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The Quaternionic Expression of Ruled Surfaces

Süleyman Şenyurt[a], Abdussamet Çalışkan[a]

[a]*Faculty of Arts and Sciences, Department of Mathematics, Ordu Universty, Ordu, Turkey*

Abstract. In this paper, firstly, the ruled surface is expressed as a spatial quaternionic. Also, the spatial quaternionic definitions of the Striction curve, the distribution parameter, angle of pitch and the pitch are given. Finally, integral invariants of the closed spatial quaternionic ruled surfaces drawn by the motion of the Frenet vectors $\{t, n_1, n_2\}$ belonging to the spatial quaternionic curve α are calculated.

1. Introduction

What algebraic structure plays an analogous role for rotations in the space? The answer was discovered in 1843 by William Rowan Hamilton, [6]. Quaternions arose historically from Hamilton's essays in the mid nineteenth century to generalize complex numbers in some way that would be applicable to three-dimensional (3D) space. A feature of quaternions is closely related to 3D rotations, a fact apparent to Hamilton almost immediately but first published by Hamilton's contemporary Arthur Cayley in 1845 [3]. He struggled for years attempting to make sense of an unsuccessful algebraic system containing one real and two imaginary parts. Hamilton had a brilliant stroke of imagination, and invented in a single instant the idea of a three-part imaginary system that became the quaternion algebra [7]. The technology did not penetrate the computer animation community until the landmark Siggraph 1985 paper of Ken Shoemake [13]. The importance of Shoemake's paper is that it took the concept of the orientation frame for moving 3D objects and cameras, which require precise orientation specification, exposed the deficiencies of the then-standard Euler-angle methods, and introduced quaternions to animators as a solution. The Serret-Frenet formulae for a quaternionic curves in $\mathbb{IR}^3$ and $\mathbb{IR}^4$ are introduced by K. Bharathi and M. Nagaraj [2]. There are lots of studies that investigated quaternionic curves by using this study. One of them is Karadağ and Sivridağ's study whose they gave many characterizations for quaternionic inclined curves in $\mathbb{IR}^4$ [9]. Şenyurt et al. calculated curvature and torsion of spatial quaternionic involute curve according to the normal vector and the unit Darboux vector of Smarandache curve [12].

A surface is said to be ruled if it is generated by moving a straight line continuously in Euclidean space $\mathbb{E}^3$. Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. A practical application of ruled surfaces is that they are used in civil engineering.The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight. Among ruled surfaces, developable surfaces form an important subclass since they are useful in sheet metal design and processing

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Email addresses: senyurtsuleyman@hotmail.com (Süleyman Şenyurt), a.caliskan@odu.edu.tr (Abdussamet Çalışkan)

[1, 11]. Every developable surface can be obtained as the envelope surface of a moving plane under a one-parameter motion. Developable ruled surfaces are well-known and widely used in computer aided design and manufacture. A developable ruled surface is a surface that can be rolled on a plane, touching along the entire surface as it rolls. Such a surface has a constant tangent plane for the whole length of each ruling [11]. It is well known from H.R. Müller [10] that a closed ruled surface generated by oriented line of a rigid body has two real integral invariants; the pitch and the angle of pitch [8, 10].

In recent years there have been many studies on ruled surfaces. In some studies, the dual expression of the ruled surface has been investigated. However, the ruled surface was not studied as a quaternionic. In this study, we investigate the ruled surface as spatial quaternionic. We quaternionally calculate the integral invariants of the ruled surface.

2. Preliminaries

Real quaternion is defined by the $1, e_1, e_2, e_3$. 1 is real number, e_1, e_2, e_3 are vectors with the following properties:

$$\begin{cases} {e_1}^2 = {e_2}^2 = {e_3}^2 = e_1 \times e_2 \times e_3 = -1, e_1, e_2, e_3 \in \mathbb{R}^3 \\ e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2. \end{cases} \tag{1}$$

The 4-dimensional real Euclidean space $\mathbb{R}^4$ is identified with the space of real quaternions

$$\mathbb{K} = \left\{ q = d + ae_1 + be_2 + ce_3 | a, b, c, d \in \mathbb{R}, \vec{e}_1, e_2, e_3 \in \mathbb{R}^3 \right\}$$

in [2, 5].

Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1e_1 + b_1e_2 + c_1e_3$ and $q_2 = S_{q_2} + V_{q_2} = d_2 + a_2e_1 + b_2e_2 + c_2e_3$ be two quaternions in $\mathbb{K}$, the quaternion multiplication of q_1 and q_2 is given by

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ &+ (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3. \end{aligned}$$

When inner and cross products in $\mathbb{E}^3$ are taken into consideration, this quaternion multiplication gives way to

$$q_1 \times q_2 = S_{q_1}S_{q_2} - \langle V_{q_1}, V_{q_2} \rangle + S_{q_1}V_{q_2} + S_{q_2}V_{q_1} + V_{q_1} \wedge V_{q_1}. \tag{2}$$

The symmetric real-valued bilineer form h which is defined as

$$\begin{aligned} h &: \mathbb{K} \times \mathbb{K} \to \mathbb{R} \\ &h(q_1, q_2) = \frac{1}{2}(q_1 \times \bar{q_2} + q_2 \times \bar{q_1}) \end{aligned} \tag{3}$$

is called quaternion inner product [2]. Let q be a real quaternion. Its conjugate is $\bar{q} = S_q - V_q$. Norm of real quaternion is a real number in the form of

$$N(q) = \sqrt{h(q, q)} = \sqrt{d^2 + a^2 + b^2 + c^2}. \tag{4}$$

If $N(q) = 1$, q is called a unit quaternion. Inverse of real quaternion is $q^{-1} = \dfrac{\bar{q}}{N(q)}$. Quaternion division is noncommutative, and is defined by the (order-dependent) relations $r_1 = q_1 \times {q_2}^{-1}, r_2 = {q_2}^{-1} \times q_1$. Where r_1 is right division, r_2 is left division [5]. The three-dimensional real Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternions

$$Q = \{q \in \mathbb{K} \mid q + \bar{q} = 0\}$$

in obvious manner [2, 14]. In this case, the elements of Q are $q = ae_1 + be_2 + ce_3$. As a result, the quaternion multiplication of the two spatial quaternions is [5]

$$q_1 \times q_2 = -\langle q_1, q_2 \rangle + q_1 \wedge q_2. \tag{5}$$

Definition 2.1. Let $s \in I = [0,1]$ be the arc parameter along the smooth curve

$$\begin{aligned} \alpha : [0,1] &\rightarrow Q \\ \alpha(s) &= \sum_{n=1}^{3} \alpha_i(s) e_i. \end{aligned}$$

This is called a spatial quaternionic curve [2, 14].

Let $\alpha : [0,1] \rightarrow Q$ be a spatial quaternionic curve parametrized by arc length $s \in I$. Frenet vectors of this curve are given by [2, 14].

$$t(s) = \alpha'(s),\ n_1(s) = \frac{\alpha''(s)}{\mathbf{N}(\alpha''(s))},\ n_2(s) = t(s) \times n_1(s).$$

Let $\alpha : [0,1] \rightarrow Q$ be a spatial quaternionic curve given by arbitrary parameter $s^* \in I$. Frenet invariants of this curve are given by [14].

$$t(s^*) = \frac{\alpha'(s^*)}{\mathbf{N}(\alpha'(s^*))},\ n_1(s^*) = n_2(s^*) \times t(s^*),\ n_2(s^*) = \frac{\alpha'(s^*) \times \alpha''(s^*) + \mathbf{N}(\alpha'(s^*)) \times \mathbf{N}'(\alpha'(s^*))}{\mathbf{N}(\alpha'(s^*) \times \alpha''(s^*) + \mathbf{N}(\alpha'(s^*)) \times \mathbf{N}'(\alpha'(s^*)))},$$

$$k(s^*) = \frac{\mathbf{N}(\alpha'(s^*) \times \alpha''(s^*) + \mathbf{N}(\alpha'(s^*)) \times \mathbf{N}'(\alpha'(s^*)))}{\mathbf{N}^3(\alpha'(s^*))},\ r(s^*) = \frac{h(\alpha'(s^*) \times \alpha''(s^*), \alpha'''(s^*))}{\mathbf{N}(\alpha'(s^*) \times \alpha''(s^*) + \mathbf{N}(\alpha'(s^*)) \times \mathbf{N}'^2(\alpha'(s^*)))}.$$

