Filomat 32:15 (2018), 5265–5274
https://doi.org/10.2298/FIL1815265A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ulam Stability for Delay Fractional Differential Equations with a Generalized Caputo Derivative

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Abstract. The objective of this paper is to extend Ulam-Hyers stability and Ulam-Hyers-Rassias stability theory to differential equations with delay and in the frame of a certain class of a generalized Caputo fractional derivative with dependence on a kernel function. We discuss the conditions such delay generalized Caputo fractional differential equations should satisfy to be stable in the sense of Ulam-Hyers and Ulam-Hyers-Rassias. To demonstrate our results two examples are presented.

The fractional calculus extends the theory of differentiation and integration of integer order to real or complex order. Despite the fact that this calculus is as old as the classical one, scientists working on different areas have paid attention to it only in the last decades since good results were found out when the tools in this calculus were used to illuminate some models of real world phenomena [1–6]. A good peculiarity of this calculus is that there are many fractional operators. This enables researchers to choose the most viable operator and use it in order to obtain a better description of the complex phenomena in the real world. However, the complexity of the kernels existing in the fractional operators or the need of other fractional operators which can be used to model real world problems for better results, pushed the researchers working on this field to discover new fractional operators. In fact, these researches succeeded in conceiving new fractional operators. Among these new operators, we mention the Hadamrad fractional operators and the fractional operators generated by the local conformable derivatives [7–10]. It can be clearly observed that the fractionalizing process in these articles depends on iterating integrals to find the n^{th} order operator and then replacing n by any number α. That is, the "classical" fractionalizing process is utilized. Other types of fractional operators were also discovered. These operators involve nonsingular kernels. The theory of these operators depend on a limiting approach using the Dirac delta function so that when the order of a fractional derivative of a function approaches 0, the function is obtained, while when the order of the fractional derivative tends to 1, the usual derivative of the function is obtained [11–17].

The notion of Ulam stabiliy, which can be considered as a special type of data dependence was initiated by Ulam [18, 19]. Hyers, Aoki, Rassias and Obloza contributed in the development of this field (see [20–24] and the references therein). Meanwhile, there have been few works considering the Ulam stability of variety of classes of fractional differential equations [25–27]. (For more details on the works done on Ulam stability of fractional differential equations we refer to [26] and the references therein)

2010 *Mathematics Subject Classification*. 26A33, 34A12.
Keywords. Generalized Caputo fractional derivatives, Ulam-Hyers stability, Ulam-Hyers-Rassias stability
Received: 30 December 2017; Accepted: 14 November 2018
Communicated by Naseer Shahzad
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In this article, we aim to study the Ulam stability of solutions to problems of the following form:

$$ {}^{C}_{t_0}D^{\alpha}_{g}x(t) = f(t, x(t), x(t-\tau)), \quad t \in [t_0, t_1], \tag{1}$$

$$ x(t) = \phi(t), \quad t_0 - \tau \le t \le t_0, \tau > 0 \tag{2}$$

where ${}^{C}_{t_0}D^{\alpha}_{g}$ is the left Caputo fractional derivative of x of order α, $0 < \alpha \le 1$ with respect to the continuous function g such that $g'(t) > 0, t \in [t_0, t_1]$, $f \in C\Big([t_0, t_1] \times \mathbb{R}^2, \mathbb{R}\Big)$ and ${}^{C}_{t_0}D^{\alpha}_{g}x(t) \in C[t_0, t_1]$. Now we recall some definitions and tools. For $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ the left Riemann-Liouville fractional integral of order α of $x(t)$ with respect to the continuously differentiable and increasing function $g(t)$ has the following form [2, 5]

$$ {}_{t_0}I^{\alpha}_{g}x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \Big(g(t) - g(s)\Big)^{\alpha-1} x(s)g'(s)ds. \tag{3}$$

For $\alpha \in \mathbb{C}$, $Re(\alpha) \ge 0$, $-\infty \le t_0 < t_1 \le \infty$ the left Riemann-Liouville fractional derivative of order α of $x(t)$ with respect to the continuously differentiable and increasing function $g(t)$ has the form [2, 5]

$$ {}_{t_0}D^{\alpha}_{g}x(t) = \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^{n}\big({}_{t_0}I^{n-\alpha}_{g}x\big)(t) = \frac{\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^{n}}{\Gamma(n-\alpha)} \int_{t_0}^{t} \Big(g(t) - g(s)\Big)^{n-\alpha-1} x(s)g'(s)ds, \tag{4}$$

where $n = [\alpha] + 1$.

