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Solvability of a System of Integral Equations of Volterra Type in the Fréchet Space $L^p_{loc}(\mathbb{R}_+)$ via Measure of Noncompactness

Shahram Banaei[a]

[a]*Department of Mathematics, Bonab Branch, Islamic Azad University, Bonab, Iran.*

Abstract. The purpose of this article is to analyze the existence of solutions for a system of integral equations of Volterra type in the Fréchet space $L^p_{loc}(\mathbb{R}_+)$ and prove a fixed point theorem of Darbo-type in this space. The technique of measure of noncompactness by applying fixed point theorem is the main tool in carrying out our proof. Moreover, we present an example to show the efficiency of our results.

1. Introduction

The notion of a measure of noncompactness (MNC) was introduced by Kuratowski [12] in 1930. Darbo's fixed point theorem [10] which ensures the existence of fixed point is a significant application of this measure. Measure of noncompactness, Darbo fixed point theorem and generalizations of Darbo fixed point theorem have been successfully applied to investigate the solvability and behavior of solutions of differential equations and nonlinear integral equations (see, for example, [2, 5, 7, 9]).

Recently, many authors studied solvability of a system of integral equations in different spaces. For example: Aghajani et al. [2] generalized Darbo's theorem and applied it to study the solvability of a system of integral equations in Banach space. Allahyari et al.[4] analyzed the existence of solution for a class of systems of functional integral equations of Volterra with two variables in Banach space and Olszowy introduced a new family of measures of noncompactness in the spaces $C(\mathbb{R}_+)$ and $L^1_{loc}(\mathbb{R}_+)$, then discussed the solvability of a nonlinear functional integral equation with the initial value and differential equation of neutral type with deviated argument in [13, 14].

The aim of this work is to study the existence of solutions for a system of integral equations of Volterra type in the Fréchet space $L^p_{loc}(\mathbb{R}_+)$. The structure of this paper is as follows. In Section 2, some preliminaries, concepts and Tychonoff fixed point are recalled. Section 3 is devoted to prove a fixed point theorems of Darbo-type in the spaces $L^p_{loc}(\mathbb{R}_+)$. Finally in section 4, as an application of the results, we present an existence result for a system of nonlinear functional integral equations of Volterra type

$$x_i(t) = f_i\Big(t, x_1(t), x_2(t), \dots, x_n(t), \int_0^t k_i(t,s)x_i(s)ds\Big), \quad (1 \le i \le n) \tag{1}$$

and an example is given to illustrate our results.

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Email address: math.sh.banaei@gmail.com (Shahram Banaei)

2. Preliminaries

First, we introduce some notations and definitions which are used throughout this paper.

Let $L^p(U)$ denote the space of Lebesgue integrable functions on U ($U \subset \mathbb{R}_+$) with the standard norm

$$\|x\|_{L^p(U)} = \Big(\int_U |x(t)|^p dt \Big)^{\frac{1}{p}}.$$

We say that a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ belongs to $L^p_{loc}(\mathbb{R}_+)$ if $\chi_K f \in L^p(\mathbb{R}_+)$ for every compact set $K \subset \mathbb{R}_+$. In other word, $f \in L^p_{loc}(\mathbb{R}_+)$ if and only if $f \in L^p[0, T]$ for all $T > 0$. Let us consider the set $L^p_{loc}(\mathbb{R}_+)$ equipped with the family of seminorms $\|\chi_{[0,T]} f\|_p$ for each $T > 0$. $L^p_{loc}(\mathbb{R}_+)$ becomes a Fréchet space furnished with the distance

$$\begin{aligned} d(x, y) &= \sup\Big\{\frac{1}{2^n} \min\{1, \|\chi_{[0,n]}(x - y)\|_p\} : n \in \mathbb{N}\Big\} \\ &= \sup\Big\{\frac{1}{2^n} \min\{1, \Big(\int_0^n |x(t) - y(t)|^p dt\Big)^{\frac{1}{p}}\} : n \in \mathbb{N}\Big\}. \end{aligned}$$

A sequence (x_n) is convergent to x in $L^p_{loc}(\mathbb{R}_+)$ if and only if for each $T > 0$, (x_n) is convergent to x in $L^p_{[0,T]}(\mathbb{R}_+)$.

A nonempty subset $X \subset L^p_{loc}(\mathbb{R}_+)$ is said to be bounded if

$$\sup\Big\{\Big\|\chi_{[0,T]} f\Big\|_p = \Big(\int_0^T |f(t)|^p dt\Big)^{\frac{1}{p}} : f \in X\Big\} < \infty$$

for all $T > 0$.

