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On Jensen's Type Inequalities via Generalized Majorization Inequalities

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Abstract. In this paper, we give generalizations of Jensen's, Jensen-Steffensen's and converse of Jensen's inequalities by using generalized majorization inequalities. We also present Grüss and Ostrowski-type inequalities for the generalized inequalities.

1. Introduction

The convex functions are closely related with the theory of inequalities and many important inequalities are the consequences of convex functions.

Definition 1.1. ([21, p.1]) The function $f : [a, b] \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \tag{1}$$

holds, for all $x_1, x_1 \in [a, b]$ and each $t \in [0, 1]$. The function f is said to be strictly convex if the inequality in (1) strictly holds for each $x_1 \neq x_2$ and $t \in (0, 1)$. The function f is called concave if the reverse inequality in (1) holds.

One of the most important inequality in Mathematics and Statistics is the Jensen inequality. This inequality was given by J. Jensen in 1906 (see [21, p.43]).

Theorem 1.2. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be a positive n-tuple. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{2}$$

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where

$$W_k = \sum_{i=1}^{k} w_i, \quad k = 1, \ldots, n. \tag{3}$$

If f is strictly convex, then inequality (2) *is strict unless $x_1 = \cdots = x_n$.*

The condition "$\mathbf{w}$ is a positive n-tuple" can be replaced by "$\mathbf{w}$ is a non-negative n-tuple and $W_n > 0$". Note that the Jensen inequality (2) can be used as an alternative definition of convexity.

It is reasonable to ask whether the condition "$\mathbf{w}$ is a non-negative n-tuple" can be relaxed at the expense of restricting $\mathbf{x}$ more severely. An answer to this question was given by Steffensen [23] (see also [21, p.57]).

Theorem 1.3. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{w} = (w_1, \ldots, w_n)$ a real n-tuple such that*

$$0 \leq W_k \leq W_n, \quad k = 1, \ldots, n-1, \quad W_n > 0,$$

is satisfied, where W_k are as in (3)*, then* (2) *holds. If f is strictly convex, then inequality* (2) *is strict unless $x_1 = \cdots = x_n$.*

Inequality (2) under conditions from Theorem 1.3 is called the Jensen-Steffensen inequality.

Now we give some basic introduction to majorization:

We say that the $m-$tuple $\mathbf{x}$ majorizes the $m-$tuple $\mathbf{y}$ when the sum of k largest entries of $\mathbf{y}$ does not exceed the sum of k largest entries of $\mathbf{x}$ for all $k = 1, 2, .., m-1$ with equality for $k = m$ and we write as $\mathbf{y} \prec \mathbf{x}$. A mathematical origin of majorization is illustrated by the work of Schur [22] on Hadamard's determinant inequality. Many mathematical characterization problems are known to have solutions that involve majorization. A complete and superb reference on the subject are the books [12], [19]. The comprehensive survey by Ando [11] provides alternative derivations, generalizations and a different viewpoint.

The following theorem known as the majorization theorem and its convenient proof is given by Marshall, Olkin and Arnold in [19].

Theorem 1.4. *Let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ be two $m-$tuples such that x_i, $y_i \in [a, b]$, for $i = 1, 2, ..., m$. Then for any continuous convex function $f : [a, b] \to \mathbb{R}$ the inequality*

$$\sum_{i=1}^{m} f(y_i) \leq \sum_{i=1}^{m} f(x_i)$$

holds if and only if $\mathbf{y} \prec \mathbf{x}$.

The following theorem can be regarded as the generalization of Theorem 1.4, known as weighted majorization theorem and is proved by Fuchs in [15].

Theorem 1.5. *Let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ be two decreasing real $m-$tuples with $x_i, y_i \in [a, b]$, for $i = 1, 2, ..., m$. Let $\mathbf{w} = (w_1, ..., w_m)$ be real $m-$tuple such that*

$$\sum_{i=1}^{l} w_i y_i \leq \sum_{i=1}^{l} w_i x_i \ \text{for} \quad l = 1, 2, ..., m-1 \tag{4}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{5}$$

Then for every continuous convex function $f : [a, b] \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} w_i f(y_i) \leq \sum_{i=1}^{m} w_i f(x_i). \tag{6}$$

The following theorem is a consequence of Theorem 1.5.