Property 0.1. *[2]*

$$ {}_{t_0}I^{\alpha}_{g}\Big(g(s) - g(t_0)\Big)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\Big(g(t) - g(t_0)\Big)^{\beta+\alpha-1}, Re(\alpha) > 0, Re(\beta) > 0. \tag{5}$$

$$ {}_{t_0}D^{\alpha}_{g}\Big(g(s) - g(t_0)\Big)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\Big(g(t) - g(t_0)\Big)^{\beta-\alpha-1}, Re(\alpha) > 0, Re(\beta) > 0. \tag{6}$$

The Caputo fractional derivative of order $\alpha, Re(\alpha) \ge 0$ of $x(t)$ with function $g(t)$ is defined by [28]

$$ \begin{aligned} {}^{C}_{t_0}D^{\alpha}_{g}x(t) &= {}_{t_0}I^{n-\alpha}_{g}\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^{n}x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \Big(g(t) - g(s)\Big)^{n-\alpha-1}\Big[\Big(\frac{1}{g'(s)}\frac{d}{ds}\Big)^{n}x(s)\Big]g'(s)ds, \end{aligned} \tag{7}$$

where $n = [\alpha] + 1$, $g \in C^{n}[t_0, t_1]$, $g'(t) > 0$ on $[t_0, t_1]$ and $x \in C^{n-1}[t_0, t_1]$.

Property 0.2. *[28]*

$$ {}^{C}_{t_0}D^{\alpha}_{g}\Big(g(s) - g(t_0)\Big)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\Big(g(t) - g(t_0)\Big)^{\beta-\alpha-1}, Re(\alpha) > 0, Re(\beta) > n. \tag{8}$$

$$ {}^{C}_{t_0}D^{\alpha}_{g}\Big(g(s) - g(t_0)\Big)^{k}(t) = 0, k = 0, 1, \dots, n-1. \tag{9}$$

Remark 0.3. *It is worth to mention that if $g(t) = t$, then* (3), (4) *and* (7) *are the classical left Riemann-Liouville fractional integral, Riemann-Liouvile fractional derivative and Caputo fractional derivative respectively [2, 5]. If $g(t) = \ln t$ then* (3), (4) *and* (7) *are the left Hadamard fractional integral, fractional derivative and Caputo-Hadamard fractional derivative, respectively [7, 29–31]. The fractional operators in* (3), (4) *and* (7) *coincide with the ones in [8, 9, 32] if $g(t) = \frac{t^{\rho}}{\rho}$ and they coincide with the ones in [10] if $g(t) = \frac{(t-a)^{\rho}}{\rho}$.*

The combinations of the fractional integrals and the fractional derivatives of a function with respect to another function are given by

Theorem 0.4. *[28]*

- ${}^{C}_{t_0}D^{\alpha}_{g}\ {}_{t_0}I^{\alpha}_{g}x(t) = x(t); x \in C[t_0, t_1],\ \ \alpha > 0.$
- ${}_{t_0}I^{\alpha}_{g}\ {}^{C}_{t_0}D^{\alpha}_{g}x(t) = x(t) - \sum_{k=0}^{n-1}\Big[\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^k x\Big](t_0)\Big(g(t) - g(t_0)\Big)^k,\ \ x \in C^n[t_0, t_1],\ \ \alpha > 0.$

The Mittag-Leffler functions which play an important role in the theory of fractional differential equations are defined as

