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Dynamics of a Stochastic SIQR Epidemic Model with Saturated Incidence Rate

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Abstract. The purpose of this paper is to propose and investigate a stochastic SIQR epidemic model with saturated incidence rate. Firstly, we give some conditions to guarantee the stochastic SIQR epidemic model has a unique global positive solution. Then we verify that the disease in this model will die out exponentially if $R_0^s < 1$, while the disease will be persistent in the mean if $R_0^s > 1$. Moreover, by constructing suitable Lyapunov functions, we establish some sufficient conditions for the existence of an ergodic stationary distribution for the model. Finally, we provide some numerical simulations to illustrate the analytical results.

1. Introduction

Since Kermack and Mckendrick [1] firstly proposed an epidemic model in 1927, the study of infectious diseases via mathematical models have become important and popular. Up to now, a lot of researchers have made significant progress on both spreading and controlling of infectious diseases, such as measles, plague, influence, chickenpox, smallpox, tuberculosis, hepatitis B and so on [2–4]. One of the famous diseases model is the SIR epidemic model, which includes three compartments: the susceptible compartment S, the infected compartment I and the removed compartment R. However, for some diseases, especially most childhood diseases, quarantine to the infective individuals is a common effective control strategy. In order to describe the various disease-progression stages, an extra class, the class of quarantine to the infective individuals (denoted by Q), should be added to the system. The model is called the SIQR model, and SIQR models have been studied by some researchers [5–9].

As we all know, the incidence rate of a disease plays a key role in the study of mathematical epidemiology. In the literature, the bilinear incidence rate βSI is frequently used [10–12]. However, when the number of susceptible individuals is large, the inhibition effect due to the crowding of the infective individuals was not considered in the bilinear incidence rate. Capasso and Serio [13] introduced a saturated incidence $g(I)S = \frac{SI}{1+\alpha I}$ into the cholera epidemic model to avoid the unboundedness of the incidence rate. Saturated

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incidence may be more realistic for many cases [14–17]. For more realism and interest, in this paper, we assume that the infections are transmitted through the saturated incidence rate. Thus, the corresponding SIQR model has the following form:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S - \dfrac{\beta SI}{1+\alpha I}, \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{1+\alpha I} - (\mu+\gamma+\delta+\theta)I, \\ \dfrac{dQ}{dt} = \delta I - (\mu+\epsilon+\theta)Q, \\ \dfrac{dR}{dt} = \gamma I + \epsilon Q - \mu R, \end{cases} \tag{1}$$

where S,I,Q,R denote the numbers of susceptible, infected, quarantined and recovered individuals, respectively. All the parameters in the model are positive constants and have the following features: Λ is the constant recruitment rate of S corresponding to births and immigration; μ denotes the natural death rate; β is the transmission rate; θ denotes the disease-caused death rate of I and Q; γ and ϵ represent the recover rates from compartment I, Q to R, respectively; δ denotes the removal rate from I to Q. In system (1), the basic reproduction number is $R_0 = \frac{\beta\Lambda}{\mu(\mu+\gamma+\delta+\theta)}$. According to the theory in [18], we can obtain that the system (1) has the following properties: If $R_0 < 1$, then system (1) has an unique and globally asymptotically stable disease-free equilibrium $E_0 = (S_0, 0, 0, 0) = (\frac{\Lambda}{\mu}, 0, 0, 0)$; If $R_0 > 1$, then E_0 is unstable and system (1) has an endemic equilibrium $E^* = (S^*, I^*, Q^*, R^*)$ which is globally asymptotically stable, where

$$S^* = \frac{\alpha\Lambda + \mu + \gamma + \delta + \theta}{\mu\alpha + \beta}, \quad I^* = \frac{\mu(R_0 - 1)}{\mu\alpha + \beta}, \quad Q^* = \frac{\mu\delta(R_0 - 1)}{(\mu\alpha + \beta)(\mu + \epsilon + \theta)}$$

and

$$R^* = \frac{(\gamma(\mu + \epsilon + \theta) + \epsilon\delta)(R_0 - 1)}{(\mu\alpha + \beta)(\mu + \epsilon + \theta)}.$$

In fact, due to the uncertainty and random phenomena in the nature, epidemic models are inevitably affected by the environmental fluctuations. Therefore, it is more suitable to include stochastic perturbations in the deterministic models. Many scholars have proposed different approaches to introduce stochastic perturbations into differential equations to reveal the effects of environmental fluctuations[9, 19–25]. Nevertheless, to our knowledge, none of the aforementioned SIQR epidemic models with saturated incidence rate considers stochastic fluctuation. In this paper, we adopt the approach used in [22] to assume that the natural death rate fluctuate around some average value owing to environmental fluctuation and the intensity of stochastic perturbations for each compartments is proportional to their subpopulations respectively. Thus we propose the following stochastic SIQR epidemic model with saturated incidence rate:

$$\begin{cases} dS = \left[\Lambda - \mu S - \dfrac{\beta SI}{1+\alpha I}\right]dt + \sigma_1 S dB_1(t), \\ dI = \left[\dfrac{\beta SI}{1+\alpha I} - (\mu+\gamma+\delta+\theta)I\right]dt + \sigma_2 I dB_2(t), \\ dQ = \left[\delta I - (\mu+\epsilon+\theta)Q\right]dt + \sigma_3 Q dB_3(t), \\ dR = \left[\gamma I + \epsilon Q - \mu R\right]dt + \sigma_4 R dB_4(t), \end{cases} \tag{2}$$

where $B_i(t)$ is standard one-dimensional independent Brownian motion and $\sigma_i^2 > 0$ represents the intensity of the white noise for $i = 1, 2, 3, 4$.