The symbol $\mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)}$ stands for the family of nonempty bounded subset of $L^p_{loc}(\mathbb{R}_+)$ and $\mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$ denote its subfamily consisting of all relatively compact sets.

Definition 2.1. *[8] A family of functions $\{\mu_m\}_{m\in\mathbb{N}}$, where $\mu_m : \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)} \longrightarrow \mathbb{R}_+$, is said to be a family of measures of noncompactness in $L^p_{loc}(\mathbb{R}_+)$ if it satisfies the following conditions:*

1° *The family* $\ker\{\mu_m\} = \{X \in \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)} : \mu_m(X) = 0 \ for \ T > 0\}$ *is nonempty and* $\ker \mu_m \subseteq \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$, *for any* $m \in \mathbb{N}$.

2° $X \subset Y \Longrightarrow \mu_m(X) \leq \mu_m(Y)$.

3° $\mu_m(\overline{X}) = \mu_m(X)$ *for* $T \geq 0$.

4° $\mu_m(ConvX) = \mu_m(X)$ *for* $T \geq 0$.

5° $\mu_m(\lambda X + (1 - \lambda)Y) \leq \lambda\mu_m(X) + (1 - \lambda)\mu_m(Y)$, *for* $\lambda \in [0, 1]$ *and* $T \geq 0$.

6° *If* $\{X_n\}$ *is a sequence of closed sets from* $\mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)}$ *such that* $X_{n+1} \subset X_n$, *for* $n = 1, 2, \cdots$ *and if* $\lim_{n\to\infty} \mu_m(X_n) = 0$ *for each* $T \geq 0$ *then* $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

We say that a family of measures of noncompactness is regular [3], if it additionally satisfies the following conditions:

7° $\mu_m(X \cup Y) = max\{\mu_m(X), \mu_m(Y)\}$.

8° $\mu_m(X + Y) \leq \mu_m(X) + \mu_m(Y)$.

9° $\mu_m(\lambda X) = |\lambda|\mu_m(X)$ for $\lambda \in \mathbb{R}_+$.

10° $\ker\{\mu_m\} = \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$.

Now, we recall Tychonoff fixed point theorem that is basic for our main results.

Theorem 2.2. *([1]) Let E be a Hausdorff locally convex linear topological space, C a convex subset of E and $F : C \longrightarrow E$ a continuous mapping such that*

$$F(C) \subseteq A \subseteq C$$

with A compact. Then F has at least one fixed point.

3. A Fixed Point Theorem in $L^p_{loc}(\mathbb{R}_+)$

In this section, we recall a family of measures of noncompactness in the Fréchet space $L^p_{loc}(\mathbb{R}_+)$ and prove a Darbo-type fixed point theorem. First we characterize the compact subsets of $L^p_{loc}(\mathbb{R}_+)$.

Theorem 3.1. *([11]) Let $\mathcal{F}$ be a bounded subset in $L^p_{loc}(\mathbb{R}_+)$, $1 \le p < \infty$. Then $\mathcal{F}$ is relatively compact if and only if for every $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\Big(\int_0^T |f(t) - f(t+h)|^p dt\Big)^{\frac{1}{p}} \le \varepsilon$$

for all $f \in \mathcal{F}$ and $|h| < \delta$.

Let X be a bounded subset of the space $L^p_{loc}(\mathbb{R}_+)$, $1 \le p < \infty$ and $T > 0$. For $x \in X$, and $\varepsilon > 0$. Let us denote

$$\omega^T(x, \varepsilon) = \sup\{\Big(\int_0^T |x(t+h) - x(t)|^p dt\Big)^{\frac{1}{p}} : |h| < \varepsilon\},$$
$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\},$$

$$\mu^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon).$$

We have the following fact.

Theorem 3.2. *([6]) The family of mappings $\{\mu^T\}_{T>0}$, where $\mu^T : \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)} \longrightarrow \mathbb{R}_+$ is a family of measures of noncompactness on $L^p_{loc}(\mathbb{R}_+)$ and $\ker\{\mu^T\} = \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$.*

Now, we give a fixed point theorem for continuous operators in the Fréchet space $L^p_{loc}(\mathbb{R}_+)$.