Definition 0.5. *[2, 5]*

$$E_{\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)};\quad \alpha \in \mathbb{C},\quad Re(\alpha) > 0. \tag{10}$$

The general Mittag-Leffler function with two parameters generalizes the one in Definition 0.5 and has the form

Definition 0.6. *[2, 5]*

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)},\ \ \alpha \in \mathbb{C}, \beta \in \mathbb{C},\ \ Re(\alpha) > 0, Re(\beta) > 0. \tag{11}$$

Remark 0.7. *We have the following*

$$E_{\alpha,1}(t) = E_{\alpha}(t),$$
$$E_{1,1}(t) = E_1(t) = e^t.$$

Below, we present a Gronwall inequality [33] in the frame of the fractional integral (3) that will play a significant role in the rest of this article

Theorem 0.8. *Let u and v be two integrable functions and w be a continuous function with domain $[t_0, t_1]$. Let $g \in C^1[t_0, t_1]$ with $g' > 0, \forall t \in [t_0, t_1]$. Assume that*

- *u and v are nonnegative,*
- *w is nonneagative and nondecreasing.*

If

$$u(t) \le v(t) + w(t)\int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1}u(s)ds,$$

Then

$$u(t) \le v(t)E_{\alpha}\Big(w(t)\Gamma(\alpha)\Big(g(t) - g(t_0)\Big)^{\alpha}\Big), \forall t \in [t_0, t_1].$$

We are going to use the following definitions of Ulam stabilities of (1) are similar to the definitions stated in [34].

Definition 0.9. *Equation (1) is said to be Ulam-Hyers stable if there exists a real number c_f such that for all $\epsilon > 0$ and for each $y(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ satisfying the inequality*

$$|\,{}^{C}_{t_0}D^{\alpha}_{g}y(t) - f(t, y(t), y(t - \tau))| \le \epsilon,\ \ t \in [t_0, t_1], \tag{12}$$

there exists a solution $x(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ of (1) satisfying

$$|y(t) - x(t)| \le c_f\epsilon,\ \ t \in [t_0 - \tau, t_1]. \tag{13}$$

Definition 0.10. *Equation (1) is said to be Ulam-Hyers-Rassias stable with respect to* $\varphi_(t) \in C([t_0, t_1], \mathbb{R}_+)$ *if there exists a real number* c_f *such that for all* $\epsilon > 0$ *and for each* $y(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ *satisfying the inequality*

$$|\,{}^{C}_{t_0}D^{\alpha}_{g} y(t) - f(t, y(t), y(t-\tau))| \le \epsilon\, \varphi(t), \quad t \in [t_0 - \tau, t_1], \tag{14}$$

there exists a solution $x(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ *of (1) satisfying*

$$|y(t) - x(t)| \le c_f\, \epsilon\, \varphi(t), \quad t \in [t_0 - \tau, t_1]. \tag{15}$$

This article is organized as follows. In section 1, we discuss the existence and uniqueness of solutions to the Cauchy problem (1)-(2). In section 2, we discuss the Ulam-Hyers stability of (1). In section 3, we study the Ulam-Hyers-Rassias stability of (1). In section 4, we present examples and in section 5 we conclude our results.

1. Existence and Uniqueness Results

Before we present the existence and uniqueness theorem, we show a lemma that plays an important role in proving the theorem.

Lemma 1.1. *$x(t)$ satisfies problem (1)-(2) if and only if $x(t)$ satisfies the integral equation*

$$x(t) = \begin{cases} \phi(t) & t \in [t_0 - \tau, t_1] \\ \phi(a) + {}_{t_0}I^{\alpha}_{g} f(t, x(t), x(t-\tau)) & t \in [t_0, t_1]. \end{cases} \tag{16}$$

Proof. (i) Sufficiency:
If $t \in [t_0 - \tau, t_0]$, it is obvious that $x(t) = \phi(t)$. For $t \in [t_0, t_1]$ applying ${}^{C}_{t_0}D^{\alpha}_{g}$ to both sides of the identity in (16), we get

$${}^{C}_{t_0}D^{\alpha}_{g} x(t) = {}^{C}_{t_0}D^{\alpha}_{g} \phi(a) + {}^{C}_{t_0}D^{\alpha}_{g}\, {}_{t_0}I^{\alpha}_{g} f(t, x(t), x(t-\tau)).$$

