Filomat 32:16 (2018), 5623–5632
https://doi.org/10.2298/FIL1816623D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Adjacency-Jacobsthal-Hurwitz Type Numbers

Ömür Deveci[a], **Yeşim Akuzüm**[a]

[a] *Department of Mathematics, Faculty of Science and Letters, Kafkas University 36100, Turkey*

Abstract. In this paper, we define the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind. Then we give the exponential, combinatorial, permanental and determinantal representations and the Binet formulas of the adjacency-Jacobsthal-Hurwitz numbers of the first and second kind by the aid of the generating functions and the generating matrices of the sequences defined.

1. Introduction

It is well-known that Jacobsthal sequence $\{J_n\}$ is defined recursively by the equation

$$J_{n+1} = J_n + 2J_{n-1}$$

for $n > 0$, where $J_0 = 0$, $J_1 = 1$.

In [5], Deveci and Artun defined the adjacency-Jacobsthal sequence as follows:

$$J_{m,n}(mn+k) = J_{m,n}(mn-n+k+1) + 2J_{m,n}(k)$$

for $k \geq 1$, $m \geq 2$ and $n \geq 4$ with initial constants $J_{m,n}(1) = \cdots = J_{m,n}(mn-1) = 0$ and $J_{m,n}(mn) = 1$.

It is easy to see that the characteristic polynomial of the adjacency-Jacobsthal sequence is

$$f(x) = x^{mn} - x^{mn-n+1} - 2.$$

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [10], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

2010 *Mathematics Subject Classification*. Primary 11K31; Secondary 11C20, 15A15
Keywords. adjacency-Jacobsthal-Hurwitz sequence, Hurwitz matrix, representation
Received: 12 June 2017; Revised: 02 December 2017; Accepted: 04 December 2017
Communicated by Ljubiša D.R. Kočinac
This Project was supported by the Commission for the Scientific Research Projects of Kafkas University, the Project No. 2016-FM-23
Email addresses: odeveci36@hotmail.com (Ömür Deveci), yesim_036@hotmail.com (Yeşim Akuzüm)

$$A=\left[a_{i,j}\right]_{k\times k}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix}=\begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geqslant 0$.

Let an *nth* degree real polynomial q be given by

$$q(x)=c_0x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n.$$

In [9], the Hurwitz matrix $H_n=\left[h_{i,j}\right]_{n\times n}$ associated to the polynomial q was defined as shown:

$$H_n=\begin{bmatrix} c_1 & c_3 & c_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ c_0 & c_2 & c_4 & & & & \vdots & \vdots & \vdots \\ 0 & c_1 & c_3 & & & & \vdots & \vdots & \vdots \\ \vdots & c_0 & c_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & c_1 & & \ddots & & c_n & \vdots & \vdots \\ \vdots & \vdots & c_0 & & & \ddots & c_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & c_{n-2} & c_n & \vdots \\ \vdots & \vdots & \vdots & & & & c_{n-3} & c_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & c_{n-4} & c_{n-2} & c_n \end{bmatrix}.$$

Recently, many authors have studied number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper [3, 4, 6–8, 11, 12, 14, 16–19]. In this paper, we define the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind by using Hurwitz matrix for characteristic polynomial of the adjacency-Jacobsthal sequence of order $4m$. Then we develop some their properties such as the generating function, exponential representations, the generating matrices and the combinatorial representations. Also, we give relationships among the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind and the permanents and the determinants of certain matrices which are produced by using the generating matrices of the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind. Finally, we obtain the Binet formulas for the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind by the aid of the roots of characteristic polynomials of the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind.

