+2\mu+1}}^{2^{n+2\mu+2}} i^{q/p-2} E_i^q(f)_p\right)^{1/q}, \end{aligned}$$

then by Lemma 4.9, $I_p \ll I_q$ is visible.

Finally, by Lemma 4.12, we obtain that

$$\int_0^1 \alpha(t)\omega_{k_1}^\theta(f;t)_p \mathrm{d}t \ll I_p \ll I_q \ll \int_0^1 \alpha^*(t)\omega_{k_3}^\theta(f;t)_p \mathrm{d}t < \infty$$

follows with $k_1 \geq \frac{\sigma}{\theta}$.

This proves the second inequality of (4), consequently

$$B(q,\theta,\alpha^*) \subset B(p,\theta,\alpha). \tag{20}$$

Thus, (19) and (20) completes the proof of Theorem 3.1 with $\{a_n\} \in$ MVBVS. ■

5.2. Proof of Theorem 3.2

First, we prove $S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p$. We separate two cases:

(i) If $1 < p \le 2$, by Lemma 4.11, we easily know $S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p$ holds.

(ii) If $p \ge 2$, then by Lemma 4.8, Jackson's theorem and properties of $\omega_k(f;\delta)_p$, we obtain

$$\left(\sum_{i=n+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \ll E_{n^*}(f)_p \ll \omega_k\left(f;\frac{1}{n}\right)_p, \tag{21}$$

where

$$n^* = \begin{cases} m, & \text{if } n = 2^{\nu+1}m, 2^{\nu+1}m+1, \cdots, 2^{\nu+1}m+2^{\nu+1}-2 \\ m, & \text{if } n = 2^{\nu+1}m-1. \end{cases}$$

By (9), Lemma 4.8 and Hölder's inequality, we easily obtain

$$\begin{aligned} a_i^p i^{p-1} &\ll \left(\sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} \frac{a_j}{j+1}\right)^p i^{p-1} \ll \left(\sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} a_j\right)^p i^{-1} \\ &\ll \left(\sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} j^{-1/p} a_j\right)^p \ll \sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} a_j^p \left(\sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} j^{-1/(p-1)}\right)^{p-1} \\ &\ll i^{p-2} \sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} a_j^p \ll \sum_{j=[i/(2^{\nu+1})]}^{2^\nu i} j^{p-2} a_j^p \ll E^p_{[i/(2^{2\nu+2})]}(f)_p. \end{aligned}$$

Putting this into the following sum and applying Lemma 4.10, we find the following estimates:

$$\begin{aligned} n^{-k}\left(\sum_{i=1}^{n} a_i^p i^{(k+1)p-2}\right)^{1/p} &\ll n^{-k}\left(\sum_{i=1}^{n} E^p_{[i/(2^{2\nu+2})]}(f)_p i^{kp-1}\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} E_i^p(f)_p\right)^{1/p} \ll \omega_k\left(f;\frac{1}{n}\right)_p. \end{aligned} \tag{22}$$

The inequalities (21) and (22) verify $S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p$ for $2 \le p < \infty$, thus it is proved for any $1 < p < \infty$.

Next, we prove that $\omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n)$. We separate two cases:

(i) If $2 \le p < \infty$, by Lemma 4.11, we easily know $\omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n)$ holds.

(ii) If $1 < p \leq 2$, then we use Lemma 4.5 and Lemma 4.10, thus an elementary calculation, we obtain that

$$\begin{aligned}
\omega_k\left(f;\frac{1}{n}\right)_p &\ll n^{-k}\left(\sum_{i=1}^{n} E_i^p(f)_p i^{kp-1}\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n}(i+1)^{p-1}\left(\sum_{j=i+1}^{2^{\nu+1}(i+1)}|\Delta a_i|\right)^p i^{kp-1} + \sum_{i=1}^{n} i^{kp-1}\sum_{j=i+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}\left(\sum_{j=i+1}^{2^{\nu+1}(i+1)}|\Delta a_i|\right)^p\right)^{1/p} + n^{-k}\left(\sum_{i=1}^{n} i^{kp-1}\sum_{j=i+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \\
&:= \Omega_1 + \Omega_2.
\end{aligned}$$