Theorem 1.6. *Let* $x, y : [a, b] \to [\alpha, \beta]$ *be decreasing continuous functions and* $w : [a, b] \to \mathbb{R}$ *be continuous function. If*

$$\int_a^v w(t)\, y(t)\, dt \leq \int_a^v w(t)\, x(t)\, dt \quad \text{for every} \quad v \in [a, b], \tag{7}$$

and

$$\int_a^b w(t)\, y\,(t)\ dt = \int_a^b w\,(t)\ x\,(t)\ dt \tag{8}$$

hold, then for every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$, *we have*

$$\int_a^b w(t)\, f\,(y(t))\ dt \leq \int_a^b w(t)\, f\,(x(t))\ dt. \tag{9}$$

For some more recent results, related to generalizations and refinements of majorization theorem, see [1–6, 9, 16] and some of the references in them.
In our main results we will use generalized result for n-convex function, therefore here we recall the definition of n-convexity (see for example [21]).

Definition 1.7. The divided difference of order n, $n \in \mathbb{N}$, of the function $f : [a, b] \to \mathbb{R}$ at mutually different points $x_0, x_1, ..., x_n \in [a, b]$ is defined recursively by

$$[x_i; f] = f(x_i), \quad i = 0, ..., n$$
$$[x_0, ..., x_n; f] = \frac{[x_1, ..., x_n; f] - [x_0, ..., x_{n-1}; f]}{x_n - x_0}.$$

The value $[x_0, ..., x_n; f]$ is independent of the order of the points $x_0, ..., x_n$.

This definition may be extended to include the case in which some or all the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$[\underbrace{x, ..., x}_{j\text{-times}}; f] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

Definition 1.8. A function $f : [a, b] \to \mathbb{R}$ is n-convex, $n \geq 0$, if for all choices of $n + 1$ distinct points $x_i \in [a, b]$, $i = 0, ..., n$, the inequality

$$[x_0, x_1, ..., x_n; f] \geq 0$$

holds.

Theorem 1.9. ([21, p. 16]) *Let* $f : [\alpha, \beta] \to \mathbb{R}$ *be a function such that* $f^{(n)}$ *exists, then* f *is* n*-convex if and only if* $f^{(n)} \geq 0$.

From Definition 1.8, it follows that 2-convex functions are just convex functions. Furthermore, 1-convex functions are increasing functions and 0-convex functions are nonnegative functions. Consider the Green function G defined on $[a, b] \times [a, b]$ by

$$G(t, s) = \begin{cases} \frac{(t-b)(s-a)}{b-a}, & a \leq s \leq t; \\ \frac{(s-b)(t-a)}{b-a}, & t \leq s \leq b. \end{cases} \tag{10}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $f : [a,b] \to \mathbb{R}$, $f \in C^2([a,b])$, we can easily show by integrating by parts that the following is valid

$$f(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) + \int_a^b G(x,s) f''(s) ds, \tag{11}$$

where the function G is defined as above in (10) ([24]).

The following generalized Montgomery identity via Taylor's formula is given in [7, 10].

Proposition 1.10. *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$ and $a < b$. Then the following identity holds*

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!\,(k+2)} \frac{(x-a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!\,(k+2)} \frac{(x-b)^{k+2}}{b-a}$$
$$+ \frac{1}{(n-1)!} \int_a^b T_n(x,s) f^{(n)}(s)\, ds, \tag{12}$$

where

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1}, & a \le s \le x, \\ -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1}, & x < s \le b. \end{cases} \tag{13}$$

In case $n = 1$ the sum $\sum_{k=0}^{n-2} \ldots$ is empty, so the identity (12) reduces to the well-known Montgomery identity

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P(x,s) f'(s)\, ds,$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s) = \begin{cases} \frac{s-a}{b-a}, & a \le s \le x, \\ \frac{s-b}{b-a}, & x < s \le b. \end{cases}$$

The following generalizations of majorization theorem by Montgomery identity and Green function are given in [8]. To make the calculations simple they used the following notations.

$$\triangle(w_i, x_i, y_i, f) = \sum_{i=1}^m w_i f(y_i) - \sum_{i=1}^m w_i f(x_i), \tag{14}$$

where w_i, x_i, and f are as defined in Theorem 1.5. Also

$$\Lambda(w,x,y,f) = \int_a^b w(u) f(y(u)) du - \int_a^b w(u) f(x(u)) du, \tag{15}$$

where w, x, y and f are as defined in Theorem 1.6.