Theorem 3.3. *Let Ω be a nonempty, closed and convex subset of a Fréchet space $L^p_{loc}(\mathbb{R}_+)$ and $\{\mu^T\}_{T>0}$ is a family of measures of noncompactness on $L^p_{loc}(\mathbb{R}_+)$. Let $F_i : \Omega^n \longrightarrow \Omega$ $(0 \le i \le n)$ be a continuous operator such that*

$$\mu^T(F(X_1, X_2, \dots, X_n) \le k_T \max_{1 \le i \le n} \mu^T(X_i), \tag{2}$$

where $X_i \in \mathfrak{M}_{L^p_{loc}}$ and $k_T \in [0, 1)$ for all $T > 0$. Then there exist $x_1, x_2, \dots, x_n \in \Omega$ such that

$$F_i(x_1^*, x_2^*, \dots, x_n*) = x_i^* \tag{3}$$

for all $i = 1, 2, \dots, n$.

Proof. Consider the operator $\widetilde{F} : \Omega^n \longrightarrow \Omega^n$ defined by

$$\widetilde{F}(x_1, x_2, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n)).$$

Also, $\widetilde{\mu^T}(X) := \max_{1\le i\le n}\{\mu_i^T(X_i),\}$ is a family of measures of noncompactness in the space Ω^n where X_i, $i = 1, 2, \dots, n$ denote the natural projections of X. Now, by induction, we define a sequence $\{\Omega_m\}$ such that $\Omega_0 = \Omega^n$ and $\Omega_m = Conv(\widetilde{F}(\Omega_{m-1})), m \ge 1$. Then we have $\widetilde{F}\Omega_0 = \widetilde{F}\Omega^n \subseteq \Omega^n = \Omega_0, \Omega_1 = Conv(\widetilde{F}\Omega_0) \subseteq \Omega^n = \Omega_0$, and by continuing this process we obtain

$$\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots .$$

If there exists an integer $N \ge 0$ such that $\widetilde{\mu^T}(\Omega_N) = 0$ for all $T > 0$, then Ω_N is relatively compact and since $\widetilde{F}\Omega_N \subseteq Conv(\widetilde{F}\Omega_N) = \Omega_{N+1} \subseteq \Omega_N$, thus Tychonoff fixed point theorem implies that $\widetilde{F}$ has a fixed point. So there exists $T_1 > 0$ such that $\widetilde{\mu^{T_1}}(\Omega_n) \ne 0$ for $n \ge 0$. By our assumptions, we get

$$\widetilde{\mu^{T_1}}(\Omega_{n+1})) = \widetilde{\mu^{T_1}}(Conv(\widetilde{F}\Omega_n)) = \widetilde{\mu^{T_1}}(\widetilde{F}\Omega_n) \le k_{T_1}\widetilde{\mu^{T_1}}(\Omega_n). \tag{4}$$

Since $k_{T_1} \in [0, 1)$, so $\widetilde{\mu^{T_1}}(\Omega_n)$ is a positive decreasing sequence of real numbers. thus, there is a $r \ge 0$ such that $\widetilde{\mu^{T_1}}(\Omega_n) \longrightarrow r$ as $n \longrightarrow \infty$. On the other hand, in view of (4) we obtain

$$\limsup_{n\longrightarrow\infty} \widetilde{\mu^{T_1}}(\Omega_{n+1})) \le \limsup_{n\longrightarrow\infty} k_{T_1}\widetilde{\mu^{T_1}}(\Omega_n).$$

This show that $r \le k_{T_1} r$. Consequently $r = 0$. Hence we deduce that $\widetilde{\mu^{T_1}}(\Omega_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since the sequence (Ω_n) is nested, in view of axiom (6°) of Definition 2.1 we derive that the set $\Omega_\infty = \bigcap_{n=1}^{\infty}\Omega_n$ is nonempty, closed and convex subset of the set Ω^n. Moreover, the set Ω_∞ is invariant under the operator $\widetilde{F}$ and belongs to $Ker\mu_T$. Now, using Tychonoff fixed point theorem implies that $\widetilde{F}$ has a fixed point in set Ω^n. □

4. Application

In this section, we present an existence result for a system of large class nonlinear functional integral equations of Volterra type in the spaces $L^p_{loc}(\mathbb{R}_+)$.