2. The Main Results

It is readily seen that Hurwitz matrix for characteristic polynomial of the adjacency-Jacobsthal sequence of order $4m$, $H_{4m}^{J} = \left[h_{i,j}\right]_{4m\times 4m}$ is defined by

$$h_{i,j} = \begin{cases} 1 & \text{if } i = 2\imath \text{ and } j = \imath \text{ for } 1 \leq \imath \leq 2m, \\ -1 & \text{if } i = 1 + 2\imath \text{ and } j = 2 + \imath \text{ for } 1 \leq \imath \leq 2m-1, \\ -2 & \text{if } i = 2\imath \text{ and } j = 2m + \imath \text{ for } 1 \leq \imath \leq 2m, \\ 0 & \text{otherwise.} \end{cases}$$

By the aid of the matrix H_{4m}^{J}, we define the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind, respectively by:

$$J_m^1(4m+k) = J_m^1(2m+k) - 2J_m^1(k) \text{ for } k \geq 1 \text{ and } m \geq 4, \tag{1}$$

where

$$J_m^1(1) = 1, J_m^1(2) = \cdots = J_m^1(2m) = 0, J_m^1(2m+1) = 1, J_m^1(2m+2) = \cdots = J_m^1(4m) = 0$$

and

$$J_m^2(4m+k) = J_m^2(k) - 2J_m^2(2m+k) \text{ for } k \geq 1 \text{ and } m \geq 4, \tag{2}$$

where

$$J_m^2(1) = 1, J_m^2(2) = \cdots = J_m^2(4m-1) = 0, J_m^2(4m) = 1.$$

Clearly, the generating functions of the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind are given by

$$g^1(x) = \frac{1}{1 - x^{2m} + 2x^{4m}}$$

and

$$g^2(x) = \frac{1 + 3x^{2m}}{1 + 2x^{2m} - x^{4m}},$$

respectively. It can be readily established that the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind have the following exponential representations, respectively:

$$g^1(x) = \exp\left(\sum_{i=1}^{\infty} \frac{\left(x^{2m}\right)^i}{i}\left(1 - 2x^{2m}\right)^i\right)$$

and

$$g^2(x) = \left(1 + 3x^{2m}\right)\exp\left(\sum_{i=1}^{\infty} \frac{\left(x^{2m}\right)^i}{i}\left(x^{2m} - 2\right)^i\right).$$

By equations (1) and (2), we can write the following companion matrices, respectively:

$$C_m^1 = \begin{array}{c} \qquad\qquad (2m)\,\text{th} \\ \qquad\qquad \downarrow \\ \left[\begin{array}{cccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{array}\right]_{4m\times 4m} \end{array}$$

and

$$C_m^2 = \begin{array}{c} \left[\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & \cdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \ldots & 0 & -2 & 0 & \cdots & 0 \end{array}\right]_{4m\times 4m.} \\ \qquad \uparrow \\ \qquad (2m+1)\,\text{th} \end{array}$$

The companion matrices C_m^1 and C_m^2 are called the adjacency-Jacobsthal-Hurwitz matrices of the first and second kind, respectively. For detailed information about the companion matrices, see [13, 15]. Let $J_m^1(\alpha)$ and $J_m^2(\alpha)$ be denoted by $J_m^{1,\alpha}$ and $J_m^{2,\alpha}$. By mathematical induction on α, we derive

$$\left(C_m^1\right)^\alpha = \left[\begin{array}{ccccccc} J_m^{1,\alpha+1} & J_m^{1,\alpha+2} & \cdots & J_m^{1,\alpha+2m} & -2J_m^{1,\alpha-2m+1} & \cdots & -2J_m^{1,\alpha-1} & -2J_m^{1,\alpha} \\ J_m^{1,\alpha} & J_m^{1,\alpha+1} & \cdots & J_m^{1,\alpha+2m-1} & -2J_m^{1,\alpha-2m} & \cdots & -2J_m^{1,\alpha-2} & -2J_m^{1,\alpha-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ J_m^{1,\alpha-4m+2} & J_m^{1,\alpha-4m+3} & \cdots & J_m^{1,\alpha-2m+1} & -2J_m^{1,\alpha-6m+2} & \cdots & -2J_m^{1,\alpha-4m} & -2J_m^{1,\alpha-4m+1} \end{array}\right]_{4m\times 4m} \tag{3}$$