Theorem 1.11. ([8]) *Suppose all the assumptions of Theorem 1.5 are valid. Also let $n \in N$, $f : I \to \mathbb{R}$ be function such that $f^{(n-1)}(n > 3)$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$, n is even, f is $n-$convex and $G(.,s)$ be as defined in (10) then for all $s \in [a,b]$, the following inequalities hold.*

(i)

$$\triangle(w_i, x_i, y_i, f) \geq \frac{f(b)-f(a)}{b-a}\triangle(w_i, x_i, y_i, id) + \frac{f'(a)-f'(b)}{b-a}\times \int_a^b \triangle(w_i, x_i, y_i, G(.,s))ds + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \triangle(w_i, x_i, y_i, G(.,s))\times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds, \tag{16}$$

(ii)

$$\triangle(w_i, x_i, y_i, f) \geq \frac{f(b)-f(a)}{b-a}\triangle(w_i, x_i, y_i, id) + \frac{f'(b)-f'(a)}{b-a}\times \int_a^b \triangle(w_i, x_i, y_i, G(.,s))ds + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \triangle(w_i, x_i, y_i, G(.,s))\times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds. \tag{17}$$

2. Generalizations of Jensen's Inequality

First we introduce some notations which we will use in the rest of paper.

$$\nabla(\overline{x}, f) := \frac{1}{W_m}\sum_{i=1}^{m} w_i f(x_i) - f(\overline{x}),$$

where w_i, x_i and f are as given in Theorem 1.5. Also if $W = \int_a^b w(t)dt$ and $\overline{x} = \frac{\int_a^b x(t)w(t)dt}{W}$, we denote

$$\Upsilon(\overline{x}, f) := \frac{1}{W}\int_a^b w(t)f(x(t))dt - f(\overline{x}),$$

where w, x and f are as given in Theorem 1.6.

We give our first main result in the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $\mathbf{x} = (x_1, ..., x_m)$ be $m-$tuple with $x_i \in [a, b]$ and $\mathbf{w} = (w_1, ...w_m)$ be positive real $m-$tuple, $W_m = \sum_{i=1}^{m} w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$ and G be the Green function as defined in (10).*

(i) If $\mathbf{x}$ is decreasing $m-$tuple and $f : [a, b] \to \mathbb{R}$ is $2n-$convex function, then the following inequalities hold.

$$\nabla(\bar{x}, f) \geq \frac{f'(a)-f'(b)}{b-a}\int_a^b \nabla(\bar{x}, G(.,s))ds + \sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b \nabla(\overline{x}, G(.,s))\times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds, \tag{18}$$

$$\nabla(\bar{x}, f) \geq \frac{f'(b) - f'(a)}{b-a} \int_a^b \nabla(\bar{x}, G(.,s))ds + \sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!} \int_a^b \nabla(\bar{x}, G(.,s)) \times$$
$$\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \tag{19}$$

(ii) If the inequalities (18) and (19) hold and the functions L_1 and L_2 defined by

$$L_1(.) = \frac{f'(a) - f'(b)}{b-a} \int_a^b G(.,s)ds + \sum_{k=2}^{2n-2} \frac{k}{(k-1)!} \int_a^b G(.,s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds, \tag{20}$$

$$L_2(.) = \frac{f'(b) - f'(a)}{b-a} \int_a^b G(.,s)ds + \sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!} \int_a^b G(.,s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds, \tag{21}$$

are convex, then the right hand sides of (18) and (19) are non-negative and

$$f(\bar{x}) \leq \frac{1}{W_m} \sum_{i=1}^{m} w_i f(x_i), \tag{22}$$

holds in both cases.

Proof. (i) Let k be the largest number from $\{1, ..., m\}$ such that $x_k \geq \bar{x}$, then as $\mathbf{x}$ is decreasing m-tuple so we have $x_l \geq \bar{x}$ for $l = 1, 2, ..., k$ and $x_l \leq \bar{x}$ for $l = k+1, k+2, ..., m$.

Now as $x_l \geq \bar{x}$ for $l = 1, 2, ..., k$, so we have

$$\sum_{i=1}^{l} w_i \bar{x} \leq \sum_{i=1}^{l} w_i x_i \quad \text{for} \quad l = 1, 2, ..., k. \tag{23}$$

Similarly as $x_l \leq \bar{x}$ for $l = k+1, k+2, ..., m$, so we have

$$\sum_{i=k+1}^{j} w_i x_i \leq \sum_{i=k+1}^{j} w_i \bar{x} \quad \text{for } j = k+1, k+2, ..., m.$$

Hence

$$\sum_{i=1}^{j} w_i x_i = \sum_{i=1}^{m} w_i x_i - \sum_{i=j+1}^{m} w_i x_i \geq \sum_{i=1}^{m} w_i \bar{x} - \sum_{i=j+1}^{m} w_i \bar{x} = \sum_{i=1}^{j} w_i \bar{x}, \quad \text{for} \quad j = k+1, k+2, ..., m. \tag{24}$$