Definition 4.1. *A function $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to have the Carathéodory property if*

- (i) *For all $x \in \mathbb{R}^n$ the function $t \to f(t, x)$ is measurable on $\mathbb{R}_+$.*
- (ii) *For almost all $t \in \mathbb{R}_+$ the function $x \to f(t, x)$ is continuous on $\mathbb{R}^n$.*

Lemma 4.2. *[5] Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \le p \le \infty$. If $\{f_n\}$ is convergent to $f \in L^p(\Omega)$ in the L_p-norm, then there is a subsequence $\{f_{n_k}\}$ which converges to f a.e., and there is $g \in L_p(\Omega)$, $g \ge 0$, such that*

$$|f_{n_k}(x)| \le g(x), \qquad a.e. x \in \Omega \tag{5}$$

Theorem 4.3. *(Minkowki's Inequality for Integrals)[5]. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let f be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. If $f \ge 0$ and $1 \le p < \infty$, then*

$$\Big[\int\Big(\int f(x, y)d\nu(y)\Big)^p d\mu(x)\Big]^{\frac{1}{p}} \le \int\Big(\int f(x, y)^p d\mu(x)\Big)^{\frac{1}{p}} d\nu(y).$$

We will consider the Equation (1) under the following assumptions:

(i) $f_i : \mathbb{R}_+ \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ $(1 \le i \le n)$ satisfies the Carathéodory conditions, there exists $\lambda \in [0,1)$ and $a \in L^p_{loc}(\mathbb{R}_+)$ such that

$$|f_i(t, x_1, x_2, \ldots, x_{n+1}) - f_i(s, y_1, y_2, \ldots .y_{n+1})| \le |a(t) - a(s)| + \lambda \max_{1\le k\le n}\{|x_k - y_k|\} \quad + |x_{n+1} - y_{n+1}|, \tag{6}$$

for any $x_k, y_k \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}_+$.

(ii) $f_i(., 0, 0, \ldots, 0) \in L^p_{loc}(\mathbb{R}_+)$ $(1 \le i \le n)$.

(iii) $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ $(1 \le i \le n)$ is measurable function and there exist $g, b \in L^p_{loc}$ such that $|k(t, s)| \le g(t)$ for all $t, s \in \mathbb{R}_+$ and

$$ess \sup_{s\in[0,T]} \int_0^T |k_i(t, s)|dt \le b(T),$$

and

$$ess \sup_{t\in[0,T]} \int_0^T |k_i(t, s)|ds \le b(T).$$

for all $T > 0$ and $1 \le i \le n$.

(iv) There exists a positive increasing function r such that

$$\lambda r(T) + \max_{1\le i\le n}\{\|f_i(., 0, 0, \ldots, 0)\|_{L^p[0,T]}\} + b(T)r(T) \le r(T), \tag{7}$$

Remark 4.4. *Under the hypothesis (iii) the linear operator* $K_i : L^p[0, T] \to L^p[0, T]$, $(1 \le i \le n)$ *defined by*

$$(K_i x)(t) = \int_0^t k_i(t, s)x(s)ds \tag{8}$$

is a continuous linear operator and $\|K_i x\|_{L^p[0,T]} \le b(T)\|x\|_{L^p[0,T]}$ *for all* $T > 0$.

Theorem 4.5. *Under assumptions* (i)-(iv), *the Equation* (1) *has at least a solution in the space* $L^p_{loc}(\mathbb{R}_+)$.

Proof. In the first step, we define the operator $F_i : \left\{L^p_{loc}(\mathbb{R}_+)\right\}^n \to L^p_{loc}(\mathbb{R}_+)$, $(1 \le i \le n)$ by

$$F_i(x_1, \ldots, x_n)(t) = f_i(t, x_1(t), \ldots, x_n(t), \int_0^t k_i(t, s)x_i(s)ds).$$

Fix $i \in \{1, 2, \ldots, n\}$. In view of the Carathéodory conditions, we infer that $F_i(x_1, \ldots, x_n)$ is measurable for any $x_1, \ldots, x_n \in L^p_{loc}(\mathbb{R}_+)$.

Now, we show that $F_i(x_1, \ldots, x_n) \in L^p_{loc}(\mathbb{R}_+)$ for any $x_1, \ldots, x_n \in L^p_{loc}(\mathbb{R}_+)$. For this purpose, we only need to prove that $F_i(x_1, \ldots, x_n) \in L^p[0, T]$ for all $T > 0$. Let us fix $T > 0$. Then, applying assumptions (i)-(iv), we have