Let $\alpha(s)$ be a curve parametrized by arclength function s. Then for the unit speed spatial quaternionic curve α with frame vectors the following Frenet equations are given by [14]

$$t'(s) = k(s)n_1(s),\ {n_1}'(s) = -k(s)t(s) + r(s)n_2(s),\ {n_2}'(s) = -r(s)n_1(s). \tag{6}$$

Definition 2.2. A ruled surface in $\mathbb{R}^3$ is a surface which contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\begin{aligned} \varphi : I \times \mathbb{R} &\rightarrow \mathbb{R}^3 \\ (s,v) &\rightarrow \vec{\varphi}(s,v) = \vec{\alpha}(s) + v\vec{X}(s). \end{aligned} \tag{7}$$

where we call α the anchor curve, X the generator vector of ruled surface [4].

The instantaneous Pfaffian vector is given by

$$d\vec{X} = X^T \Omega a = \vec{w} \wedge \vec{X}. \tag{8}$$

Here $\vec{X}$ generator vector of ruled surface, Ω skew-symmetric matrix. Also, the relation between a and Ω is

$$da = \Omega a \tag{9}$$

from which, $a = (\vec{a_1}, \vec{a_2}, \vec{a_3})$. The Steiner rotation vector of the closed motion is given by $\vec{D}(s) = \oint_{(\alpha)} \vec{w}(s) ds$ where the integration is taken along the closed curve α. Also, the Steiner translation vector of the closed motion is $\vec{V}(s) = \oint_{(\alpha)} d\vec{\alpha}(s) ds$. If α is a closed curve, then this surface is called closed ruled surface. Moreover the distribution parameter (drall), the pitch L_X, and the angle of pitch λ_X of the closed ruled surface, respectively, are defined by [5]

$$P_X = \frac{\langle X \times X',\ \alpha' \rangle}{\|X'\|^2},\ L_X = \langle V, X \rangle, \lambda_X = \langle D, X \rangle.$$

3. The Quaternionic Expression of Ruled Surfaces

In this study, firstly, the ruled surface is expressed as a spatial quaternionic. Also, the spatial quaternionic definitions of the Striction curve, the distribution parameter, angle of pitch and the pitch are given. Finally, geometrical properties and integral invariants of closed ruled surface which is given as a spatial quaternionic are calculated.

Parametric expression of the spatial quaternion expression of a ruled surface is

$$\begin{aligned} \varphi : I \times \mathbb{R} &\rightarrow Q \\ (s,v) &\rightarrow \overrightarrow{\varphi}(s,v) = \overrightarrow{\alpha}(s) + v\overrightarrow{X}(s), \end{aligned}$$

where α spatial quaternionic curve and X quaternionic vector. In the present text, spatial quaternionic ruled surface term will be used instead of the spatial quaternionic expression of ruled surface.

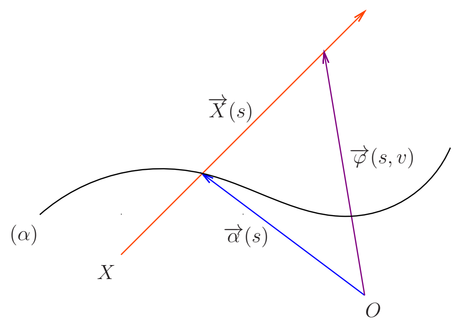

Figure 1: The spatial quaternionic expression of ruled surface

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \rightarrow Q$ be spatial quaternionic curve on the spatial quaternionic ruled surface, t be tangent vector field of α, X be generator vector and N be normal vector field of the surface. Also t, X, N are the vectors which are satisfying* (1)*. Then, $\{t, X, N\}$ establishes an orthonormal frame field along the curve α and variation of this frame along the curve is*

$$\begin{bmatrix} t' \\ X' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \sigma & \beta \\ -\sigma & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix} \begin{bmatrix} t \\ X \\ N \end{bmatrix} \tag{10}$$

where γ, β and σ are real valued functions.