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is rightly continuous and increasing while $\mathscr{F}_0$ contains all $\mathbb{P}-$null sets). If $f(t)$ is an integral function on $[0, \infty)$, we denote $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(r)dr$. The subsequent part of this

paper is as follows: In section 2, we prove the existence and uniqueness of a global positive solution in system (2). In section 3, we obtain sufficient conditions for extinction of the disease. In section 4, we also establish sufficient conditions for the persistence in the mean of the disease. In section 5, we investigate the existence of an ergodic stationary distribution, which means that the disease will prevail and can not die out in the population. In section 6, some numerical simulations are presented to illustrate the theoretical results. The paper ends with conclusions in Section 7.

2. Existence and Uniqueness of Positive Solution

In order to analyze the dynamical behavior of an epidemic model, the first concern is whether the solution is global and positive. In this section, we shall use the Lyapunov function method used in [11] to prove the existence and uniqueness of a global positive solution in system (2).

Theorem 2.1. *For any given initial value* $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, *there is a unique positive solution* $(S(t), I(t), Q(t), R(t))$ *of system (2) on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^4_+$ *with probability one, namely* $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ *for all* $t \geq 0$ *almost surely (a.s.).*

Proof. Since the coefficients of system (2) satisfy the local Lipschitz condition, we know that, for any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there is a unique local solution $(S(t), I(t), Q(t), R(t))$ on $t \in [0, \tau_e]$, where τ_e is the explosion time [26]. To show this solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficiently large such that each component of $(S(0), I(0), Q(0), R(0))$ all lies in the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the following stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I(t), Q(t), R(t)\} \leq \frac{1}{m} \text{ or } \max\{S(t), I(t), Q(t), R(t)\} \geq m\right\}.$$

Throughout this paper, we let $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_m is an increasing function as $m \to \infty$. We also let $\tau_\infty = \lim_{m\to\infty} \tau_m$. Then $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ a.s. for all $t \geq 0$. That is to say, in order to show this assertion, we only need to prove $\tau_\infty = \infty$ a.s. If the assertion is false, then there is a pair of constants $T > 0$ and $\bar{\epsilon} \in (0, 1)$ such that $\mathbb{P}\{\tau_m \leq T\} \geq \bar{\epsilon}$ for each integer $m \geq m_0$. Let us define a C^2-function from $\mathbb{R}^4_+$ to $\mathbb{R}_+$ by

$$U(S, I, Q, R) = (S - 1 - \ln S) + (I - 1 - \ln I) + (Q - 1 - \ln Q) + (R - 1 - \ln R).$$

According to the general Itô formula (see, for example, Theorem 4.2.1 of [26]), we have

$$dU(S, I, Q, R) = LU(S, I, Q, R)dt + \sigma_1(S - 1)dB_1(t) + \sigma_2(I - 1)dB_2(t) + \sigma_3(Q - 1)dB_3(t) + \sigma_4(R - 1)dB_4(t),$$

where $LU : \mathbb{R}^4_+ \to \mathbb{R}_+$ is defined by

$$\begin{aligned}
LU(S, I, Q, R) &= \left(1 - \frac{1}{S}\right)\left(\Lambda - \mu S - \frac{\beta SI}{1 + \alpha I}\right) + \left(1 - \frac{1}{I}\right)\left(\frac{\beta SI}{1 + \alpha I} - (\mu + \gamma + \delta + \theta)I\right) \\
&\quad + \left(1 - \frac{1}{Q}\right)(\delta I - (\mu + \epsilon + \theta)Q) + \left(1 - \frac{1}{R}\right)(\gamma I + \epsilon Q - \mu R) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&= \Lambda + 4\mu + \gamma + \delta + \epsilon + 2\theta - \mu(S + R) - (\mu + \theta)(I + Q) - \frac{\Lambda}{S} + \frac{\beta I}{1 + \alpha I} - \frac{\beta S}{1 + \alpha I} \\
&\quad - \frac{\delta I}{Q} - \frac{\gamma I}{R} - \frac{\epsilon Q}{R} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq \Lambda + 4\mu + \gamma + \delta + \epsilon + 2\theta + \frac{\beta}{\alpha} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} := K
\end{aligned}$$

and K is a positive constant. Thus,

$$dU(S, I, Q, R) \leq Kdt + \sigma_1(S - 1)dB_1(t) + \sigma_2(I - 1)dB_2(t) + \sigma_3(Q - 1)dB_3(t) + \sigma_4(R - 1)dB_4(t).$$

For any $m \geq m_0$, integrating the above inequality on both sides from 0 to $\tau_m \wedge T$ and taking expectation yield

$$\begin{aligned}\mathbb{E}U(S(\tau_m \wedge T), I(\tau_m \wedge T), Q(\tau_m \wedge T), R(\tau_m \wedge T)) &\leq U(S(0), I(0), Q(0), R(0)) + \mathbb{E}\int_0^{\tau_m \wedge T} K dt \\ &\leq U(S(0), I(0), Q(0), R(0)) + KT \\ &< \infty.\end{aligned}$$

Let $\Omega_m = \{\tau_m \leq T\}$. Then we have $\mathbb{P}(\Omega_m) \geq \bar{\epsilon}$. Note that, for every $\omega \in \Omega_m$, one of the components in $(S(\tau_m \wedge T), I(\tau_m \wedge T), Q(\tau_m \wedge T), R(\tau_m \wedge T))$ equals either m or $\frac{1}{m}$. Consequently,

$$\begin{aligned}U(S(0), I(0), Q(0), R(0)) + KT &\geq \mathbb{P}\{\tau_m \leq T\} \min\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\} \\ &\geq \bar{\epsilon} \min\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\}.\end{aligned}$$

Letting $m \to \infty$ leads to the contradiction

$$\infty > U(S(0), I(0), Q(0), R(0)) + KT = \infty.$$

This completes the proof.

