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Hermite-Hadamard Type Inequalities for n-Times Differentiable Convex Functions via Riemann-Liouville Fractional Integrals

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Abstract. In this paper, we establish an identity for n-times differentiable functions via Riemann-Liouville fractional integrals. By using this new identity, we have some new results about trapezoid type inequalities for n-times differentiable convex functions via Riemann-Liouville fractional integrals. The results, given here extended the results given the previous works.

1. Introduction

Let $I \subseteq \mathbb{R}$ be a non-empty interval and $f : I \to \mathbb{R}$. f is said to be a convex function on I, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality (1) is reversed, then f is said to be concave on I.

The most important inequality in the theory of convex functions is Hermite-Hadamard's inequality in below [3, 4]. If f is a convex function on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

If f is concave on $[a,b]$, then the inequality (2) is reversed. It is worth noting that Hadamard's inequality can be seen as a refinement of the concept of convexity. Hadamard's inequality for convex functions has been renewed in recent years and there has been clear diversifications, generalizations (see, for example, [5, 7–12, 14–19]) and the references cited therein.

In [1], Dragomir and Agarwal proved the following results connected with the right part of (2).

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Theorem 1.1. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} \left(|f'(a)| + |f'(b)| \right). \tag{3}$$

Theorem 1.2. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$ and $|f'|^{p/(p-1)}$ is convex on $[a, b]$ for $p > 1$. Then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{2(p+1)^{1/p}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}. \tag{4}$$

In [11] Pearce and Pećarić improved the inequality (4) as the following result.

Theorem 1.3. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$ and $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$. Then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}. \tag{5}$$

In the following, we recall some essential definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can see [2, 6, 13].

Definition 1.4. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a^+} f$ and $J^\alpha_{b^-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \quad \text{and} \quad J^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a^+} f(x) = J^0_{b^-} f(x) = f(x)$.

In [16], Sarıkaya et al. gave some Hermite-Hadamard type inequalities related to the fractional integrals inequalities for convex functions by using the following identity.

Lemma 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a) \right] = \frac{b-a}{2} \int_0^1 \left[(1-t)^\alpha - t^\alpha \right] f'(ta + (1-t)b)\, dt. \tag{6}$$

Lemma 1.6. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable mapping on (a, b) with $a < b$. If $f'' \in L[a, b]$, then the following equality for fractional integrals holds*

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[{}_{RL}J^\alpha_{a^+} f(b) + {}_{RL} J^\alpha_{b^-} f(a) \right] \tag{7}$$

$$= \frac{(b-a)^2}{2} \int_0^1 \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} f''(ta + (1-t)b)\, dt. \tag{8}$$

For various studies and results on fractional integral inequalities, see [5, 7, 8, 10, 12, 14, 16, 17].

In this study, we have obtained a new lemma for Hermite-Hadamard inequalities involving Riemann-Liouville fractional for n-times differentiable functions. By using this new lemma, we have some new results about trapezoid type inequalities for n-times differentiable convex functions via Riemann-Liouville fractional integrals. Also, we have some applications to special means of positive numbers.

2. Main Results

Lemma 2.1. *Let $n \geq 1$ and the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is n-times differentiable. If $a, b \in I$ with $a < b$ and $f^{(n)} \in L[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right] \tag{9}$$

$$= \frac{b-a}{2} S(\alpha, n-1)\int_0^1 \left[(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right] f^{(n)}(ta+(1-t)b)\,dt$$

where $S(\alpha,n) = \frac{(b-a)^n}{(\alpha+1)(\alpha+2)...(\alpha+n)}$.

Proof. We will use the mathematical induction principle. The case $n = 1$ is Lemma 1.5. Suppose (9) holds for $n - 1$, i.e.

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] + \sum_{k=1}^{n-2} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]$$

$$= \frac{b-a}{2} S(\alpha, n-2)\int_0^1 \left[(-1)^{n-2}(1-t)^{\alpha+n-2} - t^{\alpha+n-2}\right] f^{(n-1)}(ta+(1-t)b)\,dt.$$

Now integrating by parts, we have

$$\frac{b-a}{2} S(\alpha, n-1)\int_0^1 \left[(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right] f^{(n)}(ta+(1-t)b)\,dt$$

$$= S(\alpha, n-1)\frac{f^{(n-1)}(a)+(-1)^{n-1} f^{(n-1)}(b)}{2}$$

$$+\frac{b-a}{2} S(\alpha, n-2)\int_0^1 \left[(-1)^{n-2}(1-t)^{\alpha+n-2} - t^{\alpha+n-2}\right] f^{(n-1)}(ta+(1-t)b)\,dt$$

$$= S(\alpha, n-1)\frac{f^{(n-1)}(a)+(-1)^{n-1} f^{(n-1)}(b)}{2} + \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]$$

$$+\sum_{k=1}^{n-2} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]$$

$$= \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right].$$

This completes the proof of Lemma 1.6. □

Remark 2.2. 1. It is taken $n = 1$ to (9), we obtain (6).