Using (23) and (24) we get that

$$\sum_{i=1}^{l} w_i \bar{x} \leq \sum_{i=1}^{l} w_i x_i, \text{ for all } l = 1, 2, ..., m-1$$

and obviously

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} w_i x_i.$$

The conditions (4) and (5) are satisfied for $\bar{\mathbf{x}} = (\overline{x}, ..., \overline{x})$ and $\mathbf{y} = (x_1, ..., x_m)$. Also

$$\nabla(\bar{\mathbf{x}}, id) = 0,$$

therefore substituting $\mathbf{y} = (x_1, ..., x_m)$ and $\mathbf{x} = (\overline{x}, ..., \overline{x})$ in Theorem 1.11 (i) we get (18). Proceeding similarly and using Theorem 1.11(ii), we obtain (19).

(ii) We may write the right hand side of (18) as

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i L_1(x_i) - L_1(\overline{x}).$$

Since L_1 is convex so by Jensen's inequality, we have

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i L_1(x_i) - L_1(\overline{x}) \geq 0.$$

Hence (22) holds. Analogously, we obtain (22) for L_2. □

In the following theorem we give integral version of Theorem 2.1.

Theorem 2.2. *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $x : [a, b] \to \mathbb{R}$ be continuous function such that $x([a, b]) \subseteq I$, $w : [a, b] \to \mathbb{R}$ be positive continuous function with $w(a) \neq w(b)$, $W = \int_a^b w(t)dt$, $\overline{x} = \frac{\int_a^b x(t)w(t)dt}{W}$ and G be the Green function as defined in (10).*

(i) If x is decreasing and $f : [a, b] \to \mathbb{R}$ is $2n-$convex functions, then the following inequalities hold.

$$\Upsilon(\bar{x}, f) \geq \frac{f'(a) - f'(b)}{b - a} \int_a^b \Upsilon(\bar{x}, G(., s))ds + \sum_{k=2}^{2n-2} \frac{k}{(k-1)!} \int_a^b \Upsilon(\overline{x}, G(., s)) \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds, \tag{25}$$

$$\Upsilon(\overline{x}, f) \geq \frac{f'(b) - f'(a)}{b - a} \int_a^b \Upsilon(\overline{x}, G(., s))ds + \sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!} \int_a^b \Upsilon(\overline{x}, G(., s)) \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \tag{26}$$

(ii) If the inequalities (25) and (26) hold and the functions L_1 and L_2 defined as in (20) and (21) respectively are convex, then the right hand sides of (25) and (26) are non-negative and

$$f(\overline{x}) \leq \frac{\int_a^b w(t) f(x(t))\, dt}{W}, \tag{27}$$

holds in both cases.

Remark 2.3. If we take $x(t) = t$, $w(t) = 1$, in the inequality (25) and (26) then we obtain the generalizations of Hermite-Hadamard inequality.

3. Generalizations of Jensen-Steffensen's Inequality

Theorem 3.1. *Let* $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(2n-1)}$ *is absolutely continuous,* $\mathbf{x} = (x_1, ..., x_m) \in [a,b]^m$ *be decreasing* $m-$*tuple. Let* $\mathbf{w} = (w_1, ..., w_m)$ *be real* $m-$*tuple such that* $0 \le W_k \le W_m$ $(k = 1, 2, ..., m)$, $W_m > 0$ *where* $W_k = \sum_{i=1}^{k} w_i$, $\overline{x} = \frac{1}{W_m} \sum_{i=1}^{m} w_i x_i$ *and G be the Green function as defined in (10).*

(i) Then for $2n-$*convex function* f*, the inequalities (18) and (19) hold.*

(ii) If the inequalities (18) and (19) hold and the functions L_1 *and* L_2 *defined as in (20) and (21) are convex, then the right hand sides of (18) and (19) are non-negative and (22) holds.*

Proof. (i) Let k be the largest number $\{1, 2, ..., m\}$ such that $x_k \ge \overline{x}$ then $x_l \ge \overline{x}$ for $l = 1, ..., k$, and we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1} (x_i - x_{i+1}) W_i \ge 0$$

and so we obtain

$$\sum_{i=1}^{l} w_i \overline{x} = W_l \overline{x} \le W_l x_l \le \sum_{i=1}^{l} x_i w_i. \tag{28}$$

Also for $l = k+1, ..., m$ we have $x_{k+1} < \overline{x}$, therefore

$$x_l (W_m - W_l) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m} (x_{i-1} - x_i)(W_m - W_{i-1}) \ge 0.$$