Applying Lemma 4.4 and Hölder's inequality, we can estimate that

$$\begin{aligned}
\Omega_1 &\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}\left(\sum_{j=[(i+1)/(2^\nu)]}^{2^{2\nu+1}(i+1)}\frac{a_j}{j+1}\right)^p\right)^{1/p} \ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-2}\left(\sum_{j=[(i+1)/(2^\nu)]}^{2^{2\nu+1}(i+1)} a_j\right)^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-3}\sum_{j=[(i+1)/(2^\nu)]}^{2^{2\nu+1}(i+1)} a_j^p\right)^{1/p} \ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-3}\sum_{j=i}^{2^{3\nu+1}(i+1)} a_j^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{2^{3\nu+1}(n+1)} i^{(k+1)p-2}a_i^p\right)^{1/p} \ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p + \sum_{i=n+1}^{2^{3\nu+1}(n+1)} i^{(k+1)p-2}a_i^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{1/p} + n^{-k}\left(\sum_{i=n+1}^{2^{3\nu+1}(n+1)} i^{(k+1)p-2}a_i^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{1/p} + \left(\sum_{i=n+1}^{2^{3\nu+1}(n+1)} i^{p-2}a_i^p\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{1/p} + \left(\sum_{i=n+1}^{\infty} i^{p-2}a_i^p\right)^{1/p}.
\end{aligned} \tag{23}$$

Now, we estimate Ω_2. Using Lemma 4.4, we can get that

$$\begin{aligned}
\Omega_2 &= n^{-k}\left(\sum_{i=1}^{n} i^{kp-1}\sum_{j=i+1}^{n} a_i^p i^{p-2} + \sum_{i=1}^{n} i^{kp-1}\sum_{j=n+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{j=2}^{n} j^{p-2}a_j^p\sum_{i=1}^{j} i^{kp-1} + n^{kp}\sum_{j=n+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \\
&\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{1/p} + \left(\sum_{i=n+1}^{\infty} i^{p-2}a_i^p\right)^{1/p}.
\end{aligned} \tag{24}$$

Utilizing the inequalities (23) and (24), we proves $\omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n)$ for $1 < p \leq 2$, and thus it is verified for any $1 < p < \infty$.

The proof of Theorem 3.2 is complete. ∎

5.3. *Proof of Theorem 3.3*

We start the proof with the following equivalence:

$$J := \int_0^1 t^{-r\theta-1}\omega_k^\theta(f;t)_p\,\mathrm{d}t \asymp \sum_{n=1}^{\infty} n^{r\theta-1}\omega_k^\theta\left(f;\frac{1}{n}\right)_p. \tag{25}$$

Applying Theorem 3.2, we can obtain that

$$\begin{aligned} J &\ll \sum_{n=1}^{\infty} n^{r\theta-1}\omega_k^\theta\left(f;\frac{1}{n}\right)_p \\ &\ll \sum_{n=1}^{\infty} n^{r\theta-1}\left(n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{1/p} + \left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{1/p}\right)^\theta \\ &\ll \sum_{n=1}^{\infty} n^{(r-k)\theta-1}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{\theta/p} + \sum_{n=1}^{\infty} n^{r\theta-1}\left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{\theta/p}. \end{aligned} \tag{26}$$

Case (i): $\theta \geq 1$

Sufficiency. We distinguish two cases listed under (A) and (B):