The result is obtained then by using Property 0.2 and the first assertion of Theorem 0.4.
(ii) Necessity:
Once more it is clear that $x(t) = \phi(t)$ if $t \in [t_0 - \tau, t_0]$. For $t \in [t_0, t_1]$, applying ${}_{t_0}I^{\alpha}_{g}$ to both sides of equation (1) one gets

$${}_{t_0}I^{\alpha}_{g}\, {}^{C}_{t_0}D^{\alpha}_{g} x(t) = {}_{t_0}I^{\alpha}_{g} f(t, x(t), x(t-\tau)).$$

The result is then reached by the second assertion of Theorem 0.4. □

Now we can state the existence uniqueness theorem.

Theorem 1.2. *Let*

1. $f \in C([t_0, t_1] \times \mathbb{R}^2, \mathbb{R})$ *and* $\phi \in C[t_0 - \tau, t_0]$
2. $|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L\Big(|u_1 - v_1| + |u_2 - v_2|\Big)$
3. $\dfrac{2L(g(t_1) - g(t_0))^{\alpha}}{\Gamma(\alpha + 1)} < 1.$

Then the system (1)-(2) has a unique solution in $C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$.

Proof. Define the space

$$X = C([t_0 - \tau, t_1], \mathbb{R})$$

and the operator

$$Tx(t) = \begin{cases} \phi(t) & t \in [t_0 - \tau, t_0] \\ \phi(a) + {}_{t_0}I^{\alpha}_{g} f(t, x(t), x(t-\tau)) & t \in [t_0, t_1]. \end{cases}$$

For $t \in [t_0 - \tau, t_0]$, we have $|Tx(t) - Ty(t)| = 0$ if $x, y \in C([t_0 - \tau, t_1], \mathbb{R})$. Now for $t \in [t_0, t_1]$, we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &= {}_{t_0}I_g^\alpha f(t, x(t), x(t-\tau)) - {}_{t_0}I_g^\alpha f(t, y(t), y(t-\tau)) \\
&\leq {}_{t_0}I_g^\alpha\Big(|f(t, x(t), x(t-\tau)) - f(t, y(t), y(t-\tau))|\Big) \\
&\leq {}_{t_0}I_g^\alpha\Big(L|x(t) - y(t)| + L|x(t-\tau) - y(t-\tau)|\Big) \\
&\leq L\Big(\max_{t_0-\tau\leq t\leq t_1} |x(t) - y(t)| + \max_{t_0-\tau\leq t\leq t_1} |x(t-\tau) - y(t-\tau)|\Big)\, {}_{t_0}I_g^\alpha 1 \\
&\leq \frac{2L\Big(g(t) - g(t_0)\Big)^\alpha}{\Gamma(\alpha+1)} \|x(t) - y(t)\|_{C[t_0-\tau,t_1]} \\
&\leq \frac{2L\Big(g(t_1) - g(t_0)\Big)^\alpha}{\Gamma(\alpha+1)} \|x(t) - y(t)\|_{C[t_0-\tau,t_1]} \\
&\leq \|x(t) - y(t)\|_{C[t_0-\tau,t_1]}.
\end{aligned}$$

Therefore, $\|Tx - Ty\| \leq \|x - y\|$ and since $\dfrac{2L(g(t_1) - g(t_0))^\alpha}{\Gamma(\alpha+1)} < 1$, the operator T is a contraction and thus it has a unique fixed point by Banach fixed point theorem. □

2. Ulam-Hyers Stability

Before we state the Ulam-Hyers stability of equation (1), let us state the following lemma