Hence, we conclude that

$$\sum_{i=l+1}^{m} w_i \overline{x} = (W_m - W_l)\overline{x} > (W_m - W_l) x_l \ge \sum_{i=l+1}^{m} w_i x_i. \tag{29}$$

From (28) and (29), we get

$$\sum_{i=1}^{l} w_i \overline{x} \le \sum_{i=1}^{l} x_i w_i \quad for \ \ all \ \ l = 1, 2, ...m-1.$$

Obviously the equality

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} x_i w_i$$

holds. The conditions (4) and (5) are satisfied. Also

$$\nabla(\bar{x}, id) = 0,$$

therefore using Theorem 1.11 (i) for $\mathbf{y} = (x_1, ..., x_m)$ and $\mathbf{x} = (\overline{x}, ..., \overline{x})$, we get (18). Proceeding similarly using Theorem 1.11(ii), we obtain (19).

(ii) The proof is similar to the proof of Theorem 2.1(ii). □

The integral version of above theorem is given here.

Theorem 3.2. *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $x : [a, b] \to \mathbb{R}$ be continuous decreasing function such that $x([a, b]) \subseteq I$, $w : [a, b] \to \mathbb{R}$ is either continuous or of bounded variation with $w(a) \le w(t) \le w(b)$ for all $t \in [a, b]$, $\bar{x} = \frac{\int_a^b x(t)w(t)d(t)}{\int_a^b w(t)dt}$ and G be the Green function as defined in (10).*

(i) Then for any $2n-$convex function f, the inequalities (25) and (26) hold.

(ii) If the inequalities (25) and (26) hold and the functions L_1 and L_2 defined as in (20) and (21) respectively, are convex, then the right hand sides of (25) and (26) are non-negative and (27) holds.

4. Generalization of Converse of Jensen's Inequality

Theorem 4.1. *Let $n \in \mathbb{N}$, $f : [a, b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous. Let $\mathbf{x} = (x_1, ..., x_r)$ be real $r-$tuple with $x_i \in [m, M] \subseteq [a, b]$, $i = 1, 2, ..., r$, $\mathbf{w} = (w_1, ...w_r)$ be positive $r-$tuple, $W_r = \sum_{i=1}^{r} w_i$, $\bar{x} = \frac{1}{W_r} \sum_{i=1}^{r} w_i x_i$ and G be the Green function as defined in (10).*

(i) Then for any $2n-$convex function $f : [a, b] \to \mathbb{R}$, the following inequalities hold:

$$
\begin{aligned}
&\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \le \frac{\bar{x}-m}{M-m} f(M) + \frac{M-\bar{x}}{M-m} f(m) \\
&+\frac{f'(b)-f'(a)}{b-a}\int_a^b \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{1}{W_r}\sum_{i=1}^{r} w_i G(x_i,s)\right] ds \\
&-\sum_{k=2}^{2n-2} \frac{k}{(k-1)!}\int_a^b \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{1}{W_r}\sum_{i=1}^{r} w_i G(x_i,s)\right] \times \\
&\quad \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds,
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \le \frac{\bar{x}-m}{M-m} f(M) + \frac{M-\bar{x}}{M-m} f(m) \\
&+\frac{f'(a)-f'(b)}{b-a}\int_a^b \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{1}{W_r}\sum_{i=1}^{r} w_i G(x_i,s)\right] ds \\
&-\sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!}\int_a^b \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{1}{W_r}\sum_{i=1}^{r} w_i G(x_i,s)\right] \times \\
&\quad \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds.
\end{aligned} \tag{31}
$$

(ii) If the inequalities (30) and (31) hold and the functions L_1 and L_2 defined as in (20) and (21) respectively, are convex then the inequality

$$
\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \le \frac{\bar{x}-m}{M-m} f(M) + \frac{M-\bar{x}}{M-m} f(m),
$$

holds in both cases.

Proof. (i) Putting $m = 2$, $x_1 = M$, $x_2 = m$, $w_1 = \frac{x_i - m}{M - m}$ and $w_2 = \frac{M - x_i}{M - m}$ in (18), we have

$$f(x_i) \leq \frac{x_i - m}{M - m} f(M) + \frac{M - x_i}{M - m} f(m) + \frac{f'(b) - f'(a)}{b - a} \int_a^b \left[\frac{x_i - m}{M - m} G(M, s) + \frac{M - x_i}{M - m} G(m, s) - G(x_i, s) \right] ds - \sum_{k=2}^{2n-2} \frac{k}{(k-1)!} \int_a^b \left[\frac{x_i - m}{M - m} G(M, s) + \frac{M - x_i}{M - m} G(m, s) - G(x_i, s) \right] \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b - a} ds. \tag{32}$$

Multiplying (32) with w_i, dividing by W_r and taking the summation from $i = 1$ to r, we get (30). Proceeding similarly we obtain (31).