Proof. Let $\alpha : I \subset \mathbb{R} \rightarrow Q$ be differentiable unit speed spatial quaternionic curve and $\overrightarrow{X}(s)$ be unit vector. Let X be perpendicular with t. Thus, the $\{t, X, N\}$ system establishes an orthonormal system. (1) is considered

$$\begin{aligned} h(t,X) = h(t,N) = h(X,N) = 0, \\ t \times t = N \times N = X \times X = -1, \\ t \times X = N, N \times t = X, X \times N = t. \end{aligned} \tag{11}$$

Since t is a unit vector, we can write

$$\mathbf{N}(t(s))^2 = h(t,t) = 1 \Rightarrow \frac{1}{2}(t \times \bar{t} + \bar{t} \times t) \Rightarrow t \times \bar{t} = 1. \tag{12}$$

By differentiating (12) with s, we have $t' \times \bar{t} + t \times \bar{t}' = 0$.

So, t and t' are $h-$ orthogonal. That is, $h(t',t) = 0 \Rightarrow t' \times \bar{t} + \overline{(t' \times \bar{t})} = 0$. Then, $t' \times \bar{t}$ is spatial quaternion. Let us find the variation of the system $\{t, X, N\}$ along the curve.

Considering $t'(s) \in Sp\{t, X, N\}$, we can write

$$t' = \eta t + \sigma X + \beta N. \tag{13}$$

From the above equation, $\eta = h(t', t) = 0, \sigma = h(t', X), \beta = h(t', N)$ are found. If these values are written in (13), t' is found in the form of

$$t' = \sigma X + \beta N. \tag{14}$$

Since $N(s)$ is a unit vector, we can write

$$\mathbf{N}(N(s))^2 = h(N, N) = 1 \Rightarrow \frac{1}{2}(N \times \bar{N} + \bar{N} \times N) = 1 \Rightarrow N \times \bar{N} = 1. \tag{15}$$

If the derivative is taken from (15),

$$N' \times \bar{N} + N \times \bar{N}' = 0. \tag{16}$$

By definition of quaternionic inner product, we have $h(N', N) = 0$. From the equation (11), it can be written $t \times N = -X$. If it is derived from the last equation and written in place of t', we get

$$-X' = t' \times N + t \times N' = \sigma t - \beta + t \times N'. \tag{17}$$

Considering $N'(s) \in Sp\{t, X, N\}$, it follows that

$$N' = \mu t + \gamma X + \nu N. \tag{18}$$

Taking the quaternionic inner product of both sides with N, we obtain $h(N', N) = \nu = 0$. Left-multiplying both sides of (18) by t gives

$$t \times N' = -\mu + \gamma N. \tag{19}$$

From the equations (17) and (19), X' may be written as

$$X' = -\sigma t + \beta + \mu - \gamma N. \tag{20}$$

Since X is a spatial quaternion, X' is also a spatial quaternion. Therefore, the following equation is obtained

$$X' + \bar{X}' = 0 \Rightarrow \mu = -\beta. \tag{21}$$

Considering equations $\mu = -\beta$ and $\nu = 0$, it can be written the following the equation

$$N' = -\beta t + \gamma X \tag{22}$$

and implicit in the equations (20) and (21) is that

$$X' = -\sigma t + \beta + \mu - \gamma N = -\sigma t - \gamma N. \tag{23}$$

The proof is complete. □

Definition 3.2. A plane, which passes through point of the spatial quaternionic ruled surface φ and is perpendicular to the surface normal, is called the tangent plane of the this surface.

Theorem 3.3. *Let φ be a spatial quaternionic ruled suarface. Tangent planes along a ruling coincide* $\Leftrightarrow \gamma = 0$.

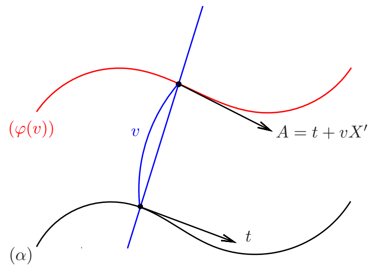

Figure 2: $\varphi(v)$ curve

Proof. If we fix the parameter v, then the spatial quaternionic curve

$$\varphi(v) : I \times \{v\} \to Q$$

can be obtained in φ. The tangent vector of this curve is

$$A = t + vX' = t + v(-\sigma t - \gamma N) = (1 - \sigma v)t - \gamma v N.$$

It can be written $A \perp X$. To have the tangent planes coincide, N must be constant. Because, in this case, each tangent plane has mutual lines and normals are the same. Then $\{A, t\}$ is linear dependent, so $\gamma = 0$. □

Definition 3.4. A spatial quaternionic ruled surface is developable surface, If its all tangent planes remains the same along a fixed generator.