Case (A): $\theta/p \geq 1$, then by Lemma 4.4, we can obtain

$$\begin{aligned} J &\ll \sum_{n=1}^{\infty} n^{r\theta+\theta-\frac{\theta}{p}-1-\frac{(r-k)\theta^2}{p}} a_n^\theta \left(\sum_{i=n}^{\infty} i^{(r-k)\theta-1}\right)^{\theta/p} + \sum_{n=1}^{\infty} n^{r\theta+\theta-\frac{\theta}{p}-1-\frac{r\theta^2}{p}} a_n^\theta \left(\sum_{i=1}^{n} i^{r\theta-1}\right)^{\theta/p} \\ &\ll \sum_{n=1}^{\infty} n^{r\theta+\theta-\theta/p-1}a_n^\theta. \end{aligned} \tag{27}$$

From the above estimate we get that $J \ll J_1$ under $\theta/p \geq 1$.

Case (B): $\theta/p < 1$, by (25), we can yields that

$$J \asymp \sum_{n=0}^{\infty} 2^{nr\theta}\omega_k^\theta\left(f;\frac{1}{2^n}\right)_p \tag{28}$$

then applying again Theorem 3.2, we obtain that

$$\begin{aligned} J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\theta/p} \\ &:= J_{11} + J_{12}. \end{aligned} \tag{29}$$

Applying Lemma 4.1, Lemma 4.4, Lemma 4.5 and Hölder's inequality, we can obtain that

$$\begin{aligned}
J_{11} &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{j=0}^{n} \sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2} a_i^p \right)^{\theta/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{j=0}^{n} \sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2} \left(\sum_{t=[i/(2^{\nu+1})]}^{2^{\nu} i} \frac{a_t}{t+1} \right)^p \right)^{\theta/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{j=0}^{n} 2^{j((k+1)p-1)} \left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} \frac{a_t}{t+1} \right)^p \right)^{\theta/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \sum_{j=0}^{n} 2^{j((k+1)\theta-\theta/p)} \left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} \frac{a_t}{t+1} \right)^{\theta} \qquad (30) \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \sum_{j=0}^{n} 2^{j((k+1)\theta-\theta/p-1)} \sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^{\theta} \ll \sum_{j=0}^{\infty} 2^{j((k+1)\theta-\theta/p-1)} \sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^{\theta} \sum_{n=j}^{\infty} 2^{n(r-k)\theta} \\
&\ll \sum_{j=0}^{\infty} 2^{j((r+1)\theta-\theta/p-1)} \sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^{\theta} \ll \sum_{j=0}^{\infty} \sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} t^{r\theta+\theta-\theta/p-1} a_t^{\theta} \\
&\ll \sum_{n=1}^{\infty} n^{r\theta+\theta-\theta/p-1} a_n^{\theta}
\end{aligned}$$

and

$$\begin{aligned}
J_{12} &\ll \sum_{n=0}^{\infty} 2^{nr\theta} \left(\sum_{j=n}^{\infty} \sum_{i=2^j}^{2^{j+1}} i^{p-2} \left(\sum_{t=[i/(2^{\nu+1})]}^{2^{\nu} i} \frac{a_t}{t+1} \right)^p \right)^{\theta/p} \qquad (31) \\
&\ll \sum_{n=0}^{\infty} 2^{nr\theta} \left(\sum_{j=n}^{\infty} 2^{j(p-1)} \left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} \frac{a_t}{t+1} \right)^p \right)^{\theta/p} \\
&\ll \sum_{n=0}^{\infty} 2^{nr\theta} \sum_{j=n}^{\infty} 2^{j(\theta-\theta/p)} \left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} \frac{a_t}{t+1} \right)^{\theta} \ll \sum_{j=0}^{\infty} 2^{j(\theta-\theta/p)} \left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} \frac{a_t}{t+1} \right)^{\theta} \sum_{n=0}^{j} 2^{nr\theta} \qquad (32) \\
&\ll \sum_{j=0}^{\infty} 2^{j((r+1)\theta-\theta/p-1)} \sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^{\theta} \ll \sum_{n=1}^{\infty} n^{r\theta+\theta-\theta/p-1} a_n^{\theta}.
\end{aligned}$$

The inequalities (30) and (32) verify $J \ll J_1$ for $\theta/p \leq 1$, and consequently we complete the proof of sufficiency under $\theta \geq 1$.