3. Extinction

When investigating epidemic models, we are interested in the threshold value which tells us when the disease will extinct and when the disease will persist in a population. In this section, we consider sufficient conditions for the extinction of the disease in system (2).

Lemma 3.1. *For any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, the solution $(S(t), I(t), Q(t), R(t))$ of system (2) has the following properties:*

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{Q(t)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{R(t)}{t} = 0 \text{ a.s.}$$

and

$$\limsup_{t\to\infty} \frac{\ln S(t)}{t} \leq 0\,, \quad \limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq 0\,, \quad \limsup_{t\to\infty} \frac{\ln Q(t)}{t} \leq 0\,, \quad \limsup_{t\to\infty} \frac{\ln R(t)}{t} \leq 0 \text{ a.s.}$$

Furthermore, if $\mu > \left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)/2$, then

$$\lim_{t\to\infty} \frac{\int_0^t S(r)dB_1(r)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{\int_0^t I(r)dB_2(r)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{\int_0^t Q(r)dB_3(r)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{\int_0^t R(r)dB_4(r)}{t} = 0 \text{ a.s.}$$

The proof of Lemma 3.1 is similar to Lemmas 2.1 and 2.2 of [27], so we omit it here.

Theorem 3.2. *Assume $\mu > \left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)/2$. Let $(S(t), I(t), Q(t), R(t))$ be a solution of system (2) with initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$. If*

$$R_0^s := \frac{\beta\Lambda}{\mu(\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2})} < 1,$$

then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq \left(\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2}\right)(R_0^s - 1) < 0 \text{ a.s.}$$

In other words, the disease will die out exponentially with probability one.

Proof. From the first two equations of system (2), we have

$$\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}=\Lambda-\mu\langle S(t)\rangle-(\mu+\gamma+\delta+\theta)\langle I(t)\rangle+\frac{\sigma_1}{t}\int_0^t S(r)dB_1(r)+\frac{\sigma_2}{t}\int_0^t I(r)dB_2(r),$$

which implies that

$$\langle S(t)\rangle=\frac{\Lambda}{\mu}-\frac{\mu+\gamma+\delta+\theta}{\mu}\langle I(t)\rangle+\varphi(t), \tag{3}$$

where $\varphi(t)$ is defined by

$$\varphi(t)=\frac{\sigma_1}{\mu t}\int_0^t S(r)dB_1(r)+\frac{\sigma_2}{\mu t}\int_0^t I(r)dB_2(r)-\frac{S(t)-S(0)}{\mu t}-\frac{I(t)-I(0)}{\mu t}.$$

Noting Lemma 3.1, we can derive $\lim_{t\to\infty}\varphi(t)=0$ a.s. Using Itô formula to system (2) yields

$$d(\ln I)=\left[\frac{\beta S}{1+\alpha I}-(\mu+\gamma+\delta+\theta)-\frac{\sigma_2^2}{2}\right]dt+\sigma_2 dB_2(t).$$

Integrating this from 0 to t and then dividing by t on both side, we get

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t} &= \frac{\beta}{t}\int_0^t\frac{S(r)}{1+\alpha I(r)}dr-\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)+\frac{\sigma_2}{t}B_2(t)\\ &\le \beta\langle S(t)\rangle-\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)+\frac{\sigma_2}{t}B_2(t)\\ &= \frac{\beta\Lambda}{\mu}-\frac{\beta(\mu+\gamma+\delta+\theta)}{\mu}\langle I(t)\rangle+\beta\varphi(t)-\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)+\frac{\sigma_2}{t}B_2(t)\\ &\le \left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(R_0^s-1)+\beta\varphi(t)+\frac{\sigma_2}{t}B_2(t).\end{aligned}$$

By the strong law of large numbers for martingales [28], one can obtain

$$\lim_{t\to\infty}\frac{B_2(t)}{t}=0\ \text{ a.s.}$$

Taking the superior limit of both sides and note that $R_0^s<1$, we have

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(R_0^s-1)<0\ \text{a.s.}$$

This implies that $\lim_{t\to\infty}I(t)=0$ a.s. In other words, the disease will die out exponentially with probability one.

Moreover, when $\lim_{t\to\infty}I(t)=0$ a.s. , it is easy to prove that $\lim_{t\to\infty}Q(t)=0$ a.s. and $\lim_{t\to\infty}R(t)=0$ a.s. by system (2). This completes the proof.

Remark 3.3. *Comparing with the basic reproduction number of deterministic model (1), the parameter R_0^s in stochastic model (2) is less than R_0, which reveals that the extinction of the disease I in stochastic model (2) is much easier than that in the corresponding deterministic model (1). Moreover, if $\sigma_2=0$, then $R_0^s=R_0$, which means that we generalize the results of deterministic system.*

4. Persistence in the Mean

In this section, we shall establish sufficient conditions for the persistence in the mean of the disease.