and

$$\left(C_m^2\right)^\alpha = \left[\begin{array}{cccccccc} J_m^{2,\alpha} & J_m^{2,\alpha-1} & \cdots & J_m^{2,\alpha-2m+1} & J_m^{2,\alpha+2m} & \cdots & J_m^{2,\alpha+2} & J_m^{2,\alpha+1} \\ J_m^{2,\alpha+1} & J_m^{2,\alpha} & \cdots & J_m^{2,\alpha-2m+2} & J_m^{2,\alpha+2m+1} & \cdots & J_m^{2,\alpha+3} & J_m^{2,\alpha+2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ J_m^{2,\alpha+4m-1} & J_m^{2,\alpha+4m-2} & \cdots & J_m^{2,\alpha+2m} & J_m^{2,\alpha+6m-1} & \cdots & J_m^{2,\alpha+4m+1} & J_m^{2,\alpha+4m} \end{array}\right]_{4m\times 4m.} \tag{4}$$

for $\alpha \geq 1$. Note that $\det\left(C_m^1\right)^\alpha = (2)^\alpha$ and $\det\left(C_m^2\right)^\alpha = (-1)^\alpha$

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \left[\begin{array}{cccc} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{array}\right].$$

Theorem 2.1. (Chen and Louck [2]) *The (i,j) entry $k_{i,j}^{(n)}(k_1,k_2,\ldots,k_v)$ in the matrix $K^n(k_1,k_2,\ldots,k_v)$ is given by the following formula:*

$$k_{i,j}^{(n)}(k_1,k_2,\ldots,k_v)=\sum_{(t_1,t_2,\ldots,t_v)}\frac{t_j+t_{j+1}+\cdots+t_v}{t_1+t_2+\cdots+t_v}\times\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v}k_1^{t_1}\cdots k_v^{t_v}, \tag{5}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+vt_v=n-i+j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v}=\frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (5) are defined to be 1 if $n=i-j$.

Now we concentrate on finding combinatorial representations for the adjacency-Jacobsthal-Hurwitz numbers of the first and second kind.

Corollary 2.2. *The following hold:*

(i) $J_m^1(n)=\sum_{(t_1,t_2\ldots,t_{4m})}\frac{t_\alpha+t_{\alpha+1}+\cdots+t_{4m}}{t_1+t_2+\cdots+t_{4m}}\times\binom{t_1+\cdots+t_{4m}}{t_1,\ldots,t_{4m}}(-2)^{t_{4m}}$ *for* $1\le\alpha\le 2m$,
where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(4m)t_{4m}=n-1$.

(ii) $J_m^1(n)=\sum_{(t_1,t_2\ldots,t_{4m})}\frac{t_{4m}}{t_1+t_2+\cdots+t_{4m}}\times\binom{t_1+\cdots+t_{4m}}{t_1,\ldots,t_{4m}}(-2)^{t_{4m}}$,
where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(4m)t_{4m}=n+4m-1$.

(iii) $J_m^2(n)=\sum_{(t_1,t_2\ldots,t_{4m})}\frac{t_\alpha+t_{\alpha+1}+\cdots+t_{4m}}{t_1+t_2+\cdots+t_{4m}}\times\binom{t_1+\cdots+t_{4m}}{t_1,\ldots,t_{4m}}(-2)^{t_{2m+1}}$ *for* $1\le\alpha\le 2m$,
where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(4m)t_{4m}=n$.

Proof. If we take $i=\alpha+1$, $j=\alpha$ such that $1\le\alpha\le 2m$ for case (i)., $i=1$, $j=4m$ for case (ii) and $i=j=\alpha$ such that $1\le\alpha\le 2m$ for case (iii) in Theorem 2.1 , then we can directly see the conclusions from equations (3) and (4). □

Definition 2.3. A $u\times v$ real matrix $M=\left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row) contains exactly two non-zero entries.

Let u_1, u_2, …,u_{mn} be row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k}\neq 0, m_{j,k}\neq 0$ and $i\neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j+m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

If M is a real matrix of order $\alpha>1$ and N is a contraction of M, then $per(M)=per(N)$ which was proved in [1].

Now we consider relationships between the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind and the permanents of certain matrices which are obtained by using the generating matrices of these sequences.