$$\begin{aligned} |F_i(x_1, \ldots, x_n)(t)| &\le |f_i(t, x_1(t), \ldots, x_n(t), \int_0^t k_i(t, s)x_i(s)ds) - f(t, 0, \ldots, 0) + f(t, 0, \ldots, 0)| \\ &\le \lambda \max_{1\le k\le n}\{|x_k(t)|\} + |f_i(t, 0, \ldots, 0)| + \left| \int_0^t k_i(t, s)x_i(s)ds \right| \end{aligned}$$

for any $x \in \mathbb{R}$ and almost all $t \in \mathbb{R}_+$. Therefore,

$$\begin{aligned}
\|F_i(x_1,\dots,x_n)\|_{L^p[0,T]} &\leq \lambda \max_{1\leq k\leq n}\{\|x_k\|_{L^p[0,T]}\} + \|f_i(.,0,\dots,0)\|_{L^p[0,T]} \\
&\quad + \Big(\int_0^T \Big|\int_0^t k_i(t,s)x_i(s)ds\Big|^p dt\Big)^{\frac{1}{p}} \\
&\leq \lambda \max_{1\leq k\leq n}\{\|x_k\|_{L^p[0,T]}\} + \|f_i(.,0,\dots,0)\|_{L^p[0,T]} \\
&\quad + \Big(\int_0^T \Big|\int_0^T \chi_{[0,t]}(s)k_i(t,s)x_i(s)ds\Big|^p dt\Big)^{\frac{1}{p}} \\
&\leq \lambda \max_{1\leq k\leq n}\{\|x_k\|_{L^p[0,T]}\} + \|f_i(.,0,\dots,0)\|_{L^p[0,T]} + b(T)\|x_i\|_{L^p[0,T]}.
\end{aligned}$$

Thus, $F_i(x_1,\dots,x_n) \in L^p_{loc}(\mathbb{R}_+)$, and F_i is well defined and if we define the subset Q of $L^p_{loc}(\mathbb{R}_+)$ by

$$Q = \{x \in L^p_{loc}(\mathbb{R}_+) : \|x\|_{L^p[0,T]} \leq r(T) \ for\ T > 0\}$$

then Q is nonempty, convex, and closed in $L^p_{loc}(\mathbb{R}_+)$. Next, observe that condition (iv) ensure that F_i transforms Q^n into Q for all $i = 1,2,\dots,n$. Now, we show that the map F is continuous. To this end, we only need to show that $F_i\ (x_1,\dots,x_n)$ is a continuous operator from $\{L^p[0,T]\}^n$ into $L^p[0,T]$ for all $T > 0$. Let $T > 0$ be fixed and $\{(x_1^m,\dots,x_n^m\}$ be an arbitrary sequence in $\{L^p[0,T]\}^n$ which converges to $(x_1,\dots,x_n) \in \{L^p[0,T]\}^n$ in the $L^p[0,T]$-norm. Since the Volterra integral operator K_i generated by k_i maps (continuously) the space $L^p[0,T]$ into itself, so Kx_n converges to Kx. By using Lemma 4.2, there is a subsequence $\{(x_1^{m_k},\dots,x_n^{m_k}\}$ which converges to $(x_1,\dots,x_n)$ a.e. ,$\{K_i x_i^{m_k}\}$ converges to $K_i x_i$ a.e. and there is $h \in L^p[0,T]$, $h \geq 0$, such that

$$\max\{|x_i^{m_k}(t)|, |K_i x_i^{m_k}(t)| : 1 \leq i \leq n\} \leq h(t). \qquad a.e.\ on\ [0,T] \tag{9}$$

Since $x_i^{m_k} \to x_i$ almost everywhere in $[0,T]$ and f satisfies the Carathéodory conditions, so

$$f_i(t, x_1^{m_k}(t),\dots,x_n^{m_k}(t), K_i x_i^{m_k}(t)) \longrightarrow f_i(t, x_1(t),\dots,x_n(t), K_i x_i(t)), \tag{10}$$

for almost all $t \in [0,T]$. From inequalities (6) and (9), we infer that

$$|f_i(t, x_1^{m_k}(t),\dots,x_n^{m_k}(t), K_i x_i^{m_k}(t))| \leq 2h(t) + |f_i(t,0,\dots,0)|, \qquad a.e.\ on\ [0,T]. \tag{11}$$

As a consequence of the Lebesgue's Dominated Convergence Theorem, (10) and (11) yield

$$\int_0^T \Big(f_i(s, x_1^{m_k}(s),\dots,x_n^{m_k}(s), K_i x_i^{m_k}(s)) - f_i(s, x_1(s),\dots,x_n(s), K_i x_i(s))\Big)^p ds \longrightarrow 0$$

and

$$\|F_i(x_1^{m_k},\dots,x_n^{m_k}) - F(x_1,\dots,x_n)\|_{L^p} \longrightarrow 0.$$