Lemma 2.1. *If a function $y(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ is a solution of the inequality (12), then $y(t)$ satisfies*

$$|y(t) - y(t_0) - {}_{t_0}I_g^\alpha f(t, y(t), y(t-\tau))| \leq \frac{\Big(g(t_1) - g(t_0)\Big)^\alpha \epsilon}{\Gamma(\alpha+1)}. \tag{17}$$

Proof. It is clear that $y(t)$ satisfies (12) if and only if there exists a function $h(t)$ such that $|h(t)| \leq \epsilon$ and

$${}_{t_0}^{C}D_g^\alpha y(t) - f(t, y(t), y(t-\tau)) = h(t), \quad t \in [t_0, t_1]. \tag{18}$$

Applying the fractional integral (3) to both sides of (18) and using Theorem 0.4 we get

$$y(t) - y(t_0) - {}_{t_0}I_g^\alpha f(t, y(t), y(t-\tau)) = {}_{t_0}I_g^\alpha h(t).$$

Thus, we have

$$\begin{aligned}
|y(t) - y(t_0) - {}_{t_0}I_g^\alpha f(t, y(t), y(t-\tau)| &\leq {}_{t_0}I_g^\alpha |h(t)| \\
&\leq {}_{t_0}I_g^\alpha \epsilon \\
&= \frac{\Big(g(t) - g(t_0)\Big)^\alpha \epsilon}{\Gamma(\alpha+1)} \\
&\leq \frac{\Big(g(t_1) - g(t_0)\Big)^\alpha \epsilon}{\Gamma(\alpha+1)}.
\end{aligned}$$

□

Theorem 2.2. *Under the hypotheses of Theorem 1.2, equation (1) is Ulam-Hyers stable.*

Proof. Let $y(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ be a solution of the inequality (12) and let $x(t)$ be the unique solution of equation (1) satisfying the condition $x(t) = y(t)$ for $t \in [t_0 - \tau, t_0]$. The we have

$$x(t) = \begin{cases} y(t) & t \in [t_0 - \tau, t_0] \\ y(t_0) + {}_{t_0}I_g^\alpha f(t, x(t), x(t-\tau)) & t \in [t_0, t_1]. \end{cases},$$

which is guaranteed by Theorem 1.2. For $t \in [t_0 - \tau, t_0]$ we have $|y(t) - x(t)| = 0$.
For $t \in [t_0, t_0 + \alpha]$, we have

$$\begin{aligned}
|y(t) - x(t)| &= |y(t) - y(t_0) - {}_{t_0}I_g^{\alpha} f(t, x(t), x(t-\tau))| \\
&= |y(t) - y(t_0) - {}_{t_0}I_g^{\alpha} f(t, y(t), y(t-\tau)) \\
&+ {}_{t_0}I_g^{\alpha} f(t, y(t), y(t-\tau)) - {}_{t_0}I_g^{\alpha} f(t, x(t), x(t-\tau))| \\
&\leq |y(t) - y(t_0) - {}_{t_0}I_g^{\alpha} f(t, y(t), y(t-\tau))| \\
&+ |{}_{t_0}I_g^{\alpha} f(t, y(t), y(t-\tau)) - {}_{t_0}I_g^{\alpha} f(t, x(t), x(t-\tau))| \\
&\leq |y(t) - y(t_0) - {}_{t_0}I_g^{\alpha} f(t, y(t), y(t-\tau))| \\
&+ |{}_{t_0}I_g^{\alpha}\Big(f(t, y(t), y(t-\tau)) - f(t, x(t), x(t-\tau))\Big)| \\
&\leq |y(t) - y(t_0) - {}_{t_0}I_g^{\alpha} f(t, y(t), y(t-\tau))| + L\, {}_{t_0}I_g^{\alpha}\Big(|y(t) - x(t)|\Big),
\end{aligned}$$

where the last inequality holds since $x(t-\tau) - y(t-\tau) = 0,$ for $t \in [t_0, t_0 + \alpha]$. Now using Lemma 2.1 and the definition of the fractional integral (3), we get

$$|y(t) - x(t)| \leq \frac{\Big(g(t_1) - g(t_0)\Big)^{\alpha} \epsilon}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} |y(s) - x(s)| ds.$$