Let $u\geqslant 4m$ be a positive integer and suppose that $M_m^{1,u}=\left[m_{i,j}^{1,u,m}\right]$ and $M_m^{2,u}=\left[m_{i,j}^{2,u,m}\right]$ are the $u\times u$ super-diagonal matrices, defined by

$$M_m^{1,u}=\begin{array}{c}\begin{array}{ccccccccccccc} & & & (2m)\,\text{th} & & & & (4m)\,\text{th} & & & & & \\ & & & \downarrow & & & & \downarrow & & & & & \end{array}\\ \left[\begin{array}{ccccccccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 0\\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0\\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2\\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \end{array}\right]_{u\times u}\end{array}$$

and

$$M_m^{2,u}=\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\
\vdots & & \ddots & \ddots & \ddots & & \ddots & & \ddots & & \ddots & & & \vdots\\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
-2 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & -2 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & \ddots & & \ddots & \vdots\\
0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0\\
1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0\\
0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \vdots\\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 1 & 0\\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0 & 1\\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 0
\end{bmatrix}_{u\times u}.$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$
$$(u-4m+1)\,\text{th} \qquad\qquad (u-2m+1)\,\text{th}$$

Theorem 2.4. *For $u \geqslant 4m$,*

$$per\left(M_m^{1,u}\right)=J_m^1(u+1) \text{ and } per\left(M_m^{2,u}\right)=J_m^2(u+4m).$$

Proof. Let us consider the matrix $M_m^{1,u}$ and the adjacency-Jacobsthal-Hurwitz sequence of the first kind. We use induction on u. Now we assume that the equation holds for $u \geq 4m$, then we show that the equation holds for $u+1$. If we expand the $per\left(M_m^{1,u+1}\right)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$per\left(M_m^{1,u+1}\right)=per\left(M_m^{1,u-2m+1}\right)-2per\left(M_m^{1,u-4m+1}\right).$$

Since $per\left(M_m^{1,u-2m+1}\right)=J_m^1(u-2m+2)$ and $per\left(M_m^{1,u-4m+1}\right)=J_m^1(u-4m+2)$, it is easy to see that $per\left(M_m^{1,u+1}\right)=J_m^1(u-2m+2)-2J_m^1(u-4m+2)=J_m^1(u+2)$. So we have the conclusion.

There is a similar proof for the matrix $M_m^{2,u}$ and the adjacency-Jacobsthal-Hurwitz sequence of the second kind. □

Let $v \geq 4m$ be a positive integer and suppose that the matrices $A_m^{1,v}=\left[a_{i,j}^{1,v,m}\right]_{v\times v}$ and $A_m^{2,v}=\left[a_{i,j}^{2,v,m}\right]_{v\times v}$ are defined, respectively, by

$$a_{i,j}^{1,v,m}=\begin{cases} 1 & \begin{array}{c}\text{if } i=\imath \text{ and } j=\imath+2m-1 \text{ for } 1\leq \imath\leq v-2m+1\\ \text{and}\\ i=\imath+1 \text{ and } j=\imath \text{ for } 1\leq\imath\leq v-2m,\end{array}\\ -1 & \text{if } i=1+\imath \text{ and } j=\imath \text{ for } v-2m+1\leq\imath\leq v-1,\\ -2 & \text{if } i=\imath \text{ and } j=4m+\imath-1 \text{ for } 1\leq\imath\leq v-4m+1,\\ 0 & \text{otherwise}\end{cases}$$

and

$$a_{i,j}^{2,v,m}=\begin{cases} 1 & \begin{array}{c}\text{if } i=\imath+2m-1 \text{ and } j=\imath+2m \text{ for } 1\leq \imath\leq v-2m\\ \text{and}\\ i=\imath+4m-1 \text{ and } j=\imath \text{ for } 1\leq\imath\leq v-4m+1,\end{array}\\ -1 & \text{if } i=\imath \text{ and } j=\imath+1 \text{ for } 1\leq\imath\leq 2m-1,\\ -2 & \text{if } i=\imath+2m-1 \text{ and } j=\imath \text{ for } 1\leq\imath\leq v-2m+1,\\ 0 & \text{otherwise.}\end{cases}$$

Then we can give the permanental representations other than the above by the following theorem.