(ii) Using similar arguments as in the proof of Theorem 2.1(ii), we get the required result. □

Remark 4.2. In Theorem 4.1, assume that $x_0, \sum_{i=1}^{r} w_i x_i \in [m, M]$ with $x_0 \neq \sum_{i=1}^{r} w_i x_i$ and $(x_i - x_0)\left(\sum_{i=1}^{r} w_i x_i - x_i\right) \geq 0, i = 1, 2, ,.., r$. If $x_0 < \sum_{i=1}^{r} w_i x_i$, then by taking $m = x_0$ and $M = \sum_{i=1}^{r} w_i x_i$, in inequalities (30) and (31), we obtain the generalizations of Giaccardi inequality. Similarly if $x_0 > \sum_{i=1}^{r} w_i x_i$, then by taking $M = x_0$ and $m = \sum_{i=1}^{r} w_i x_i$, in inequalities (30) and (31), we obtain the generalizations of Giaccardi inequality. Moreover, if we take $m = x_0 = 0$ in the generalized Giaccardi inequalities we obtain generalizations of Jensen-Petrović's inequalities.

The integral version of the above theorem can be stated as:

Theorem 4.3. *Let $n \in \mathbb{N}$, $f : [\alpha, \beta] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $x : [a, b] \to \mathbb{R}$ be continuous function such that $x([a, b]) \subseteq [m, M] \subseteq [\alpha, \beta]$, $w : [a, b] \to \mathbb{R}$ be positive bounded function with $w(a) \neq w(b)$, $W = \int_a^b w(t)dt$, $\overline{x} = \frac{\int_a^b x(t)w(t)dt}{W}$ and G be the Green function as defined in (10).*

(i) Then for any $2n-$convex function $f : [a, b] \to \mathbb{R}$, the following inequalities hold.

$$\frac{\int_a^b f(x(t))w(t)dt}{W} \leq \frac{\overline{x} - m}{M - m} f(M) + \frac{M - \overline{x}}{M - m} f(m) + \frac{f'(b) - f'(a)}{b - a} \times \int_a^b \left[\frac{\overline{x} - m}{M - m} G(M, s) + \frac{M - \overline{x}}{M - m} G(m, s) - \frac{1}{W} \int_a^b w(t) G(x(t), s) dt \right] ds - \sum_{k=2}^{2n-2} \frac{k}{(k-1)!} \int_a^b \left[\frac{\overline{x} - m}{M - m} G(M, s) + \frac{M - \overline{x}}{M - m} G(m, s) - \frac{1}{W} \int_a^b w(t) G(x(t), s) dt \right] \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b - a} ds, \tag{33}$$

$$\frac{\int_a^b f(x(t))w(t)dt}{W} \leq \frac{\overline{x}-m}{M-m}f(M) + \frac{M-\overline{x}}{M-m}f(m) + \frac{f'(a)-f'(b)}{b-a}\times$$
$$\int_a^b \Big[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W}\int_a^b w(t)\, G(x(t),s)dt\Big]ds$$
$$-\sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!}\int_a^b \Big[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W}\int_a^b w(t)\, G(x(t),s)dt\Big]\times$$
$$\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds. \tag{34}$$

(ii) If the inequalities (33) and (34) hold and the functions L_1 and L_2 defined as in (20) and (21) are convex, then the inequality

$$\frac{\int_a^b f(x(t))w(t)dt}{W} \leq \frac{\overline{x}-m}{M-m}f(M) + \frac{M-\overline{x}}{M-m}f(m),$$

holds in both cases.

5. Bounds for Identities Related to the Generalizations of Jensen's Inequality

For two Lebesgue integrable functions $\phi, \psi : [a,b] \to \mathbb{R}$, we consider Čebyšev functional

$$T(\phi,\psi) = \frac{1}{b-a}\int_a^b \phi(t)\psi(t)dt - \frac{1}{b-a}\int_a^b \phi(t)dt \frac{1}{b-a}\int_a^b \psi(t)dt. \tag{35}$$

The following results can be found in [14].

Theorem 5.1. *Let $\phi : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $\psi : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(.-a)(b-.)[\psi']^2 \in L[a,b]$. Then the inequality*

$$| T(\phi,\psi) | \leq \frac{1}{\sqrt{2}}[T(\phi,\phi)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}}\Big(\int_a^b (x-a)(b-x)[\psi'(x)]^2 dx\Big)^{\frac{1}{2}} \tag{36}$$

holds. The constant $\frac{1}{\sqrt{2}}$ in (36) is the best possible.