Definition 4.1. *[27] System (2) is said to be persistent in the mean if* $\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(r)dr > 0$ *a.s.*

Theorem 4.2. *Assume* $\mu > \left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)/2$. *Let* $(S(t), I(t), Q(t), R(t))$ *be a solution of system (2) with initial value* $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}_+^4$. *If* $R_0^s > 1$, *then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(r)dr \geq \frac{\mu(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2})}{(\beta+\alpha\mu)(\mu+\gamma+\delta+\theta)}(R_0^s-1) > 0 \text{ a.s.}$$

Proof. By Itô formula, we can get

$$d(\ln I + \alpha I) = \left[\beta S - (\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}) - \alpha(\mu+\gamma+\delta+\theta)I\right]dt + \sigma_2(1+\alpha I)dB_2(t).$$

Integrating this equality from 0 to t and then dividing by t on both sides, it follows from (3) that

$$\begin{aligned}
&\frac{\ln I(t) - \ln I(0)}{t} + \alpha\frac{I(t)-I(0)}{t}\\
=\ & \beta\langle S(t)\rangle - (\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}) - \alpha(\mu+\gamma+\delta+\theta)\langle I(t)\rangle + \frac{\sigma_2}{t}B_2(t) + \frac{\alpha\sigma_2}{t}\int_0^t I(r)dB_2(r)\\
=\ & \frac{\beta\Lambda}{\mu} - \frac{\beta(\mu+\gamma+\delta+\theta)}{\mu}\langle I(t)\rangle + \beta\varphi(t) - (\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}) - \alpha(\mu+\gamma+\delta+\theta)\langle I(t)\rangle\\
&+\frac{\sigma_2}{t}B_2(t) + \frac{\alpha\sigma_2}{t}\int_0^t I(r)dB_2(r)\\
=\ & (\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2})(R_0^s-1) - \frac{(\mu+\gamma+\delta+\theta)(\beta+\alpha\mu)}{\mu}\langle I(t)\rangle + \beta\varphi(t) + \frac{\sigma_2}{t}B_2(t)\\
&+\frac{\alpha\sigma_2}{t}\int_0^t I(r)dB_2(r).
\end{aligned}$$

Obviously, this equality can be rewritten as follows:

$$\begin{aligned}
\frac{1}{t}\int_0^t I(r)dr \ =\ & \frac{1}{l}\Bigg[(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2})(R_0^s-1) + \beta\varphi(t) + \frac{\sigma_2}{t}B_2(t) + \frac{\alpha\sigma_2}{t}\int_0^t I(r)dB_2(r)\\
&-\frac{\ln I(t) - \ln I(0)}{t} - \alpha\frac{I(t)-I(0)}{t}\Bigg],
\end{aligned}$$

where $l = \frac{1}{\mu}(\beta+\alpha\mu)(\mu+\gamma+\delta+\theta)$. By Lemma 3.1 and Theorem 3.2, taking the limit inferior on both sides of the last equality, we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(r)dr \geq \frac{\mu(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2})}{(\beta+\alpha\mu)(\mu+\gamma+\delta+\theta)}(R_0^s-1) \text{ a.s.}$$

Consequently, due to the condition $R_0^s > 1$, the proof of Theorem 4.2 is completed.

Remark 4.3. *From Theorems 3.2 and 4.2, we can conclude that when the noise is so small such that* $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$, *then the value of* R_0^s *which is below 1 or above 1 will lead to the disease to go extinct or persist in the mean. Therefore,* R_0^s *can be considered as the threshold value of the stochastic model (2).*

5. Stationary Distribution and Ergodicity

In the previous section, we have discussed the extinction and persistence in the mean of disease under the condition $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$. In this section, we get rid of the condition $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$, and focus on the existence of an ergodic stationary distribution, which means that the disease will prevail.

Let $X(t)$ be a homogeneous Markov process in $\mathbb{R}_+^n$ described by the following stochastic equation:

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t).$$

The diffusion matrix is defined as follows:

$$A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 5.1. *[29] The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded domain $D \subset \mathbb{R}^n$ with regular boundary Γ and*

A1: there is a positive number M such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2, \quad \forall x \in D,\ \forall \xi \in \mathbb{R}^n.$$

A2: there exists a nonnegative C^2-function V such that $\mathcal{L}V$ is negative for any $\mathbb{R}^n \setminus D$, where $\mathcal{L}$ denotes the differential operator defined by

$$\mathcal{L} = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Then

$$\mathbb{P}_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{\mathbb{R}^n} f(x)\pi(dx)\right\} = 1$$

for all $x \in \mathbb{R}^n$, where $f(\cdot)$ is a function integrable with respect to the measure π.

Theorem 5.2. *Assume*

$$\widehat{R}_0^s := \frac{\beta\Lambda}{\left(\mu + \frac{\sigma_1^2}{2}\right)\left(\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2}\right)} > 1.$$

Then, for any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}_+^4$, system (2) has a unique stationary distribution $\pi(\cdot)$ and it has the ergodic property.