Lemma 3.5. *A spatial quaternionic ruled surface is developable if and only if*

$$h(X \times X', \alpha') = 0.$$

Proof. Considering Theorem 3.3 and equation (18), we can write

$$\gamma = h(N', X) = 0.$$

On the other hand, from the equation (11) we know that $h(N, X) = 0$. If the derivative of this equation is taken, we obtain the following

$$\gamma = -h(N, X').$$

Normal of spatial quaternionic ruled surface is $N = \varphi_s \wedge \varphi_v$. Also it can be written

$$\varphi_s \times \varphi_v = -\langle \varphi_s, \varphi_v \rangle + \varphi_s \wedge \varphi_v = -\langle t + vX', X \rangle + \varphi_s \wedge \varphi_v = \varphi_s \wedge \varphi_v.$$

Thus, we have $N = \varphi_s \wedge \varphi_v = \varphi_s \times \varphi_v$. Using (3) and (10), we can write

$$\begin{aligned}
\gamma &= -h(X', N) = -h(X', \varphi_s \times \varphi_v) = -h(X', (t + vX') \times X) \\
&= -\frac{1}{2}\Big[X' \times \overline{(t \times X + vX' \times X)} + (t \times X + vX' \times X) \times \overline{X'}\Big] \\
&= -\frac{1}{2}\Big[X' \times \overline{(t \times X)} + (t \times X) \times \overline{X'} + v\Big((X' \times \overline{(-\sigma t - \gamma N) \times X}) + ((-\sigma t - \gamma N) \times X) \times \overline{X'}\Big)\Big] \\
&= -h(X', t \times X) - \frac{1}{2} v\Big[(-\sigma t - \gamma N) \times (\sigma N - \gamma t) + (-\sigma N + \gamma t) \times (\sigma t + \gamma N)\Big] \\
&= -h(X', \alpha' \times X). \qquad (24)
\end{aligned}$$

Similarly, it can be written

$$h(X \times X', \alpha') = h(X \times X', t) = \frac{1}{2}\Big[(X \times X') \times \bar{t} + t \times \overline{(X \times X')}\Big] = -\gamma. \qquad (25)$$

From the equations (24) and (25), the equality $h(X', \alpha' \times X) = h(X \times X', \alpha')$ is found. □

This lemma is closely related to distribution parameter of spatial quaternionic ruled surface.

Definition 3.6. Distribution parameter of φ is defined as the ratio of the shortest distance between successive generators $X, X + dX$ to the angle between successive generators.

Lemma 3.7. *Distribution parameter of φ is*

$$P_X = \frac{h(X \times X', \alpha')}{\mathbf{N}(X')^2} = \frac{1}{2}\frac{\left((X \times X') \times \overline{\alpha'} + \alpha' \times \overline{(X \times X')}\right)}{\mathbf{N}(X')^2}.$$

Proof. Common perpendicular of spatial quaternionic ruled surface is

$$X \wedge (X + dX) = X \wedge X + X \wedge dX = X \wedge dX.$$

$\langle X, X\rangle = 1 \Rightarrow \langle X, dX\rangle = 0$, then we can write $X \times dX = -\langle X, dX\rangle + X \wedge dX = X \wedge dX$. Then, common perpendicular can be written as $X \times dX$.