Theorem 2.5. *For $v \geq 4m$,*

$$per\left(A_m^{1,v}\right) = -J_m^1(v+1) \text{ and } per\left(A_m^{2,v}\right) = -J_m^2(v+4m).$$

Proof. Let us consider the matrix $A_m^{2,v}$ and the adjacency-Jacobsthal-Hurwitz sequence of the second kind. The assertion may be proved by induction on v. Let the equation be hold for $v \geq 4m$, then we show that the equation holds for $v+1$. If we expand the $per\left(A_m^{2,v+1}\right)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned} per\left(A_m^{2,v+1}\right) &= per\left(A_m^{2,v-4m+1}\right) - 2per\left(A_m^{2,v-2m+1}\right) \\ &= -J_m^2(v+1) - 2\left(-J_m^2(v+2m+1)\right) = -J_m^2(v+4m+1). \end{aligned}$$

Thus we have the conclusion.

There is a similar proof for the matrix $A_m^{1,v}$ and the adjacency-Jacobsthal-Hurwitz sequence of the first kind. □

Now we define a $v \times v$ matrix B_m^v as in the following form:

$$B_m^v = \begin{bmatrix} -1 & \cdots & \underset{\downarrow}{\overset{(v-4m)\text{ th}}{-1}} & 0 & \cdots & 0 \\ -1 & & & & & \\ 0 & & & A_m^{1,v-1} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix},$$

then we have the following result:

Corollary 2.6. *For $v > 4m+1$,*

$$perB_m^v = \sum_{i=1}^{v-1} J_m^1(i).$$

Proof. If we extend the $perB_m^v$ with respect to the first row, we obtain

$$per\ B_m^v = per\ B_m^{v-1} + perA_m^{1,v-1}.$$

From Theorem 2.4, Theorem 2.5 and induction on v, the proof follows directly. □

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $\det(M \circ K) = perM$, where $M \circ K$ denotes the Hadamard product of M and K.

Now assume that the matrices $T = \left[t_{i,j}\right]_{u \times u}$ and $S = \left[s_{i,j}\right]_{v \times v}$ are defined by

$$T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1 & -1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & 1 & -1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

Then we give relationships between the adjacency-Jacobsthal-Hurwitz sequences of the first and second kind and the determinants of the Hadamard products $M_m^{1,u} \circ T$, $A_m^{1,v} \circ T$, $M_m^{2,u} \circ S$ and $A_m^{2,v} \circ S$.

Theorem 2.7. *Let $u, v \geq 4m$, then*

$$\det\left(M_m^{1,u} \circ T\right) = J_m^1 (u+1),$$

$$\det\left(A_m^{1,v} \circ T\right) = -J_m^1 (v+1),$$

$$\det\left(M_m^{2,u} \circ S\right) = J_m^2 (u+4m)$$

and

$$\det\left(A_m^{2,v} \circ S\right) = -J_m^2 (v+4m).$$

Proof. Since $\det\left(M_m^{1,u} \circ T\right) = per\left(M_m^{1,u}\right)$, $\det\left(A_m^{1,v} \circ T\right) = per\left(A_m^{1,v}\right)$, $\det\left(M_m^{2,u} \circ S\right) = per\left(M_m^{2,u}\right)$ and $\det\left(A_m^{2,v} \circ S\right) = per\left(A_m^{2,u}\right)$ for $u, v \geq 4m$, by Theorem 2.4 and Theorem 2.5, we have the conclusion □

Now we concentrate on finding the Binet formulas for the adjacency-Jacobsthal numbers.