Proof. To prove Theorem 5.2, we only need to verify conditions $A1$ and $A2$ in Lemma 5.1 hold. First, we verify $A2$. Set

$$b := 2\left(\mu + \frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\left[\left(\frac{\beta\Lambda}{\left(\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2}\right)}\right)^{\frac{1}{2}} - \left(\mu + \frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\right].$$

Then it is easy to see that $b > 0$. Now we construct a C^2-function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$ as follows:

$$\begin{aligned} V(S, I, Q, R) &= p(-\ln S - c\ln I) + (S + I + Q + R)^{(\rho+1)} - \ln S - \ln Q - \ln R \\ &:= pV_1 + V_2 + V_3 + V_4 + V_5, \end{aligned}$$

where

$$c = \frac{\mu + \frac{\sigma_1^2}{2}}{\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2}},$$

positive constants p and ρ satisfy the following conditions

$$-pb + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \leq -2, \tag{4}$$

$$\frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right) < \mu, \tag{5}$$

and A is a constant which will be determined later. It is easy to check that

$$\liminf_{k\to\infty,(S,I,Q,R)\in\mathbb{R}_+^4\setminus U_k} V(S,I,Q,R) = +\infty,$$

where $U_k = \left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)$. In addition, $V(S,I,Q,R)$ is a continuous function. Hence, $V(S,I,Q,R)$ must have a minimum point $(\bar{S}_0,\bar{I}_0,\bar{Q}_0,\bar{R}_0)$ in the interior of $\mathbb{R}_+^4$. Therefore, we define a nonnegative C^2-function $\bar{V}$ in the following form

$$\bar{V}(S,I,Q,R) = V(S,I,Q,R) - V(\bar{S}_0,\bar{I}_0,\bar{Q}_0,\bar{R}_0).$$

Applying the general Itô formula [26], we have

$$\begin{aligned}
\mathcal{L}V_1 &= -\left(\frac{\Lambda}{S} + \frac{c\beta S}{1+\alpha I}\right) + \frac{\beta I}{1+\alpha I} + \mu + \frac{\sigma_1^2}{2} + c(\mu+\gamma+\delta+\theta) + \frac{c\sigma_2^2}{2} \\
&\leq -2\left(\frac{c\beta\Lambda}{1+\alpha I}\right)^{\frac{1}{2}} + \frac{\beta I}{1+\alpha I} + \mu + \frac{\sigma_1^2}{2} + c(\mu+\gamma+\delta+\theta) + \frac{c\sigma_2^2}{2} \\
&= -2\left[\frac{\beta\Lambda\left(\mu+\frac{\sigma_1^2}{2}\right)}{\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(1+\alpha I)}\right]^{\frac{1}{2}} + 2\left(\mu+\frac{\sigma_1^2}{2}\right) + \frac{\beta I}{1+\alpha I} \\
&= -2\left(\mu+\frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\left[\left(\frac{\beta\Lambda}{\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(1+\alpha I)}\right)^{\frac{1}{2}} - \left(\mu+\frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\right] + \frac{\beta I}{1+\alpha I}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\mathcal{L}V_2 &= (\rho+1)(S+I+Q+R)^{\rho}(\Lambda - \mu S - (\mu+\theta)I - (\mu+\theta)Q - \mu R) + \frac{1}{2}\rho(\rho+1) \\
&\quad \times(S+I+Q+R)^{\rho-1}(\sigma_1^2S^2 + \sigma_2^2I^2 + \sigma_3^2Q^2 + \sigma_4^2R^2) \\
&\leq (\rho+1)(S+I+Q+R)^{\rho}(\Lambda - \mu(S+I+Q+R)) + \frac{1}{2}\rho(\rho+1)(S+I+Q+R)^{\rho+1} \\
&\quad \times\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right) \\
&= (\rho+1)\Lambda(S+I+Q+R)^{\rho} - (\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right](S+I+Q+R)^{\rho+1} \\
&\leq A - \frac{1}{2}(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right](S+I+Q+R)^{\rho+1} \\
&\leq A - \frac{1}{2}(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right](S^{\rho+1} + I^{\rho+1} + Q^{\rho+1} + R^{\rho+1}),
\end{aligned}$$

where

$$\begin{aligned} A &= \sup_{S+I+Q+R\in(0,\infty)} \Big\{\Lambda(\rho+1)(S+I+Q+R)^{\rho} - \frac{1}{2}(\rho+1) \\ &\quad \times\Big[\mu - \frac{1}{2}\rho\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\right)\Big](S+I+Q+R)^{\rho+1}\Big\} < \infty. \end{aligned}$$

Moreover, we can obtain

$$\mathcal{L}V_3 = -\frac{\Lambda}{S} + \mu + \frac{\beta I}{1+\alpha I} + \frac{\sigma_1^2}{2}, \quad \mathcal{L}V_4 = -\frac{\delta I}{Q} + \mu + \epsilon + \theta + \frac{\sigma_3^2}{2}, \quad \mathcal{L}V_5 = -\frac{\gamma I}{R} - \frac{\epsilon Q}{R} + \mu + \frac{\sigma_4^2}{2}.$$

Hence, one has

$$\begin{aligned} \mathcal{L}\bar{V} &\le -2p\left(\mu+\frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\left[\left(\frac{\beta\Lambda}{\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(1+\alpha I)}\right)^{\frac{1}{2}} - \left(\mu+\frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\right] \\ &\quad + \frac{(p+1)\beta I}{1+\alpha I} - \frac{1}{2}(\rho+1)\Big[\mu - \frac{1}{2}\rho\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\right)\Big](S^{\rho+1}+I^{\rho+1}+Q^{\rho+1}+R^{\rho+1}) \\ &\quad - \frac{\Lambda}{S} - \frac{\delta I}{Q} - \frac{\gamma I}{R} - \frac{\epsilon Q}{R} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}. \end{aligned}$$