By utilizing the Gronwall inequality in Theorem 0.8 we obtain

$$|y(t) - x(t)| \leq \frac{\Big(g(t_1) - g(t_0)\Big)^{\alpha} \epsilon}{\Gamma(\alpha+1)} E_{\alpha}\Big(L(g(t) - g(t_0))^{\alpha}\Big).$$

Thus,

$$|y(t) - x(t)| \leq \Big[\frac{(g(t_1) - g(t_0))^{\alpha}}{\Gamma(\alpha+1)} E_{\alpha}\Big(L(g(t_0+\tau) - g(t_0))^{\alpha}\Big)\Big]\epsilon \quad \forall t \in [t_0, t_0+\tau].$$

For $t \in [t_0 + \tau, t_1]$, if we follow similar steps as above, we reach to the inequality

$$\begin{aligned}
|y(t) - x(t)| &\leq \frac{\Big(g(t_1) - g(t_0)\Big)^{\alpha} \epsilon}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} |y(s) - x(s)| ds \\
&+ \frac{L}{\Gamma(\alpha)} \int_{t_0+\tau}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} |y(s-\tau) - x(s-\tau)| ds.
\end{aligned}$$

If we set $z(t) = \sup\limits_{u \in [t_0-\tau, t_0]} |y(t+u) - x(t+u)|$, we get

$$\begin{aligned}
z(t) &\leq \frac{\Big(g(t_1) - g(t_0)\Big)^{\alpha} \epsilon}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} z(s) ds \\
&+ \frac{L}{\Gamma(\alpha)} \int_{t_0+\tau}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} z(s) ds \\
&\leq \frac{\Big(g(t_1) - g(t_0)\Big)^{\alpha} \epsilon}{\Gamma(\alpha+1)} + \frac{2L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} z(s) ds.
\end{aligned}$$

Using the Gronwall inequality in Theorem 0.8, we get

$$z(t) \leq \Big[\frac{(g(t_1) - g(t_0))^{\alpha}}{\Gamma(\alpha+1)} E_{\alpha}\Big(2L(g(t_0+\tau) - g(t_0))^{\alpha}\Big)\Big]\epsilon \quad \forall t.$$

Since $|y(t) - x(t)| \leq z(t)$, we have

$$|y(t) - x(t)| \leq \Big[\frac{(g(t_1) - g(t_0))^{\alpha}}{\Gamma(\alpha + 1)} E_{\alpha}\Big(2L(g(t_0 + \tau) - g(t_0))^{\alpha}\Big)\Big]\epsilon \ \ \forall t.$$

This was the end of the proof. □

3. Ulam-Hyers-Rassias Stability

Theorem 3.1. *Assume that*

1. $f \in C([t_0, t_1] \times \mathbb{R}^2, \mathbb{R})$ *and* $\phi \in C[t_0 - \tau, t_0]$
2. $|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L\Big(|u_1 - v_1| + |u_2 - v_2|\Big)$
3. $\dfrac{2L(g(t_1) - g(t_0))^{\alpha}}{\Gamma(\alpha + 1)} < 1.$
4. *There exists a function* $\varphi(t) \in C([t_0, t_1], \mathbb{R}^+)$ *and a real number* $\lambda_{\varphi} > 0$ *such that*

$$_{t_0}I_g^{\alpha}\varphi(t) \leq \lambda_{\varphi}\varphi(t).$$

Then equation (1) is Ulam-Hyers-Rassias stable with respect to φ.