Since any sequence $\{(x_1^m,\dots,x_n^m)\}$ converging to $(x_1,\dots,x_n) \in \{L^p[0,T]\}^n$ has a subsequence $\{(x_1^{m_k},\dots,x_n^{m_k})\}$ such that $F_i(x_1^{m_k},\dots,x_n^{m_k}) \longrightarrow F_i(x_1,\dots,x_n)$ in $L^p[0,T]$, we can conclude that F_i is a continuous operator.

In order to finish the proof, Now we show that F satisfies assumptions imposed in Theorem 3.3. The proof will be divided into two steps.

Step 1: If we define $k_{i,s} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $k_{i,s}(t) := k_i(t,s)$ for all $s \in \mathbb{R}_+$, then we show that $\omega^T(\{k_{i,s} : s \in [0,T]\}) = 0$. To do this, fix arbitrary $\varepsilon > 0$. We define the function $\vartheta_i : [0,T] \longrightarrow \mathbb{R}$ as follows

$$\vartheta_i(s) = \int_0^T |k_i(t,s)|^p dt. \tag{12}$$

Since there exists $g \in L^p_{loc}(\mathbb{R}_+)$ such that $|k_i(t,s)| \le g(t)$ for all $t,s \in [0,T]$, so ϑ_i is continuous and there exists $\delta_1 > 0$ such that $|\vartheta_i(v) - \vartheta_i(w)| < \varepsilon$ for all $v,w \in [0,T]$ with $|v-w| < \delta_1$. Moreover, there exist $s_1,\dots,s_m$ such that $[0,T] \subseteq \cup_{i=1}^m B_{\delta_1}(s_i)$. Since $\{k_{i,s_1},\dots,k_{i,s_m}\}$ is a compact subset of $L^p_{loc}(\mathbb{R}_+)$, so we have $\omega^T(\{k_{i,s_1},\dots,k_{i,s_m}\}) = 0$. In the other word there exists $\delta_2 > 0$ such that

$$\int_0^T |k_{i,s_l}(t+h) - k_{i,s_l}(t)|^p dt \le \varepsilon$$

where $|h| \le \delta_2$. for every $s \in [0,T]$ and $|h| \le \delta_2$, there exist s_{l_0} such that $|s - s_{l_0}| \le \delta_1$ and

$$\begin{aligned}
\Big(\int_0^T |k_{i,s}(t) - k_{i,s}(t+h)|^p dt\Big)^{\frac{1}{p}} \le& \Big(\int_0^1 |k_{i,s}(t) - k_{i,s_{l_0}}(t)|^p dt\Big)^{\frac{1}{p}} \\
&+ \Big(\int_0^T |k_{i,s_{l_0}}(t) - k_{i,s_0}(t+h)|^p dt\Big)^{\frac{1}{p}} \\
&+ \Big(\int_0^T |k_{i,s}(t+h) - k_{i,s_{l_0}}(t+h)|^p dt\Big)^{\frac{1}{p}} \\
\le& 2|\vartheta_i(s) - \vartheta_i(s_{l_0})|^p + \varepsilon \\
\le& 2\varepsilon^p + \varepsilon.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\omega^T(k_{i,s}, \delta_2) \le 2\varepsilon^p + \varepsilon, \\
&\omega^T(\{k_{i,s} : s \in [0,T]\}, \delta_2) \le 2\varepsilon^p + \varepsilon,
\end{aligned}$$

and

$$\mu^T(\{k_{i,s} : s \in [0,T]\}) = 0.$$

Step 2: Let $X_1, X_2, \dots, X_n$ be nonempty and bounded subsets of $L^p_{loc}(\mathbb{R}_+)$, and $T > 0$. Then F_i satisfies condition 2.