Clearly, the characteristic equations of the matrices $M_m^{1,u}$ and $M_m^{2,u}$ are

$$x^{4m} - x^{2m} + 2 = 0$$

and

$$x^{4m} + 2x^{2m} - 1 = 0,$$

respectively. It is easy to see that the above equations do not have multiple roots. Let $\left\{\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{4m}^{(1)}\right\}$ and $\left\{\beta_1^{(2)}, \beta_2^{(2)}, \ldots, \beta_{4m}^{(2)}\right\}$ be the sets of the eigenvalues of the matrices $M_m^{1,u}$ and $M_m^{2,u}$, respectively and let $V_m^{(\lambda)}$ be $(4m) \times (4m)$ Vandermonde matrix as follows:

$$V_m^{(\lambda)} = \begin{bmatrix} \left(\beta_1^{(\lambda)}\right)^{4m-1} & \left(\beta_2^{(\lambda)}\right)^{4m-1} & \cdots & \left(\beta_{4m}^{(\lambda)}\right)^{4m-1} \\ \left(\beta_1^{(\lambda)}\right)^{4m-2} & \left(\beta_2^{(\lambda)}\right)^{4m-2} & \cdots & \left(\beta_{4m}^{(\lambda)}\right)^{4m-2} \\ \vdots & \vdots & & \vdots \\ \beta_1^{(\lambda)} & \beta_2^{(\lambda)} & & \beta_{4m}^{(\lambda)} \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $\lambda = 1, 2$. Now assume that

$$W_m^{(\lambda)}(i) = \begin{bmatrix} \left(\beta_1^{(\lambda)}\right)^{\alpha+4m-i} \\ \left(\beta_2^{(\lambda)}\right)^{\alpha+4m-i} \\ \vdots \\ \left(\beta_{4m}^{(\lambda)}\right)^{\alpha+4m-i} \end{bmatrix}$$

and $V_m^{(\lambda)}(i, j)$ is a $(4m) \times (4m)$ matrix obtained from $V_m^{(\lambda)}$ by replacing the jth column of $V_m^{(\lambda)}$ by $W_m^{(\lambda)}(i)$.

Theorem 2.8. *For $\alpha \geq 1$ and $\lambda = 1, 2$,*

$$c_{i,j}^{m,\lambda,\alpha} = \frac{\det\left(V_m^{(\lambda)}(i,j)\right)}{\det\left(V_m^{(\lambda)}\right)},$$

where $\left(C_m^{\lambda}\right)^{\alpha} = \left[c_{i,j}^{m,\lambda,\alpha}\right]$.

Proof. Let us consider λ as 1. Since the equation $x^{4m} - x^{2m} + 2 = 0$ does not have multiple roots, $\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{4m}^{(1)}$ are distinct and so the matrix $M_m^{1,u}$ is diagonalizable. Then, it is readily seen that $C_m^1 V_m^{(1)} = V_m^{(1)} \Omega_m^1$, where $\Omega_m^1 = \left(\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{4m}^{(1)}\right)$. Since the Vandermonde matrix $V_m^{(1)}$ is invertible, we can write $\left(V_m^{(1)}\right)^{-1} C_m^1 V_m^{(1)} = \Omega_m^1$. Thus, we easily see that the matrix C_m^1 is similar to Ω_m^1. Then, we have $\left(C_m^1\right)^{\alpha} V_m^{(1)} = V_m^{(1)} \left(\Omega_m^1\right)^{\alpha}$ for $\alpha \geq 1$. So we obtain the following linear system of equations:

$$\begin{cases} c_{i,1}^{m,1,\alpha}\left(\beta_1^{(1)}\right)^{4m-1} + c_{i,2}^{m,1,\alpha}\left(\beta_1^{(1)}\right)^{4m-2} + \cdots + c_{i,4m}^{m,1,\alpha} = \left(\beta_1^{(1)}\right)^{\alpha+4m-i} \\ c_{i,1}^{m,1,\alpha}\left(\beta_2^{(1)}\right)^{4m-1} + c_{i,2}^{m,1,\alpha}\left(\beta_2^{(1)}\right)^{4m-2} + \cdots + c_{i,4m}^{m,1,\alpha} = \left(\beta_2^{(1)}\right)^{\alpha+4m-i} \\ \vdots \\ c_{i,1}^{m,1,\alpha}\left(\beta_{4m}^{(1)}\right)^{4m-1} + c_{i,2}^{m,1,\alpha}\left(\beta_{4m}^{(1)}\right)^{4m-2} + \cdots + c_{i,4m}^{m,1,\alpha} = \left(\beta_{4m}^{(1)}\right)^{\alpha+4m-i} \end{cases}.$$