2. It is taken $n = 2$ to (9), we obtain (7).

Proof. If we take $n = 2$ to (9), we get

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] + \left[\frac{f'(a)-f'(b)}{2}\right]$$

$$\frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''(ta+(1-t)b)\,dt$$

$$= \frac{b-a}{2(\alpha+1)}\left[f'(b)-f'(a)\right] + \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''(ta+(1-t)b)\,dt.$$

On the other hand, If we distribute on the right of (7), we get

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]$$

$$= \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} f''(ta+(1-t)b)\,dt$$

$$= \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 f''(ta+(1-t)b)\,dt + + \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''(ta+(1-t)b)\,dt$$

$$= \frac{b-a}{2(\alpha+1)}\int_a^b f''(x)dx + + \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''(ta+(1-t)b)\,dt$$

$$= \frac{b-a}{2(\alpha+1)}\left[f'(b)-f'(a)\right] + \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''(ta+(1-t)b)\,dt.$$

As it is seen from the last equation, it is obtained that it is equaled (7) to (9) for $n = 2$. □

Theorem 2.3. *Let $n \geq 1$, the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is n-times differentiable and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|$ is convex on $[a,b]$ then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2}\right]\right| \tag{10}$$

$$\leq \begin{cases} \frac{S(\alpha,n)}{2}\left[\left(1-\frac{1}{2^{\alpha+n-1}}\right)\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right)\right] & , n \text{ is odd} \\ \frac{S(\alpha,n)}{2}\left[\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right] & , n \text{ is even} \end{cases}$$

where $S(\alpha,n) = \frac{(b-a)^n}{(\alpha+1)(\alpha+2)...(\alpha+n)}$.

Proof. Let $n \geq 1$. By using the convexity of $\left|f^{(n)}\right|$ on $[a,b]$ and Lemma 1.6, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2}\right]\right| \tag{11}$$

$$\leq \frac{b-a}{2} S(\alpha,n-1)\int_0^1 \left|(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right|\left|f^{(n)}(ta+(1-t)b)\right| dt$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left| (1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right| \left| f^{(n)}(ta+(1-t)b) \right| dt \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left| -(1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right| \left| f^{(n)}(ta+(1-t)b) \right| dt \right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left| (1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right| \begin{pmatrix} t \left| f^{(n)}(a) \right| \\ +(1-t) \left| f^{(n)}(b) \right| \end{pmatrix} dt \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left| (1-t)^{\alpha+n-1} + t^{\alpha+n-1} \right| \begin{pmatrix} t \left| f^{(n)}(a) \right| \\ +(1-t) \left| f^{(n)}(b) \right| \end{pmatrix} dt \right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \int_0^{1/2} \left((1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right) \begin{pmatrix} t \left| f^{(n)}(a) \right| \\ +(1-t) \left| f^{(n)}(b) \right| \end{pmatrix} dt \\ + \int_{1/2}^1 \left(t^{\alpha+n-1} - (1-t)^{\alpha+n-1} \right) \begin{pmatrix} t \left| f^{(n)}(a) \right| \\ +(1-t) \left| f^{(n)}(b) \right| \end{pmatrix} dt \end{array} \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left((1-t)^{\alpha+n-1} + t^{\alpha+n-1} \right) \begin{pmatrix} t \left| f^{(n)}(a) \right| \\ +(1-t) \left| f^{(n)}(b) \right| \end{pmatrix} dt \right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \begin{pmatrix} \int_0^{1/2} t(1-t)^{\alpha+n-1} dt - \int_0^{1/2} t^{\alpha+n} dt \\ + \int_{1/2}^1 t^{\alpha+n} dt - \int_{1/2}^1 t(1-t)^{\alpha+n-1} dt \end{pmatrix} \left| f^{(n)}(a) \right| \\ + \begin{pmatrix} \int_0^{1/2} (1-t)^{\alpha+n} dt - \int_0^{1/2} (1-t) t^{\alpha+n-1} dt \\ + \int_{1/2}^1 (1-t) t^{\alpha+n-1} dt - \int_{1/2}^1 (1-t)^{\alpha+n} dt \end{pmatrix} \left| f^{(n)}(b) \right| \end{array} \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\int_0^1 t(1-t)^{\alpha+n-1} + t^{\alpha+n} dt \right) \left| f^{(n)}(a) \right| \\ + \left(\int_0^1 \left((1-t)^{\alpha+n} + (1-t) t^{\alpha+n-1} \right) dt \right) \left| f^{(n)}(b) \right| \end{array} \right] & , n \text{ is even} \end{cases}.$$