Theorem 5.2. *Suppose that $\phi : [a,b] \to \mathbb{R}$ is absolutely continuous with $\phi' \in L_\infty[a,b]$ and $\psi : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$. Then the inequality*

$$| T(\phi,\psi) | \leq \frac{1}{2(b-a)} \parallel \phi' \parallel_\infty \int_a^b (x-a)(b-x)d\psi(x) \tag{37}$$

holds. The constant $\frac{1}{2}$ in (37) is the best possible.

Let $\mathbf{w} = (w_1, ..., w_m)$ and $\mathbf{x} = (x_1, ..., x_m)$ be $m-$tuples with $x_i \in [a,b]$, $w_i \in \mathbb{R}$ $i = 1, ..., m$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i \in [a,b]$, $W_m \neq 0$ and the function T_n be defined as in (13). We denote

$$\widetilde{T}_{n-2}(t,s) = \begin{cases} \frac{1}{b-a}\left[\frac{(t-s)^{n-2}}{n-2} + (t-a)(t-s)^{n-3}\right], & a \leq s \leq t, \\ \frac{1}{b-a}\left[\frac{(t-s)^{n-2}}{n-2} + (t-b)(t-s)^{n-3}\right], & t < s \leq b. \end{cases} \tag{38}$$

$$\hbar(t) = \int_a^b \nabla(\overline{x}, G(., s))\tilde{T}_{n-2}(s,t)ds. \tag{39}$$

$$\aleph(t) = \int_a^b \nabla(\overline{x}, G(., s))T_{n-2}(s,t)ds. \tag{40}$$

Now, we are in the position to state the main results of this section:

Theorem 5.3. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$. Let $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,2,...,m$, $W_m = \sum_{i=1}^m w_i \neq 0$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i \in [a,b]$. Let the functions T_n, $\tilde{T}_n$, T, $\hbar$ and $\aleph$ be as defined in (13), (38), (35), (39) and (40) respectively. Then*

(i) the remainder $R_n^1(\overline{x}, f)$ defined by

$$R_n^1(\overline{x}, f) = \nabla(\bar{x}, f) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \nabla(\bar{x}, G(., s))ds$$
$$-\sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b \nabla(\overline{x}, G(., s))\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$-\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b \hbar(s)ds, \tag{41}$$

satisfies the estimation

$$\left|R_n^1(\overline{x}, f)\right| \leq \frac{1}{(n-3)!}\left(\frac{b-a}{2}\left|T(\hbar,\hbar)\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|\right)^{\frac{1}{2}}. \tag{42}$$

(ii) The remainder $R_n^2(\overline{x}, f)$ defined by

$$R_n^2(\overline{x}, f) = \nabla(\bar{x}, f) - \frac{f'(b)-f'(a)}{b-a}\int_a^b \nabla(\bar{x}, G(., s))ds$$
$$-\sum_{k=3}^{2n-2}\frac{k-2}{(k-1)!}\int_a^b \nabla(\overline{x}, G(., s))\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$-\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b \aleph(s)ds, \tag{43}$$

satisfies the estimation

$$\left|R_n^2(\overline{x}, f)\right| \leq \frac{1}{(n-3)!}\left(\frac{b-a}{2}\left|T(\aleph,\aleph)\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|\right)^{\frac{1}{2}}.$$

Proof. (i) Using (11) and (12) in the expression $\nabla(\bar{x}, f)$, we obtain

$$\nabla(\bar{x}, f) = \frac{f'(a)-f'(b)}{b-a}\int_a^b \nabla(\bar{x}, G(., s))ds + \sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\times$$
$$\int_a^b \nabla(\overline{x}, G(., s))\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$+\frac{1}{(n-3)!}\int_a^b \hbar(t)f^{(n)}(t)dt. \tag{44}$$

Comparing (41) and (44), we obtain

$$R_n^1(\overline{x}, f) = \frac{1}{(n-3)!}\int_a^b \hbar(t) f^{(n)}(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!\,(b-a)}\int_a^b \hbar(t)dt. \tag{45}$$

Now applying Theorem 5.1 for $\phi \to \hbar$ and $\psi \to f^{(n)}$ and using Čebyšev functional we obtain

$$\left|\frac{1}{b-a}\int_a^b \hbar(s) f^{(n)}(t)dt - \left(\frac{1}{b-a}\int_a^b \hbar(t)dt\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right)\right|$$
$$\leq \frac{1}{\sqrt{2}}[T(\hbar,\hbar)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right)^{\frac{1}{2}}. \tag{46}$$

Multiplying (46) with $(b-a)$ and dividing by $(n-3)!$ and using (45), we obtain (42).
(ii) Similar to the proof of (i). □

In the next theorem we obtain Grüss type inequality.