Necessity. Now, we prove that $J \gg J_1$, we start again with (28) and use Theorem 3.2, thus we obtain that

$$\begin{aligned}
J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta} \left(\sum_{i=1}^{2^n} i^{(k+1)p-2} a_i^p \right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta} \left(\sum_{i=2^n+1}^{\infty} i^{p-2} a_i^p \right)^{\theta/p} \qquad (33) \\
&:= J_{21} + J_{22}.
\end{aligned}$$

Similarly, we distinguish two cases listed under (C) and (D):

Case (C): $\theta/p \geq 1$, since A∈ MVBVS, when $2^i \leq n \leq 2^{i+1}$, by Lemma (4.1), we can obtain that

$$a_n \leq |a_n - a_{2^{i+1}}| + a_{2^{i+1}} \leq \sum_{k=n}^{2^{i+1}-1} |\Delta a_k| + a_{2^{i+1}} \leq \sum_{k=2^i}^{2^{i+1}-1} |\Delta a_k| + a_{2^{i+1}}$$
$$\ll \frac{1}{2^i}\sum_{k=2^{i-\nu}}^{2^{i+\nu}} a_k + \frac{1}{2^i}\sum_{k=2^{i-\nu}}^{2^{i+\nu+1}} a_k \ll \frac{1}{2^i}\sum_{k=2^{i-\nu}}^{2^{i+\nu+1}} a_k.$$

Form this, we can deduce that

$$\left(\frac{1}{2^i}\sum_{k=2^{i-\nu}}^{2^{i+\nu+1}} a_k\right)^{\theta} \gg \frac{1}{2^i}\sum_{k=2^i}^{2^{i+1}} a_k^{\theta}.$$

Combining with Lemma 4.4, Lemma 4.5 and Hölder's inequality, we can obtain that

$$J_{21} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n-1}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}a_i^p\right)^{\theta/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n-\nu-1}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} i^{(k+1)p-2}a_i^p\right)^{\theta/p}$$
$$\gg \sum_{n=0}^{\infty} 2^{(n-\nu-1)(r-k)\theta}\left(\sum_{j=0}^{n-\nu-1}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} i^{(k+1)p-2}a_i^p\right)^{\theta/p}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} i^{(k+1)p-2}a_i^p\right)^{\theta/p}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n} 2^{j((k+1)p-1)}\frac{1}{2^j}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} a_i^p\right)^{\theta/p}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\sum_{j=0}^{n} 2^{j((k+1)\theta-\theta/p)}\left(\frac{1}{2^j}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} a_i^p\right)^{\theta/p}$$
$$\gg \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\sum_{j=0}^{n} 2^{j((k+1)\theta-\theta/p)}\left(\frac{1}{2^j}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} a_i\right)^{\theta}$$
$$\gg \sum_{j=0}^{\infty} 2^{j((k+1)\theta-\theta/p)}\left(\frac{1}{2^j}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} a_i\right)^{\theta}\sum_{n=j}^{\infty} 2^{n(r-k)\theta}$$
$$\gg \sum_{j=0}^{\infty} 2^{j(r\theta+\theta-\theta/p)}\left(\frac{1}{2^j}\sum_{i=[2^{j-\nu}]}^{2^{j+\nu+1}} a_i\right)^{\theta} \gg \sum_{j=0}^{\infty} 2^{j(r\theta+\theta-\theta/p-1)}\sum_{i=2^j}^{2^{j+1}} a_i^{\theta} \gg \sum_{n=1}^{\infty} n^{r\theta+\theta-\theta/p-1}a_n^{\theta}.$$