Calculating the appearing integrals in (14), we have

$$\int_0^{1/2} t(1-t)^{\alpha+n-1} dt - \int_0^{1/2} t^{\alpha+n} dt + \int_{1/2}^1 t^{\alpha+n} dt - \int_{1/2}^1 t(1-t)^{\alpha+n-1} dt = \frac{1}{\alpha+n} \left(1 - \frac{1}{2^{\alpha+n-1}} \right), \tag{12}$$

$$\int_0^{1/2} (1-t)^{\alpha+n} dt - \int_0^{1/2} (1-t) t^{\alpha+n-1} dt + \int_{1/2}^1 (1-t) t^{\alpha+n-1} dt - \int_{1/2}^1 (1-t)^{\alpha+n} dt = \frac{1}{\alpha+n} \left(1 - \frac{1}{2^{\alpha+n-1}} \right), \tag{13}$$

$$\int_0^1 t(1-t)^{\alpha+n-1} + t^{\alpha+n} dt = \int_0^1 \left((1-t)^{\alpha+n} + (1-t) t^{\alpha+n-1} \right) dt = \frac{1}{\alpha+n}. \tag{14}$$

A combination of (11)-(14) gives (10). This completes the proof. □

Corollary 2.4. *In Theorem 2.3,*

1. *If one takes $n = 1$, one has [16, Theorem 3].*

2. *If one takes $n = 1$ and $\alpha = 1$, one has* (3).
3. *If one takes $\alpha = 1$, one has the following trapezoid type inequality for n-times differentiable convex functions*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx + \sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{15}$$

$$\leq \begin{cases} \frac{(b-a)^n}{2(n+1)!}\left[\left(1-\frac{1}{2^n}\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\right] & , n \text{ is odd} \\ \frac{(b-a)^n}{2(n+1)!}\left[\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right] & , n \text{ is even} \end{cases}.$$

Theorem 2.5. *Let $n \geq 1$, the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is n-times differentiable and $a < b$. If $\left|f^{(n)}\right|^q \in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{16}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha,n-1)\left[\begin{array}{l} \left(\frac{1}{p(\alpha+n-1)+1}\left(1-\frac{1}{2^{p(\alpha+n-1)}}\right)\right)^{\frac{1}{p}} \\ \times\left[\left(\frac{\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]\end{array}\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha,n-1)\left(\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}} & , n \text{ is even} \end{cases}$$

where $S(\alpha,n) = \frac{(b-a)^n}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Let $n \geq 1$. By using the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, Hölder inequality and Lemma 1.6, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|$$