Then, for each $i, j = 1, 2, \ldots, 4m$, we derive $c_{i,j}^{m,1,\alpha}$ as

$$\frac{\det\left(V_m^{(1)}(i,j)\right)}{\det\left(V_m^{(1)}\right)}.$$

There is a similar proof for $\lambda = 2$. □

As an immediate consequence of this we have

Corollary 2.9. *For $\alpha \geq 1$,*

$$J_m^1(\alpha) = \frac{\det\left(V_m^1(k+1,k)\right)}{\det\left(V_m^{(1)}\right)} \text{ for } 1 \leq k \leq 2m\,,$$

$$J_m^1(\alpha) = -\frac{\det\left(V_m^1(1,4m)\right)}{2\det\left(V_m^{(1)}\right)},$$

and

$$J_m^2(\alpha) = \frac{\det\left(V_m^2(k,k)\right)}{\det\left(V_m^{(2)}\right)} \text{ for } 1 \leq k \leq 2m.$$

References

[1] R.A. Brualdi, P.M. Gibson, Convex polyhedra of doubly stochastic matrices I: applications of permanent function, J. Combin. Theory, Series A 22 (1977) 194–230.
[2] W.Y.C. Chen, J.D. Louck, The combinatorial power of the companion matrix, Linear Algebra Appl. 232 (1996) 261–278.
[3] O. Deveci, The Pell-circulant sequences and their applications, Maejo Int. J. Sci. Technol. 10 (2016) 284–293.
[4] O. Deveci, E. Karaduman, The Lehmer sequences in finite groups, Ukrainian Math. J. 68 (2016) 193–202.
[5] O Deveci, G. Artun, The adjacency-Jacobsthal numbers, submitted.

[6] G.B. Djordjevic, Generalizations of the Fibonacci and Lucas polynomials, Filomat 23:3 (2009) 291–301.
[7] D.D. Frey, J.A. Sellers, Jacobsthal numbers and alternating sign matrices, J. Integer Seq. 3, Article 00.2.3, 2000.
[8] N. Gogin, A.A. Myllari, The Fibonacci-Padovan sequence and MacWilliams transform matrices, Programing and Computer Software, published in Programmirovanie 33:2 (2007) 74–79.
[9] A. Huwitz, Ueber die Bedingungen unter welchen eine gleichung nur Wurzeln mit negative reellen teilen besitzt, Math. Ann. 46 (1895) 273–284.
[10] D. Kalman, Generalized Fibonacci numbers by matrix methods, Fibonacci Quart. 20 (1982) 73–76.
[11] E. Kilic, The Binet formula, sums and representations of generalized Fibonacci p-numbers, European J. Combin. 29 (2008) 701–711.
[12] E.G. Kocer, N. Tuglu, A.P. Stakhov, On the m-extension of the Fibonacci and Lucas p-numbers, Chaos, Solitons Fractals 40 (2009) 1890–1906.
[13] P. Lancaster, M. Tismenetsky, The Theory of Matrices (2nd ed.): with Applications, Elsevier, 1985.
[14] G-Y. Lee, k-Lucas numbers and associated bipartite graphs, Linear Algebra Appl. 320 (2000) 51–61.
[15] R. Lidl, H. Niederreiter, Introduction to Finite Fields and their Applications, Cambridge Univ Press, 1994.
[16] K. Lü, J. Wang, k-Step Fibonacci sequence modulo m, Util. Math. 71 (2006) 169–177.
[17] A.G. Shannon, L. Bernstein, The Jacobi-Perron algorithm and the algebra of recursive sequences, Bull. Australian Math. Soc. 8 (1973) 261–277.
[18] A.G. Shannon, A.F. Horadam, P.G. Anderson, The auxiliary equation associated with plastic number, Notes Number Theory Disc. Math. 12 (2006) 1-12.
[19] D. Tasci, M.C. Firengiz, Incomplete Fibonacci and Lucas p-numbers, Math. Comput. Modell. 52 (2010) 1763–1770.