Let $X_1,\dots,X_n$ be a nonempty and bounded subset of $L^p_{loc}(\mathbb{R}_+)$, and assume that $T > 0$ and $\varepsilon > 0$ are chosen arbitrarily. Let $t,h \in [0,T]$, with $|h| < \varepsilon$ and $x \in X$, we obtain

$$\begin{aligned}
|F_i(x_1,\dots,x_n)(t) - F_i(x_1,\dots,x_n)(t+h)| \le& \Big| f_i\Big(t, x_1(t),\dots,x_n(t), \int_0^t k_i(t,s)x_i(s)ds\Big) \\
&- f_i\Big(t+h, x_1(t+h),\dots,x_n(t+h) \\
&, \int_0^{t+h} k_i(t+h,s)x_i(s)ds\Big)\Big| \\
\le& |a(t) - a(t+h)| + \lambda \max_{1\le k\le n} |x_k(t) - x_k(t+h)| \\
&+ \Big|\int_0^t k_i(t,s)x_i(s)ds - \int_0^t k_i(t+h,s)x_i(s)ds\Big| \\
&+ \Big|\int_t^{t+h} k_i(t+h,s)x_i(s)ds\Big|
\end{aligned}$$

Thus,

$$
\begin{aligned}
\Big(\int_0^T |F_i(x_1,\dots,x_n)(t+h)-F_i(x_1,\dots,x_n)(t)|^p dt\Big)^{\frac{1}{p}} \leq\ & \Big(\int_0^T |a(t)-a(t+h)|^p dt\Big)^{\frac{1}{p}} \\
& +\Big(\int_0^T \lambda \max_{1\leq k\leq n} |x_k(t)-x_k(t+h)|^p dt)^{\frac{1}{p}}\Big) \\
& +\Big(\int_0^T |\int_0^t k_i(t,s)-k_i(t+h,s)||x_i(s)|ds|^p dt\Big)^{\frac{1}{p}} \\
& +\Big(\int_0^T |\int_t^{t+h} k_i(t+h,s)|x_i(s)|ds|^p dt\Big)^{\frac{1}{p}} \\
\leq\ & \omega^T(a,\varepsilon)+\lambda \max_{1\leq k\leq n}\{\omega^T(x_k,\varepsilon)\}+\int_0^T |x_i(s)| \\
& \Big(\int_0^T |k_i(t,s)-k_i(t+h,s)|^p dt\Big)^{\frac{1}{p}} ds \\
& +\Big(\int_0^T |\int_t^{t+h} |g(t)||x_i(s)|ds|^p dt\Big)^{\frac{1}{p}} \\
\leq\ & \omega^T(a,\varepsilon)+\lambda \max_{1\leq k\leq n}\{\omega^T(x,\varepsilon)\} \\
& +T\|x_i\|_{L^p[0,T]}\omega^T(\{k_{i,s}:s\in[0,T]\},\varepsilon) \\
& +h\|x_i\|_{L^p[0,T]}\|g\|_{L^p[0,T]}
\end{aligned}
$$

By using the above estimate we have

$$
\begin{aligned}
\omega^T(F(X_1\times\ldots\times X_n),\varepsilon) \leq\ & \omega^T(a,\varepsilon)+\lambda \max_{1\leq k\leq n}\{\omega^T(X_k,\varepsilon)\}+Tr(T)\omega^T(\{k_{i,s}:s\in[0,T]\},\varepsilon) \\
& +hr(T)\|g\|_{L^p[0,T]}
\end{aligned}
$$

Since the singleton $\{a\}$ is a compact set and $\mu^T(\{k_{i,s} : s\in[0,T]\}) = 0$, so we have $\omega^T(a,\varepsilon)\to 0$ and $\omega^T(\{k_{i,s}: s\in[0,T]\},\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Then we obtain

$$
\mu^T(F_i(X_1\times\ldots\times X_n))\leq \lambda \max_{1\leq k\leq n}\{\mu^T(X_k)\}, \tag{13}
$$

Obviously, F_i satisfies condition 2 and thus by Theorem 3.3, there exist $x_1^*,\dots x_n^*\in L^p_{loc}(\mathbb{R}_+)$ that are solutions of the system of integral Equation (1), and the proof is complete. □

Example 4.6. *Consider the following functional integral equation*

$$
x_i(t)=t^3+(\frac{1}{2i}\sum_{j=1}^{i}|x_j(t)|)+\int_0^t e^{-2(t+s)}x(s)ds, \quad (1\leq i\leq n). \tag{14}
$$