$$\leq \frac{b-a}{2} S(\alpha,n-1)\int_0^1 \left|(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right|\left|f^{(n)}(ta+(1-t)b)\right|dt$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha,n-1)\left[\int_0^1 \left|(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right|\left|f^{(n)}(ta+(1-t)b)\right|dt\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha,n-1)\left[\int_0^1 \left|-(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right|\left|f^{(n)}(ta+(1-t)b)\right|dt\right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha,n-1)\left[\begin{array}{l} \int_0^{1/2}\left((1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right)\left|f^{(n)}(ta+(1-t)b)\right|dt \\ +\int_{1/2}^1\left(t^{\alpha+n-1} - (1-t)^{\alpha+n-1}\right)\left|f^{(n)}(ta+(1-t)b)\right|dt \end{array}\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha,n-1)\left[\int_0^1\left((1-t)^{\alpha+n-1} + t^{\alpha+n-1}\right)\left|f^{(n)}(ta+(1-t)b)\right|dt\right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{l} \left(\int\limits_0^{1/2} \left((1-t)^{(\alpha+n-1)} - t^{(\alpha+n-1)} \right)^p dt \right)^{\frac{1}{p}} \\ \times \left(\int\limits_0^{1/2} t \left| f^{(n)}(a) \right|^q + (1-t) \left| f^{(n)}(b) \right|^q dt \right)^{\frac{1}{q}} \\ + \left(\int\limits_{1/2}^{1} \left(t^{(\alpha+n-1)} - (1-t)^{(\alpha+n-1)} \right)^p dt \right)^{\frac{1}{p}} \\ \times \left(\int\limits_{1/2}^{1} t \left| f^{(n)}(a) \right|^q + (1-t) \left| f^{(n)}(b) \right|^q dt \right)^{\frac{1}{q}} \end{array} \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\int\limits_0^{1} \left((1-t)^{\alpha+n-1} + t^{\alpha+n-1} \right)^p dt \right)^{\frac{1}{p}} \\ \left(\int\limits_0^{1} t \left| f^{(n)}(a) \right|^q + (1-t) \left| f^{(n)}(b) \right|^q dt \right)^{\frac{1}{q}} \end{array} \right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{l} \left(\int\limits_0^{1/2} (1-t)^{p(\alpha+n-1)} - t^{p(\alpha+n-1)} dt \right)^{\frac{1}{p}} \\ \times \left(\frac{\left| f^{(n)}(a) \right|^q + 3\left| f^{(n)}(b) \right|^q}{8} \right)^{\frac{1}{q}} \\ + \left(\int\limits_{1/2}^{1} t^{p(\alpha+n-1)} - (1-t)^{p(\alpha+n-1)} dt \right)^{\frac{1}{p}} \\ \times \left(\frac{3\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q}{8} \right)^{\frac{1}{q}} \end{array} \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\int\limits_0^{1} \left((1-t)^{\alpha+n-1} + t^{\alpha+n-1} \right)^p dt \right)^{\frac{1}{p}} \\ \times \left(\frac{\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q}{2} \right)^{\frac{1}{q}} \end{array} \right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{l} \left(\frac{1}{p(\alpha+n-1)+1} \left(1 - \frac{1}{2^{p(\alpha+n-1)}} \right) \right)^{\frac{1}{p}} \\ \times \left[\left(\frac{\left| f^{(n)}(a) \right|^q + 3\left| f^{(n)}(b) \right|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q}{8} \right)^{\frac{1}{q}} \right] \end{array} \right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left(\frac{\left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q}{2} \right)^{\frac{1}{q}} & , n \text{ is even} \end{cases}$$

This completes the proof. □

Corollary 2.6. *In Theorem 2.5,*

1. *If one takes $n = 1$ and $\alpha = 1$, one has the following trapezoid type inequality for convex functions*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int\limits_a^b f(x)\, dx \right| \tag{17}$$

$$\leq \frac{b-a}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^p} \right)^{\frac{1}{p}} \left[\left(\frac{\left| f'(a) \right|^q + 3\left| f'(b) \right|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3\left| f'(a) \right|^q + \left| f'(b) \right|^q}{8} \right)^{\frac{1}{q}} \right].$$

2. *If one takes $\alpha = 1$, one has the following trapezoid type inequality for n-times differentiable convex functions*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx + \sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{18}$$

$$\leq \begin{cases} \frac{(b-a)^n}{2n!}\left[\begin{array}{c}\left(\frac{1}{pn+1}\left(1-\frac{1}{2^{pn}}\right)\right)^{\frac{1}{p}} \\ \times\left[\left(\frac{|f^{(n)}(a)|^q+3|f^{(n)}(b)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{8}\right)^{\frac{1}{q}}\right]\end{array}\right] & , n \text{ is odd} \\ \frac{(b-a)^n}{2n!}\left(\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\right)^{\frac{1}{q}} & , n \text{ is even}\end{cases}.$$

3. *If one takes $n = 1$, one has the following trapezoid type inequalities for convex functions via fractional integrals*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \tag{19}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right],$$

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \tag{20}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(1-\frac{1}{2^{\alpha p}}\right)^{1/p}\left(\frac{1+3^{1/q}}{8^{1/q}}\right)\left(|f'(a)|+|f'(b)|\right).$$

Proof. In need of proof of the inequality (20), we conceive the inequality (19). Let $a_1 = \frac{|f'(a)|}{8}$, $b_1 = \frac{3|f'(b)|}{8}$, $a_2 = \frac{3|f'(a)|}{8}$, $b_2 = \frac{|f'(b)|}{8}$ in (19). Using the fact that $\sum_{i=1}^n (a_i+b_i)^r \leq \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r$ for $0 < r < 1$, $a_1, a_2, \ldots a_n > 0$ and $b_1, b_2, \ldots, b_n > 0$, we get the required result. This completes the proof. □

Remark 2.7. In (19) we obtain [19, Theorem 3.2, for $s = 1$], In (20) we obtain [19, Corollary 3.3, for $s = 1$].