Consider the compact subset

$$D = \left\{\epsilon_1 \le S \le \frac{1}{\epsilon_1}, \epsilon_1 \le I \le \frac{1}{\epsilon_1}, \epsilon_2 \le Q \le \frac{1}{\epsilon_2}, \epsilon_2 \le R \le \frac{1}{\epsilon_2}\right\},$$

where ϵ_1, ϵ_2 are sufficiently small positive constants such that the following conditions hold

$$-\frac{\Lambda}{\epsilon_1} + 2p\left(\mu+\frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \le -1, \tag{6}$$

$$-p\bar{b} + (p+1)\beta\epsilon_1 + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \le -1, \tag{7}$$

$$-\frac{\delta}{\epsilon_1} + 2p\left(\mu+\frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \le -1, \tag{8}$$

$$-\frac{\gamma}{\epsilon_1} + 2p\left(\mu+\frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \le -1, \tag{9}$$

$$\begin{aligned} &-\frac{(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\right)\right]}{2\epsilon_1^{\rho+1}} + 2p\left(\mu+\frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A \\ &+ 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \le -1, \end{aligned} \tag{10}$$

$$\begin{aligned} &-\frac{(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\right)\right]}{2\epsilon_2^{\rho+1}} + 2p\left(\mu+\frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A \\ &+ 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \le -1, \end{aligned} \tag{11}$$

where

$$\bar{b} = 2\left(\mu + \frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\left[\left(\frac{\beta\Lambda}{\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(1+\alpha\epsilon_1)}\right)^{\frac{1}{2}} - \left(\mu + \frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\right].$$

With sufficiently small ϵ_1, we obtain that condition (7) holds due to (4). Then

$$\mathbb{R}_+^4 \setminus D = D_1 \cup D_2 \cup \cdots \cup D_8,$$

with

$$\begin{aligned}
D_1 = &\ \{(S,I,Q,R) \in \mathbb{R}_+^4, 0 < S < \epsilon_1\}, \quad D_2 = \{(S,I,Q,R) \in \mathbb{R}_+^4, 0 < I < \epsilon_1\},\\
D_3 = &\ \{(S,I,Q,R) \in \mathbb{R}_+^4, I \geq \epsilon_1, 0 < Q < \epsilon_2\}, \quad D_4 = \{(S,I,Q,R) \in \mathbb{R}_+^4, I \geq \epsilon_1, 0 < R < \epsilon_2\},\\
D_5 = &\ \left\{(S,I,Q,R) \in \mathbb{R}_+^4, S > \frac{1}{\epsilon_1}\right\}, \quad D_6 = \left\{(S,I,Q,R) \in \mathbb{R}_+^4, I > \frac{1}{\epsilon_1}\right\},\\
D_7 = &\ \left\{(S,I,Q,R) \in \mathbb{R}_+^4, Q > \frac{1}{\epsilon_2}\right\}, \quad D_8 = \left\{(S,I,Q,R) \in \mathbb{R}_+^4, R > \frac{1}{\epsilon_2}\right\}.
\end{aligned}$$

Next, we show the negativity of $\mathcal{L}\bar{V}$ from the following eight cases:

Case 1. If $(S,I,Q,R) \in D_1$, then (6) implies that

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq -\frac{\Lambda}{S} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}\\
&\leq -\frac{\Lambda}{\epsilon_1} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \leq -1.
\end{aligned}$$

Case 2. If $(S,I,Q,R) \in D_2$, then it from (7) that

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq -2p\left(\mu + \frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\left[\left(\frac{\beta\Lambda}{\left(\mu+\gamma+\delta+\theta+\frac{\sigma_2^2}{2}\right)(1+\alpha I)}\right)^{\frac{1}{2}} - \left(\mu + \frac{\sigma_1^2}{2}\right)^{\frac{1}{2}}\right]\\
&\quad + \frac{(p+1)\beta I}{1+\alpha I} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}\\
&\leq -p\bar{b} + (p+1)\beta\epsilon_1 + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \leq -1.
\end{aligned}$$

Case 3. If $(S,I,Q,R) \in D_3$, then

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} - \frac{\delta I}{Q} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}\\
&\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} - \frac{\delta\epsilon_1}{\epsilon_2} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}.
\end{aligned}$$

Choosing

$$\epsilon_2 = \epsilon_1^2, \tag{12}$$

we obtain

$$\mathcal{L}\bar{V} \leq -\frac{\delta}{\epsilon_1} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2} \leq -1,$$

which follows from (8).

Case 4. If $(S, I, Q, R) \in D_4$, then it follows from (9) and (12) that

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} - \frac{\gamma I}{R} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} - \frac{\gamma\epsilon_1}{\epsilon_2} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -\frac{\gamma}{\epsilon_1} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -1.
\end{aligned}$$

Case 5. If $(S, I, Q, R) \in D_5$, then (10) implies that

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} - \frac{1}{2}(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right] S^{\rho+1} \\
&\quad + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -\frac{(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right]}{2\epsilon_1^{\rho+1}} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A \\
&\quad + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -1.
\end{aligned}$$

Case 6. If $(S, I, Q, R) \in D_6$, then one has

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} - \frac{1}{2}(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right] I^{\rho+1} \\
&\quad + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -\frac{(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right]}{2\epsilon_1^{\rho+1}} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A \\
&\quad + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2}.
\end{aligned}$$

In view of (10), we have $\mathcal{L}\bar{V} \leq -1$.