Eq. (14) is a special case of Eq. (1) *with*

$$
f_i(t,x_1,x_2,...,x_{n+1})=t^3+(\frac{1}{2i}\sum_{j=1}^{i}|x_j(t)|)+x_{n+1}, \quad (1\leq i\leq n),
$$

$$
k_i(t,s)=e^{-2(t+s)}.
$$

Let us put $a(t) = t^3$ and $\lambda = \frac{1}{2i}$, $(1 \le i \le n)$. We show that the assumptions of Theorem 4.5 are satisfied. Indeed, we have

$$\begin{aligned}|f_i(t, x_1, x_2, ..., x_{n+1}) - f_i(s, y_1, y_2, ..., y_{n+1})| &= |(t^3 + (\frac{1}{2i}\sum_{j=1}^{i}|x_j(t)| + x_{n+1}) \\ &\quad -(s^3 + (\frac{1}{2i}\sum_{j=1}^{i}|y_j(s)| + y_{n+1})| \\ &\le |t^3 - s^3| + (\frac{1}{2i}\sum_{j=1}^{i}|x_j(t) - y_j(s)| \\ &\quad +|x_{n+1} - y_{n+1}| \\ &\le |t^3 - s^3| + (\frac{1}{2i}\max_{1\le j\le n}\{\sum_{j=1}^{i}|x_j(t) - y_j(s)|\} \\ &\quad +|x_{n+1} - y_{n+1}|, \quad (1 \le i \le n).\end{aligned}$$

Moreover, the function f is continuous on the set $\mathbb{R}_+ \times \mathbb{R}^{n+1}$ and condition (i) and (ii) hold.
Obviously, k is measurable function and if we define $g(t) = e^{-2t}$ and $b(T) = \frac{1-e^{-T}}{2}$ we obtain

$$ess \sup_{s\in[0,T]} \int_0^T |k(t, s)|dt = ess \sup_{s\in[0,T]} \int_0^T e^{-2(t+s)}dt \le \frac{1 - e^{-T}}{2} = b(T),$$

for all $T > 0$ and condition (iii) holds. It is also easy to verify that there exists a function r satisfies the inequality in condition (iv), i.e.

$$\lambda r(T) + \max_{1\le i\le n} \|f_i(.,0,0,\ldots,0)\|_{L^p[0,T]} + b(T)r(T) = \frac{1}{2i}r(T) + \frac{T^4}{4} + \frac{1 - e^{-T}}{2}r(T) \le r(T).$$

Consequently, all the conditions of Theorem 4.5 are satisfied. This implies that the functional integral Eq. (14) has at least one solution which belongs to the space $L^p_{loc}(\mathbb{R}_+)$.

References

[1] R. Agarwal, M. Meehan and D. O'Regan, Fixed point theory and applications, Cambridge University Press, 2004.
[2] A. Aghajani, R. Allahyari and M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, Journal of Computational and Applied Mathematics 260 (2014) 68-77.
[3] R.R. Akmerov, M.I. Kamenski, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, Measures of noncompactness and condensing operators, Birkhäuser Verlag, Basel, 1992.
[4] R. Allahyari, R. Arab and A. Shole Haghighi, On the existence of solutions for a class of systems of functional integral equations of Volterra type in two variables, Journal of IJST 39 (2015), 407-415.
[5] R. Allahyari, R. Arab and A. Shole Haghighi, Measure of noncompactness in a sobolev space and integro-differential equations, Bull. Aust. Math. Soc 3 (2016), 497-506.
[6] Sh. Banaei and M. B. Ghaemi, A family of measures of noncompactness in the $L^p_{loc}(\mathbb{R}_+)$ space and its application to functional Volterra integral equation , Journal of Mathematial Analysis 8 (2017), 52-63.
[7] Sh. Banaei, M. B. Ghaemi and R. Saadati, An extension of Darbo's theorem and its application to system of neutral differential equations with deviating argument, Miskolc Mathematical Notes 18 (2017), 83-94.
[8] J. Banaś and K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Dekker, New York, 1980.
[9] J. Banaś, D. O'Regan and K. Sadarangani, On solutions of a quadratic Hammerstein integral equation on an unbounded interval, Dynam. Systems Appl 18 (2009), 251-264.
[10] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova. 24 (1955), 84-92.
[11] H. Ha-Olsen and H. Holden, The kolmogrov-riese compactness theorem, Expositiones Mathematicae 28 (2010), 385-394.
[12] K. Kuratowski, Sur les espaces complets, Fund. Math 15 (1930), 301-309.
[13] L. Olszowy, A Family of Measures of Noncompactness in the Space $L^1_{loc}(\mathbb{R}_+)$ and its Application to Some Nonlinear Volterra Integral Equation, Mediterr. J. Math 11 (2014), 687-701.
[14] L. Olszowy, On existence of solutions of a neutral differential equation with deviating argument, Collect. Math 61 (2010), 37- 47.