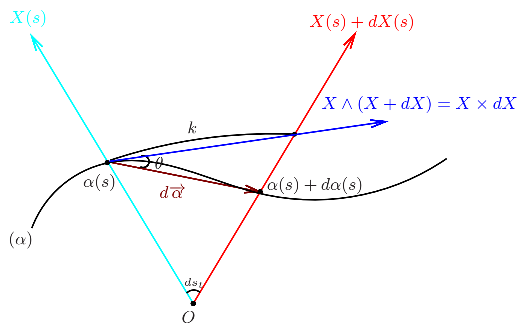

Figure 3: Distribution parameter of spatial quaternionic ruled surface

Using the figure (3), we get

$$\cos\theta = \frac{k}{\mathbf{N}(d\alpha)} = \frac{h(X \times dX, d\alpha)}{\mathbf{N}(X \times dX)\mathbf{N}(d\alpha)}$$

and from the norm property of the quaternion multiplication $\mathbf{N}(X \times dX) = \underbrace{\mathbf{N}(X)}_{=1}\mathbf{N}(dX)$, k is found as

$$k = \frac{h(X \times dX, d\alpha)}{\mathbf{N}(dX)} = \frac{h(X \times X', d\alpha)}{\mathbf{N}(X')}. \tag{26}$$

The length of an arc of the spherical indicatrix curve formed by (X) and subtending an angle between successive generators $X, X + dX$ equals. Thus,

$$\alpha_X(s) = X(s) \Rightarrow \frac{d\alpha_X}{ds_X}\frac{ds_X}{ds} = X'(s) \Rightarrow t_X\frac{ds_X}{ds} = X'(s), \Rightarrow \frac{ds_X}{ds} = \mathbf{N}(X'(s)) \Rightarrow ds_X = \mathbf{N}(X')ds. \tag{27}$$

From the definition of distribution parameter, we get

$$P_X = \frac{k}{ds_X} = \frac{h(X \times X', \alpha')}{\mathbf{N}(X')^2} = \frac{1}{2}\frac{\left((X \times X') \times \overline{\alpha'} + \alpha' \times \overline{(X \times X')}\right)}{\mathbf{N}(X')^2}. \tag{28}$$

□

Theorem 3.8. *A spatial quaternionic ruled surface is developable if and only if its distribution parameter is zero.*

Proof. Taking into consideration Lemma 3.5 and Lemma 3.7, it can be seen easily. □

Definition 3.9. If there is a curve which meets quaternionally perpendicular to each one of the rulings, then this curve is called an orthogonal trajectory of a spatial quaternionic ruled surface.

3.1. Position Vector of the Striction Curve Belonging to Spatial Quaternionic Ruled Surface

The striction point on a spatial quaternionic ruled surface φ is the foot of the common perpendicular between two successive generators (or ruling). Striction curve is the set of all striction points.

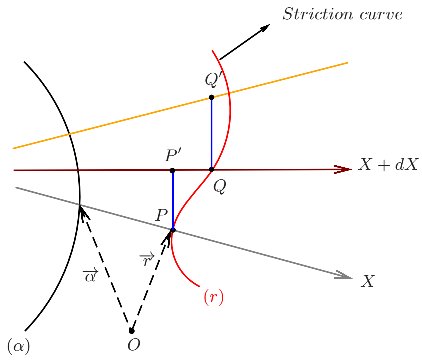

Figure 4: Spatial quaternionic striction curve

Let r be position vector of the striction curve. From the figure (4), it can be written

$$\vec{r}(s) = \vec{\alpha}(s) + u\vec{X}(s) \tag{29}$$

Let us find u in this equation:
P, P' and Q, Q' are the feet of the common perpendicular between successive generators. The common perpendicular between X and $X + dX$ was $X \times X'ds$. In case of limit, $\overrightarrow{PQ}$ and $\overrightarrow{PP'}$ will overlap and will be tangent of the striction curve. Thus, $h(X, PQ)$ and $h(X + X'ds, PQ)$ equal to zero, that is,

$$\begin{aligned} h(X + X'ds, PQ) = 0 \quad &\Rightarrow \quad \frac{1}{2}\Big[(X + X'ds) \times \overline{PQ} + PQ \times \overline{(X + X'ds)}\Big] = 0 \\ &\Rightarrow \quad \underbrace{\frac{1}{2}\Big[X \times \overline{PQ} + PQ \times \overline{X}\Big]}_{=0} + \frac{1}{2}ds\Big[X' \times \overline{PQ} + PQ \times \overline{X'}\Big] = 0 \Rightarrow h(X', PQ) = 0. \end{aligned}$$