Proof. Let $y(t) \in C([t_0 - \tau, t_1], \mathbb{R}) \cap C^1([t_0, t_1], \mathbb{R})$ be a solution of the inequality (14). We have

$$-\epsilon\ \varphi(t) \leq \ {}^{C}_{t_0}D_g^{\alpha}y(t) - f(t, y(t), y(t - \tau)) \leq \epsilon\ \varphi(t)$$

Applying the integral operator in (3) to get

$$-\epsilon\ {}_{t_0}I_g^{\alpha}\varphi(t) \leq y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau)) \leq \epsilon\ {}_{t_0}I_g^{\alpha}\varphi(t).$$

Using the fourth condition, we obtain

$$-\epsilon\lambda_{\varphi}\varphi(t) \leq y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau)) \leq \epsilon\lambda_{\varphi}\varphi(t).$$

Thus,

$$|y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau)) \leq \epsilon\lambda_{\varphi}\varphi(t).$$

Choose $x(t)$ such that

$$\begin{aligned} {}^{C}_{t_0}D_g^{\alpha}x(t) = f(t, x(t), x(t - \tau)),\ \ t \in [t_0, t_1],\ 0 < \alpha \leq 1, \\ x(t) = y(t), \quad t_0 - \tau \leq t \leq t_0, \tau > 0 \end{aligned}$$

If $t \in [t_0 - \tau, t_0]$, we have $|x(t) - y(t)| = 0$. If $t \in [t_0, t_0 + \tau]$ on using Lemma 1.1, we can write

$$\begin{aligned}
|y(t) - x(t)| &= |y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, x(t), x(t - \tau))| \\
&= |y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau)) \\
&+ \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau)) - \ {}_{t_0}I_g^{\alpha}f(t, x(t), x(t - \tau))| \\
&\leq |y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau))| \\
&+ |\ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau)) - \ {}_{t_0}I_g^{\alpha}f(t, x(t), x(t - \tau))| \\
&\leq |y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau))| \\
&+ |\ {}_{t_0}I_g^{\alpha}\Big(f(t, y(t), y(t - \tau)) - f(t, x(t), x(t - \tau))\Big)| \\
&\leq |y(t) - y(t_0) - \ {}_{t_0}I_g^{\alpha}f(t, y(t), y(t - \tau))| + L\ {}_{t_0}I_g^{\alpha}\Big(|y(t) - x(t)|\Big) \\
&\leq \epsilon\lambda_{\varphi}\varphi(t) + L\ {}_{t_0}I_g^{\alpha}\Big(|y(t) - x(t)|\Big) \\
&\leq \epsilon\lambda_{\varphi}\varphi(t)E_{\alpha}\Big(L(g(t_0 + \tau) - g(t_0))^{\alpha}\Big),
\end{aligned}$$

where the last inequality holds because of the Gronwall inequality in Theorem 0.8. For $t \in [t_0 + \tau, t_1]$, if we use similar arguments, we obtain the inequality

$$\begin{aligned} |y(t) - x(t)| &\leq \epsilon\lambda_\varphi\varphi(t) + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} |y(s) - x(s)| ds \\ &+ \frac{L}{\Gamma(\alpha)} \int_{t_0+\tau}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} |y(s-\tau) - x(s-\tau)| ds. \end{aligned}$$

Again, setting $z(t) = \sup_{u\in[t_0-\tau,t_0]} |y(t+u) - x(t+u)|$, we get

$$\begin{aligned} z(t) &\leq \epsilon\lambda_\varphi\varphi(t) + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} z(s) ds \\ &+ \frac{L}{\Gamma(\alpha)} \int_{t_0+\tau}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} z(s) ds \\ &\leq \epsilon\lambda_\varphi\varphi(t) + \frac{2L}{\Gamma(\alpha)} \int_{t_0}^{t} g'(s)\Big(g(t) - g(s)\Big)^{\alpha-1} z(s) ds. \end{aligned}$$

Using The Gronwall inequality in Theorem 0.8 yields

$$z(t) \leq \epsilon\lambda_\varphi\varphi(t) E_\alpha\Big(2L(g(t_1) - g(t_0+\tau))^\alpha\Big)\Big]\epsilon \ \ \forall t.$$

Because $|y(t) - x(t)| \leq z(t)$, we have

$$|y(t) - x(t)| \leq \epsilon\lambda_\varphi\varphi(t) E_\alpha\Big(2L(g(t_1) - g(t_0+\tau))^\alpha\Big) \ \ \forall t.$$