Case 7. If $(S, I, Q, R) \in D_7$, according to (12) we can see that

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} - \frac{1}{2}(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right] Q^{\rho+1} \\
&\quad + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -\frac{(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right]}{2\epsilon_2^{\rho+1}} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A \\
&\quad + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2},
\end{aligned}$$

which together with (11) yields that $\mathcal{L}\bar{V} \leq -1$.

Case 8. If $(S, I, Q, R) \in D_8$, the conditions (11) and (12) lead to

$$\begin{aligned}
\mathcal{L}\bar{V} &\leq 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta I}{1+\alpha I} - \frac{1}{2}(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right] R^{\rho+1} \\
&\quad + A + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -\frac{(\rho+1)\left[\mu - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\right]}{2\epsilon_2^{\rho+1}} + 2p\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{(p+1)\beta}{\alpha} + A \\
&\quad + 3\mu + \epsilon + \theta + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\leq -1.
\end{aligned}$$

Based on the above discussions, it follows that

$$\mathcal{L}\bar{V} \leq -1, \quad \forall (S, I, Q, R) \in \mathbb{R}_+^4 \setminus D,$$

which proves that condition $A2$ holds.

To verify $A1$, note that the diffusion matrix of system (2) is

$$A((S, I, Q, R)) = \begin{pmatrix} \sigma_1^2 S^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 I^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 Q^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 R^2 \end{pmatrix}.$$

For any $(S, I, Q, R) \in D$, there is a constant $M = \min\{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 Q^2, \sigma_4^2 R^2\} > 0$ such that

$$\sum_{i,j=1}^{4} a_{ij}\xi_i\xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 + \sigma_3^2 Q^2 \xi_3^2 + \sigma_4^2 R^2 \xi_4^2 \geq M|\xi|^2$$

for any $\xi \in \mathbb{R}_+^4$. This shows that $A1$ also holds. According to Lemma 5.1, the proof is completed.

Remark 5.3. *From Theorem 5.2, we can observe that, if $\widehat{R}_0^s > 1$, then system (2) has a unique ergodic stationary distribution, which means that the disease will prevail. Notice that $\widehat{R}_0^s < R_0^s$. Meanwhile, taking attention to the expression of $\widehat{R}_0^s$, we can control the disease outbreak by environmental white noise.*

6. Numerical Simulations

In this section, the Milstein's Higher Order Method developed in [30] is used to solve system (2) numerically. According to this method, we can get the following discretization equation of system (2):

$$\begin{cases}
S_{k+1} = S_k + \left[\Lambda - \mu S_k - \dfrac{\beta S_k I_k}{1+\alpha I_k}\right]\Delta t + \sigma_1 S_k \sqrt{\Delta t}\xi_{1,k} + \dfrac{\sigma_1^2}{2} S_k \Delta t(\xi_{1,k}^2 - 1), \\
I_{k+1} = I_k + \left[\dfrac{\beta S_k I_k}{1+\alpha I_k} - (\mu + \gamma + \delta + \theta) I_k\right]\Delta t + \sigma_2 I_k \sqrt{\Delta t}\xi_{2,k} + \dfrac{\sigma_2^2}{2} I_k \Delta t(\xi_{2,k}^2 - 1), \\
Q_{k+1} = Q_k + \left[\delta I_k - (\mu + \epsilon + \theta) Q_k\right]\Delta t + \sigma_3 Q_k \sqrt{\Delta t}\xi_{3,k} + \dfrac{\sigma_3^2}{2} Q_k \Delta t(\xi_{3,k}^2 - 1), \\
R_{k+1} = R_k + \left[\gamma I_k + \epsilon Q_k - \mu R_k\right]\Delta t + + \sigma_4 R_k \sqrt{\Delta t}\xi_{4,k} + \dfrac{\sigma_4^2}{2} R_k \Delta t(\xi_{4,k}^2 - 1),
\end{cases}$$

where time increment $\Delta t > 0$, and $\xi_{1,k}, \xi_{2,k}, \xi_{3,k}, \xi_{4,k}$ are independent Gaussian random variables which follows $N(0, 1)$.

The following numerical simulations are given to show the effectiveness of the main results. As mentioned in the introduction of this paper, system (2) can be used to describe some diseases in the situation that the natural death rate fluctuate around some average value owing to environmental fluctuation and the intensity of stochastic perturbations for each compartments is proportional to their subpopulations. However, we would like to point out that the values of the parameters of system (2) and the initial values in the following numerical simulations are chosen for illustration purposes and are not taken from real life data for any diseases.