Also, if it is derived from (29) to s, $r'(s)$ can be written instead of PQ,

$$\begin{aligned} h(X', PQ) \quad &= \quad h(X', r') = 0 \Rightarrow h(X', t + uX') = 0 \Rightarrow \frac{1}{2}\Big[X' \times \overline{(t + uX')} + (t + uX') \times \overline{X'}\Big] = 0 \\ &\Rightarrow \quad \underbrace{\frac{1}{2}\Big[X' \times \bar{t} + t \times \overline{X'}\Big]}_{h(X',t)} + u\underbrace{X' \times \overline{X'}}_{\mathbf{N}(X')^2} = 0 \Rightarrow u = -\frac{h(X', t)}{\mathbf{N}(X')^2}, \\ h(X', PQ) \quad &= \quad h(X', r') = 0 \Rightarrow h(X', t + u'X + uX') = h(X', t + uX') = 0 \Rightarrow u = -\frac{h(X', t)}{\mathbf{N}(X')^2}. \end{aligned}$$

If this value is substituted in the equation (29), the following result is obtained

$$\vec{r}(s) = \vec{\alpha}(s) - \frac{h(X', t)}{\mathbf{N}(X')^2}\vec{X}(s). \tag{30}$$

3.2. The Integral Invariants of Closed Spatial Quaternionic Ruled Surface

In this subsection, the spatial quaternionic definitions of the angle of pitch and the pitch are given. We express vectorial moment as a spatial quaternionic. Integral invariants of the closed spatial quaternionic ruled surfaces drawn by the motion of the Frenet vectors belonging to the spatial quaternionic curve α are calculated.

An orthogonal trajectory of closed spatial quaternionic ruled surface is defined by differential equation

$$h(X,d\varphi)=0 \Rightarrow h(X,d\alpha+dv\,X+v\,dX)=0 \Rightarrow -h(X,d\alpha)=dv.$$

Definition 3.10. For given closed spatial quaternionic ruled surface, the magnitude of

$$L_x=-\oint_{(\alpha)} h(d\alpha,X)=\oint_{(\alpha)} dv$$

is called the pitch of this surface.

Definition 3.11. Let φ be a closed spatial quaternionic ruled surface and let $V_1(s)$ denote the unit tangent vector of orthogonal trajectory at $\alpha(s)$. The angle between $V_1(s)$ and $V_1(s+p)$ is called the angle of pitch of φ where $V_1(s+p)$ is the tangent vector of the orthogonal trajectory at $\alpha(s+p)$ and p is the period of the closed spatial quaternionic curve α and denoted by λ_X.

Theorem 3.12. *Let φ, X and X^* be the spatial quaternionic ruled surface, the directrix of this surface and the vectorial moment of X, respectively. Then there exists a point Z, such that $\overrightarrow{X^*}=\overrightarrow{z}\times\overrightarrow{X}$.*

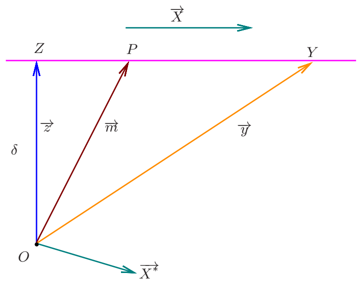

Figure 5: Vectorial moment

Proof. The vectorial equation of the generatrix to the spatial quaternionic ruled surface given by

$$(\overrightarrow{m}-\overrightarrow{y})\wedge\overrightarrow{X}=0 \Rightarrow \overrightarrow{m}\wedge\overrightarrow{X}-\overrightarrow{y}\wedge\overrightarrow{X}=0.$$

The vectorial moment $\overrightarrow{X^*}$ is

$$\overrightarrow{m}\wedge\overrightarrow{X}=\overrightarrow{y}\wedge\overrightarrow{X}=\overrightarrow{X^*}.$$

If $\overrightarrow{X^*}$ is independent of the choice of point P, Z can be taken as the foot of the perpendicular which is drawn from the point O to the generator. From the equation (5), it can be written

$$\overrightarrow{z}\times\overrightarrow{X}=\underbrace{-\langle\overrightarrow{z},\overrightarrow{X}\rangle}_{=0}+\underbrace{\overrightarrow{z}\wedge\overrightarrow{X}}_{\overrightarrow{X^*}}.$$