Theorem 2.8. *Let $n \geq 1$, the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is n-times differentiable and $a < b$. If $|f^{(n)}|^q \in L[a,b]$ and $|f^{(n)}|^q$ is convex on $[a,b]$ for $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{21}$$

$$\leq \begin{cases} \frac{b-a}{(\alpha+1)} S(\alpha, n-1)\left(1-\left(\frac{1}{2}\right)^{\alpha+n-1}\right)\left(\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\right)^{\frac{1}{q}} & , n \text{ is odd} \\ \frac{b-a}{(\alpha+1)} S(\alpha, n-1)\left(\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\right)^{\frac{1}{q}} & , n \text{ is even}\end{cases}$$

where $S(\alpha,n) = \frac{(b-a)^n}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}$.

Proof. Let $n \geq 1$. By using the convexity of $|f^{(n)}|^q$ on $[a,b]$, power mean inequality and Lemma 1.6, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] + \sum_{k=1}^{n-1} S(\alpha,k)\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|$$

$$\leq \frac{b-a}{2} S(\alpha, n-1) \int_0^1 \left|(-1)^{n-1}(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| \left|f^{(n)}(ta+(1-t)b)\right| dt$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left|(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| \left|f^{(n)}(ta+(1-t)b)\right| dt\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\int_0^1 \left|-(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| \left|f^{(n)}(ta+(1-t)b)\right| dt\right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\int_0^1 \left|(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| dt\right)^{1-\frac{1}{q}} \\ \times \left(\int_0^1 \left|(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| \left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\int_0^1 \left|-(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| dt\right)^{1-\frac{1}{q}} \\ \times \left(\int_0^1 \left|-(1-t)^{\alpha+n-1} - t^{\alpha+n-1}\right| \left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}} \left(1-\left(\frac{1}{2}\right)^{\alpha+n-1}\right)^{1-\frac{1}{q}} \\ \times \left(\begin{array}{c} \left|f^{(n)}(a)\right|^q \int_0^{1/2} t(1-t)^{\alpha+n-1} - t^{\alpha+n} dt \\ + \left|f^{(n)}(b)\right|^q \int_0^{1/2} (1-t)^{\alpha+n} - (1-t)\, t^{\alpha+n-1} dt \\ + \left|f^{(n)}(a)\right|^q \int_{1/2}^1 t^{\alpha+n} - t(1-t)^{\alpha+n-1} dt \\ \left|f^{(n)}(b)\right|^q \int_{1/2}^1 (1-t)\, t^{\alpha+n-1} - (1-t)^{\alpha+n} dt \end{array}\right)^{\frac{1}{q}} \end{array}\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}} \\ \left(\begin{array}{c} \left|f^{(n)}(a)\right|^q \int_0^1 t(1-t)^{\alpha+n-1} + t^{\alpha+n} dt \\ + \left|f^{(n)}(b)\right|^q \int_0^1 (1-t)^{\alpha+n} + (1-t)\, t^{\alpha+n-1} dt \end{array}\right)^{\frac{1}{q}} \end{array}\right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{2} S(\alpha, n-1) \left[\begin{array}{c} \left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}} \left(1-\left(\frac{1}{2}\right)^{\alpha+n-1}\right)^{1-\frac{1}{q}} \\ \times \left(\left(\frac{1}{\alpha+1}\right)\left(1-\left(\frac{1}{2}\right)^{\alpha+n-1}\right)\left(\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}} \end{array}\right] & , n \text{ is odd} \\ \frac{b-a}{2} S(\alpha, n-1) \left[\left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}} \left(\left(\frac{1}{\alpha+n}\right)\left(\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|\right)\right)^{\frac{1}{q}}\right] & , n \text{ is even} \end{cases}$$

$$\leq \begin{cases} \frac{b-a}{\alpha+1} S(\alpha, n-1) \left(1-\left(\frac{1}{2}\right)^{\alpha+n-1}\right)\left(\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}} & , n \text{ is odd} \\ \frac{b-a}{\alpha+1} S(\alpha, n-1) \left(\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|}{2}\right)^{\frac{1}{q}} & , n \text{ is even} \end{cases}.$$