Theorem 5.4. *Let $n \in \mathbb{N}$, $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $f^{(n+1)} \geq 0$ on $[a, b]$ and let T_n, $\tilde{T}_n$, T, $\hbar$ and $\aleph$ be as defined in (13), (38), (35), (39) and (40) respectively. Then*

(i) the remainder $R_n^1(\overline{x}, f)$ defined by (41) satisfies the estimation

$$\left|R_n^1(\overline{x}, f)\right| \leq \frac{\|\hbar'\|_\infty}{(n-3)!}\left[\frac{(b-a)\left(f^{(n-1)}(b) + f^{(n-1)}(a)\right)}{2} - \left\{f^{(n-2)}(b) - f^{(n-2)}(a)\right\}\right]. \tag{47}$$

(ii) the remainder $R_n^2(\overline{x}, f)$ defined by (43) satisfies the estimation

$$\left|R_n^2(\overline{x}, f)\right| \leq \frac{\|\aleph'\|_\infty}{(n-3)!}\left[\frac{(b-a)\left(f^{(n-1)}(b) + f^{(n-1)}(a)\right)}{2} - \left\{f^{(n-2)}(b) - f^{(n-2)}(a)\right\}\right]$$

Proof. (i) Since (45) holds and applying Theorem 5.2 for $f \to \hbar$ and $g \to f^{(n)}$ and using Čebyšev functional, we get

$$\left|\frac{1}{b-a}\int_a^b \hbar(t) f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \hbar(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right|$$
$$\leq \frac{1}{2(b-a)}\|\hbar'\|_\infty \int_a^b (t-a)(b-t) f^{(n+1)}(t)dt. \tag{48}$$

Since

$$\int_a^b (t-a)(b-t) f^{(n+1)}(t)dt = (b-a)\left[f^{(n-1)}(b) + f^{(n-1)}(a)\right] - 2\left[f^{(n-2)}(b) - f^{(n-2)}(a)\right].$$

Therefore, from (45) and (48), we deduce (47).
(ii) Similar to the proof of (i). □

Here, the symbol $L_p[a, b]$ $(1 \leq p < \infty)$ denotes the space of $p-$power integrable functions on the interval $[a, b]$ equipped with the norm

$$\| f \|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \quad \text{for all } f \in L_p[a, b],$$

and space of essentially bounded functions on $[a,b]$, denoted by $L_\infty[a,b]$, with the norm

$$\| f \|_\infty = ess \sup_{t\in[a,b]} |f(t)|.$$

Now we present the Ostrowski type inequalities related to the generalized Jensen's inequalities.

Theorem 5.5. *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p[a,b]$, $\mathbf{x} = (x_1,...,x_m) \in [a,b]^m$ be $m-$tuple, $\mathbf{w} = (w_1,...,w_m)$ be real $m-$tuple, $W_m = \sum_{i=1}^m w_i \neq 0$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i \in [a,b]$ and G be the Green function as defined in (10). Let (p,q) be a pair of conjugate exponents, that is, $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q} = 1$. Then the following inequalities hold.*

(i)

$$\left|\nabla(\bar{x}, f) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \nabla(\bar{x}, G(.,s))ds - \sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\times \right.$$
$$\left.\int_a^b \nabla(\overline{x}, G(.,s))\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\right|$$
$$\le \frac{1}{(n-3)!} \parallel f^{(n)} \parallel_p \left\|\nabla\left(\bar{x}, G(.,s)\right)\tilde{T}_{n-2}(s,t)\right\|_q, \tag{49}$$

(ii)

$$\left|\nabla(\bar{x}, f) - \frac{f'(b)-f'(a)}{b-a}\int_a^b \nabla(\bar{x}, G(.,s))ds - \sum_{k=3}^{2n-2}\frac{k-2}{(k-1)!}\times \right.$$
$$\left.\int_a^b \nabla(\overline{x}, G(.,s))\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\right|$$
$$\le \frac{1}{(n-3)!} \parallel f^{(n)} \parallel_p \left\|\nabla(\bar{x}, G(.,s))T_{n-2}(s,t)\right\|_q. \tag{50}$$

The constants on the right of (49) and (50) are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. The arguments of the proof is similar to the proof of Theorem 9 in [9]. □

Remark 5.6. We can give integral version of Theorems 5.3, 5.4 and 5.5.

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