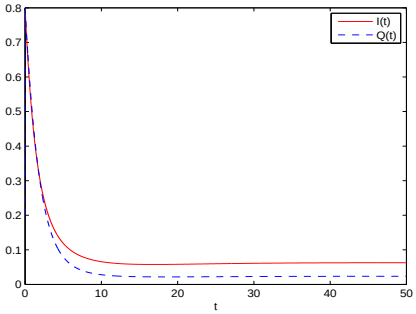

Figure 1: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$

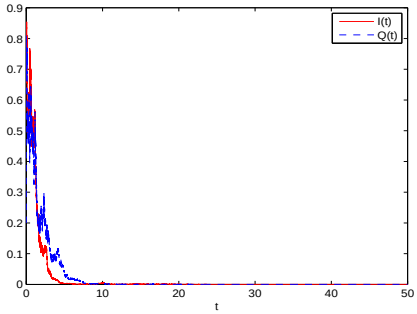

Figure 2: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.6$

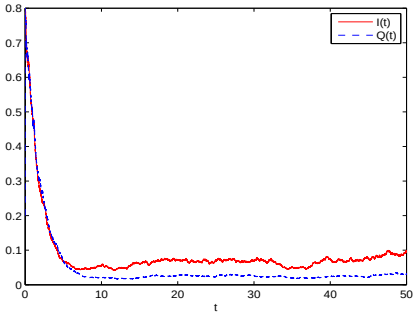

Figure 3: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$

In system (2), we choose the parameters $\Lambda = 0.6, \beta = 0.5, \alpha = 0.1, \mu = 0.3, \gamma = 0.1, \epsilon = 0.3, \theta = 0.2, \delta = 0.3$ and the initial value $S(0) = 0.7$ (see, for example, [16]), $I(0) = 0.8$ (see, for example, [16]), $Q(0) = 0.2, R(0) = 0.2$. Firstly, letting the environmental noise intensities be $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, we know that the stochastic system (2) degenerates into the deterministic system (1). Now we can calculate that $R_0 = 1.1111 > 1$ and so there is a globally asymptotically stable endemic equilibrium $S^* = 1.81, I^* = 0.06, Q^* = 0.02, R^* = 0.04$. Fig. 1 confirms this fact. Secondly, letting the environmental noise intensities be $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.6$, we obtain that $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$ and $R_0^s = 0.9259 < 1$, which satisfy the conditions of Theorem 3.2. This means that the disease will go to extinction exponentially in stochastic system (2), which is shown in Fig. 2. Comparing Fig. 1 with Fig. 2, it reveals that the extinction of the disease I in stochastic system (2) is much easier than that in the corresponding deterministic system (1). Thirdly, in Fig. 3, letting the environmental noise intensities be $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$, we get $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$ and $R_0^s = 1.1050 > 1$, and so the conditions of Theorem 4.2 hold. That is to say, the disease will be persistent in the mean. Finally, we choose the environmental noise intensities be $\sigma_1 = \sigma_2 = \sigma_3 = 0.04$ and $\sigma_4 = 0.8$. Then $\widehat{R}_0^s = 1.1072 > 1$, which means

that the conditions of Theorem 5.2 hold. Therefore, system (2) has a unique ergodic stationary distribution. Figs. 4 and 5 illustrate this fact.

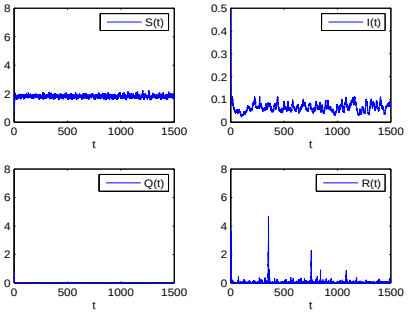

Figure 4: $\sigma_1 = \sigma_2 = \sigma_3 = 0.04, \sigma_4 = 0.8$

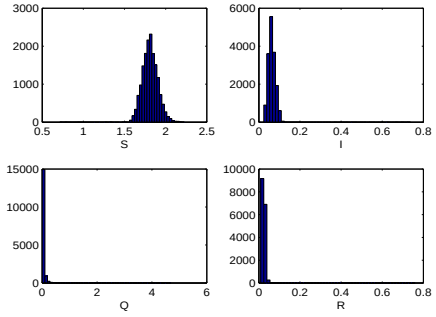

Figure 5: Relative frequency density

7. Conclusions

In this paper, we studied a stochastic SIQR epidemic model with the saturated incidence rate. We showed that the stochastic SIQR epidemic model has a unique global positive solution and derived two corresponding parameters R_0^s and $\widehat{R}_0^s$ which can be used to regulate the disease dynamics as follows:

(i) If $R_0^s < 1$ and $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$, then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq \left(\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2}\right)(R_0^s - 1) < 0 \text{ a.s.}$$

and

$$\lim_{t\to\infty} Q(t) = 0 \text{ a.s.} \quad \lim_{t\to\infty} R(t) = 0 \text{ a.s.}$$

That is to say, the disease will go to extinction exponentially.

(ii) If $R_0^s > 1$ and $\max\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) < 2\mu$, then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(r)dr \geq \frac{\mu(\mu + \gamma + \delta + \theta + \frac{\sigma_2^2}{2})}{(\beta + \alpha\mu)(\mu + \gamma + \delta + \theta)}(R_0^s - 1) > 0 \text{ a.s.}$$

This means that the disease will be persistent in the mean a.s.

(iii) If $\widehat{R}_0^s > 1$ (Note that $\widehat{R}_0^s < R_0^s$ implies that $R_0^s > 1$), then the stochastic SIQR epidemic model has a unique ergodic stationary distribution. It reveals that the disease will prevail.

From the results mentioned above, we can observe that the noise can suppress the disease outbreak. Note that the incidence rate considered in this paper is saturated. However, in some practice situations, the incidence rate may not be saturated. Therefore, it would be interesting to know what happens if system (2) is with nonlinear incidence rate of the form $f(S, I, Q, R)$? We leave this for future work.

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