This was the end of the proof. □

4. Examples

Example 4.1. *Consider the Cauchy problem*

$$ {}_1^C D_{\sqrt{t}}^{\frac{1}{2}} x(t) = \frac{\sqrt{t}\Big(\sin(x(t)) + \cos(x(t-1))\Big)}{200}, \quad t \in [1,4], \tag{19}$$

$$x(t) = 1, \quad t \in [0,1]. \tag{20}$$

Since $f(t,u,v) = \frac{\sqrt{t}(\sin u + \cos v)}{200}$ *and* f *is continuous,*

$$\begin{aligned} |f(t,u_2,v_2) - f(t,u_1,v_1)| &\leq \frac{1}{200}\sqrt{t}\Big(|\sin u_2 - \sin u_1| + |\cos v_2 - \cos v_1| \\ &\leq \frac{\sqrt{t}}{200}\Big(|u_2 - u_1| + |v_2 - v_1|\Big) \\ &\leq \frac{1}{100}\Big(|u_2 - u_1| + |v_2 - v_1|\Big). \end{aligned}$$

Thus the Lipschitz constant is $L = \frac{1}{100}$. *Moreover we have*

$$\begin{aligned} \frac{2L}{\Gamma(\alpha+1)}\Big(\frac{g(t) - g(t_0)}{\rho}\Big)^\alpha &= \frac{1}{50}\frac{1}{\Gamma(\frac{3}{2})}\sqrt{\sqrt{t} - 1} \\ &\leq \frac{1}{25\sqrt{\pi}} < 1. \end{aligned}$$

Thus, according to Theorem 2.2, (19) is Ulam-Hyers stable.

Example 4.2. *Consider the Cauchy problem*

$$ {}_{1}^{C}D_{t^2}^{\frac{1}{3}}x(t) = \frac{|x(t)| + |x(t-\frac{1}{2})|}{200}, \quad t \in [1,3], \tag{21}$$

$$ x(t) = t, \quad t \in [\frac{1}{2}, 1]. \tag{22}$$

It is obvious since $f(t,u,v) = \frac{|u|+|v|}{200}$, *f is continuous and* $L = \frac{1}{200}$. *Now*

$$\begin{aligned} \frac{2L}{\Gamma(\alpha+1)}\Big(g(t) - g(1)\Big)^{\alpha} &= \frac{1}{100\Gamma(\frac{4}{3})}\Big(t^2 - 1^2\Big)^{\frac{1}{3}} \\ &\leq \frac{1}{50\Gamma(\frac{4}{3})} < 1. \end{aligned}$$

Let $\varphi(t) = t^2 - 1$. *then we have*

$$ {}_{1}I_{t^2}^{\frac{1}{3}}\varphi(t) = \frac{\Gamma(2)}{\Gamma(\frac{7}{3})}(t^2-1)^{\frac{4}{3}} \leq \frac{1}{\Gamma(\frac{7}{3})}(t^2-1)^{\frac{1}{3}}\varphi(t) \leq \frac{2}{\Gamma(\frac{7}{3})}\varphi(t).$$

Letting $\lambda_{\varphi} = \dfrac{2}{\Gamma(\frac{7}{3})}$, *we have all conditions of Theorem 3.1 satisfied. Thus (19) is Ulam-Hyers-Rassias stable with respect to* $\varphi(t) = t^2 - 1$.

5. Conclusion

In this paper, we studied the Ulam stability of solutions of initial value problems that incorporate a certain type of generalized Caputo fractional derivative that generalizes the classical Caputo fractional derivative and well known Caputo-type fractional derivatives. We stated the conditions under which these solutions are stable in the frame of Ulam-Hyers and Ulam-Hyers-Rassias. Thus, our results is a generalization of the previous studies on Ulam stability when the Caputo or Caputo-Hadamard fractional derivatives are involved. We remark that it would be interesting to study the qualitative properties of differential equations that include fractional operators with nonsingular kernels and the stability of solutions is one of these properties.

6. Aknowlegement

The third author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

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