This proof is completed. □

Corollary 2.9. *In Theorem 2.8,*

1. *If one takes $n = 1$ and $\alpha = 1$, one has* (5).
2. *If one takes $\alpha = 1$, one has the following trapezoid type inequality for n-times differentiable convex functions*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx+\sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{22}$$

$$\leq\begin{cases}\frac{(b-a)^n}{2n!}\left(1-\left(\frac{1}{2}\right)^n\right)\left(\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}} & ,\ n \text{ is odd}\\ \frac{(b-a)^n}{2n!}\left(\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}} & ,\ n \text{ is even}\end{cases}.$$

3. *If one takes $n = 1$, one has the following trapezoid type inequalities for convex functions via fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\leq\frac{b-a}{(\alpha+1)}\left(1-\left(\frac{1}{2}\right)^\alpha\right)\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{23}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\leq\frac{b-a}{(\alpha+1)2^{\frac{1}{q}}}\left(1-\left(\frac{1}{2}\right)^\alpha\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{24}$$

Proof. In need of proof of the inequality (24), *we conceive the inequality* (23) *and the same technique in the proof of Corollary 2.6.* □

Remark 2.10. In (19) we obtain [15, Corollary 2.4] and [16, Theorem 3].

Remark 2.11. If one takes $n = 2$ in the inequalities (15) and (18) respectively, one has the following inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{b-a}{2}\left[\frac{f'(a)-f'(b)}{2}\right]\right|\leq\frac{(b-a)^2}{12}\left[\left|f''(a)\right|+\left|f''(b)\right|\right], \tag{25}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{b-a}{2}\left[\frac{f'(a)-f'(b)}{2}\right]\right|\leq\frac{(b-a)^2}{4}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{26}$$

3. Applications to Special Means

Let us recall the following special means of positive numbers a, b with $a < b$.

1. The arithmetic mean:

$$A(a,b) := \frac{a+b}{2}.$$

2. The harmonic mean:

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

3. The logarithmic mean:

$$L(a,b) := \frac{b-a}{\ln b-\ln a}, \quad a \neq b.$$

4. The n-logarithmic mean:

$$L_n(a,b) := \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}}.$$

Now we apply the inequalities (25) and (26) to establish some inequalities for special means

Proposition 3.1. *For $a,b \in \mathbb{R}_+$, $a < b$ and $m \in \mathbb{N}$, $m > 1$, the following inequality holds*

$$\left|A\left(a^{m+1},b^{m+1}\right) - L_{m+1}^{m+1}\left(a^{m+1},b^{m+1}\right) + \frac{(b-a)(m+1)}{2}A\left(a^m,-b^m\right)\right|$$

$$\leq \frac{m(m+1)(b-a)^2}{6}A\left(a^{m-1},b^{m-1}\right).$$

Proof. For the function $f(x) = x^{m+1}$, $x \in \mathbb{R}_+$ and $m \in \mathbb{N}$, $m > 1$, $\left|f''(x)\right| = m(m+1)x^{m-1}$ is a convex function on $\mathbb{R}_+$. Applying the inequality (25), the required result is obtained. □

Proposition 3.2. *For $a,b \in \mathbb{R}_+$, $a < b$ and $m,q \in \mathbb{N}$, $m,q > 1$, the following inequality holds*

$$\left|A\left(a^{m+\frac{1}{q}},b^{m+\frac{1}{q}}\right) - L_{m+\frac{1}{q}}^{m+\frac{1}{q}}\left(a^{m+\frac{1}{q}},b^{m+\frac{1}{q}}\right) + \frac{(b-a)\left(m+\frac{1}{q}\right)}{2}A\left(a^{m-1+\frac{1}{q}},-b^{m-1+\frac{1}{q}}\right)\right|$$

$$\leq \frac{\left(m-1+\frac{1}{q}\right)^q\left(m+\frac{1}{q}\right)^q(b-a)^2}{4}A\left(a^{(m-2)q+1},b^{(m-2)q+1}\right).$$

Proof. For the function $f(x) = x^{m+\frac{1}{q}}$, $x \in \mathbb{R}_+$ and $m,q \in \mathbb{N}$, $m,q > 1$, $\left|f''(x)\right|^q = \left(m-1+\frac{1}{q}\right)^q\left(m+\frac{1}{q}\right)^q x^{(m-2)q+1}$ is a convex function on $\mathbb{R}_+$. Applying the inequality (26), the required result is obtained. □

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