Similarly, we can obtain that

$$J_{22} \gg \sum_{n=1}^{\infty} n^{r\theta+\theta-\theta/p-1}a_n^{\theta}.$$

Case (D): $\theta/p < 1$, applying (25), Theorem 3.2 and Lemma 4.4, we can obtain that

$$\begin{aligned} J &\gg \sum_{n=1}^{\infty} n^{(r-k)\theta-1}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{\theta/p} + \sum_{n=1}^{\infty} n^{r\theta-1}\left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{\theta/p} \\ &\gg \sum_{n=1}^{\infty} n^{r\theta+\theta-\frac{\theta}{p}-1-\frac{(r-k)\theta^2}{p}} a_n^\theta \left(\sum_{i=n}^{\infty} i^{(r-k)\theta-1}\right)^{\theta/p} + \sum_{n=1}^{\infty} n^{r\theta+\theta-\frac{\theta}{p}-1-\frac{r\theta^2}{p}} a_n^\theta \left(\sum_{i=1}^{n} i^{r\theta-1}\right)^{\theta/p} \\ &\gg \sum_{n=1}^{\infty} n^{r\theta+\theta-\theta/p-1} a_n^\theta = J_1. \end{aligned} \tag{34}$$

The inequality (34) verify $J \gg J_1$ for $\theta/p < 1$, and consequently we complete the proof of necessity under $\theta \ge 1$.

Case (ii): $0 < \theta < 1$, in this case, we easily know that $\theta/p < 1$.

Necessity. Necessity can be proved by (34).

Sufficiency. Applying (28), Lemma 4.1, Lemma 4.4, Lemma 4.5, we can obtain that

$$\begin{aligned} J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\theta/p} \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}a_i^p\right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}a_i^p\right)^{\theta/p} \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}\left(\sum_{t=[i/(2^{\nu+1})]}^{2^\nu i} \frac{a_t}{t+1}\right)^p\right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}\left(\sum_{t=[i/(2^{\nu+1})]}^{2^\nu i} \frac{a_t}{t+1}\right)^p\right)^{\theta/p} \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{kp-2}\left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t\right)^p\right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{-2}\left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t\right)^p\right)^{\theta/p} \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\left(\sum_{j=0}^{n} 2^{j(kp-1)}\left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t\right)^p\right)^{\theta/p} + \sum_{n=0}^{\infty} 2^{nr\theta}\left(\sum_{j=n}^{\infty} 2^{-j}\left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t\right)^p\right)^{\theta/p} \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\sum_{j=0}^{n} 2^{j(k\theta-\theta/p)}\left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t\right)^\theta + \sum_{n=0}^{\infty} 2^{nr\theta}\sum_{j=n}^{\infty} 2^{-j\theta/p}\left(\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t\right)^\theta \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\theta}\sum_{j=0}^{n} 2^{j(k\theta-\theta/p)}\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^\theta + \sum_{n=0}^{\infty} 2^{nr\theta}\sum_{j=n}^{\infty} 2^{-j\theta/p}\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^\theta \\ &\ll \sum_{j=0}^{\infty} 2^{j(k\theta-\theta/p)}\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^\theta \sum_{n=j}^{\infty} 2^{n(r-k)\theta} + \sum_{j=0}^{\infty} 2^{-j\theta/p}\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^\theta \sum_{n=0}^{j} 2^{nr\theta} \\ &\ll \sum_{j=0}^{\infty} 2^{j(r\theta-\theta/p)}\sum_{t=[2^{j-\nu-1}]}^{2^{j+\nu+1}} a_t^\theta \ll \sum_{n=0}^{\infty}\sum_{i=[2^{n-\nu-1}]}^{2^{n+\nu+1}} i^{(r\theta-\theta/p)}a_i^\theta \\ &\ll \sum_{n=1}^{\infty} n^{r\theta-\theta/p}a_n^\theta \ll J_2. \end{aligned}$$

This ends our proof of Theorem 3.3. ■

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