Thus, $\overrightarrow{X^*}=\overrightarrow{z}\times\overrightarrow{X}$. □

Theorem 3.13. *The angle of pitch and the pitch of the closed spatial quaternionic ruled surface, λ_x and L_x, are equal to the projection of the generator X on the Steiner rotation vector D and the Steiner translation vector V*

$$\lambda_x=h(\overrightarrow{D},\overrightarrow{X}),\ L_x=h(\overrightarrow{V},\overrightarrow{X}).$$

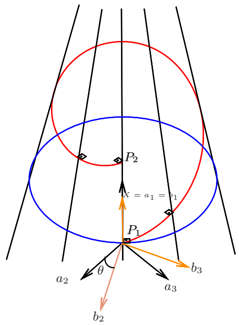

Figure 6: The pitch of the closed spatial quaternionic ruled surface

Proof. Let φ, $a_1 = X$ and a_2 be the spatial quaternionic ruled surface, the directrix of this surface and with perpendicular a_1, respectively. Let us take $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\}$ orthogonal system. The first and last positions of generator after a full rotation are the same. Therefore, $\overrightarrow{b_1} = \overrightarrow{a_1}$.

From the figure (6), we can write

$$\overrightarrow{a_2} = \overrightarrow{b_2}\cos\theta - \overrightarrow{b_3}\sin\theta,\ \overrightarrow{a_3} = \overrightarrow{b_2}\sin\theta + \overrightarrow{b_3}\cos\theta.$$

Here $\{\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}\}$ is fixed orthonormal system. If derivative is taken from the above equations according to s and $d\theta$ are solved, it becomes

$$\begin{aligned} da_2 &= db_2\cos\theta - db_3\sin\theta + \overbrace{(-b_2\sin\theta - b_3\cos\theta)}^{-a_3}d\theta, \\ da_3 &= db_2\sin\theta + db_3\cos\theta + \underbrace{(b_2\cos\theta - b_3\sin\theta)}_{a_2}d\theta \end{aligned}$$

and

$$\begin{aligned} h(da_2, a_3) &= -h(a_3, a_3)d\theta \Rightarrow h(da_2, a_3) = -d\theta,\ h(da_3, a_2) = h(a_2, a_2)d\theta \Rightarrow h(da_3, a_2) = d\theta, \\ -d\theta &= h(da_2, a_3) = -h(da_3, a_2). \end{aligned} \tag{31}$$

From the equations (31), the angle of pitch of closed spatial quaternionic ruled surface is found as

$$\lambda_X = -\oint d\theta = \oint h(da_2, a_3) = -\oint h(da_3, a_2).$$

Using the equation (9), we obtain

$$\begin{aligned} h(da_2, a_3) &= \frac{1}{2}\left(da_2 \times \overline{a_3} + a_3 \times \overline{da_2}\right), \quad da_2 = -w_3a_1 + w_1a_3 \\ &= \frac{1}{2}\left((-w_3a_1 + w_1a_3) \times \overline{a_3} + a_3 \times (\overline{-w_3a_1 + w_1a_3})\right) \\ &= \frac{1}{2}\left(2w_1(a_3 \times \overline{a_3}) + w_3(a_1 \times a_3) + w_3(a_3 \times a_1)\right) \\ &= \frac{1}{2}\Big[2w_1(a_3 \times \overline{a_3}) + w_3(-\underbrace{\langle a_1, a_3\rangle}_{0} + \underbrace{a_1 \wedge a_3}_{-a_2}) + w_3(-\underbrace{\langle a_3, a_1\rangle}_{0} + \underbrace{a_3 \wedge a_1}_{a_2})\Big] \\ &= w_1 \end{aligned}$$

and

$$\begin{aligned} -h(da_3, a_2) &= -\frac{1}{2}\left(da_3 \times \overline{a_2} + a_2 \times \overline{da_3}\right), \quad da_3 = w_2 a_1 - w_1 a_2 \\ &= -\frac{1}{2}\left((w_2 a_1 - w_1 a_2) \times \overline{a_2}) + a_2 \times (\overline{w_2 a_1 - w_1 a_2})\right) \\ &= w_1. \end{aligned}$$

Therefore, the angle of pitch is obtained as

$$h(da_2, a_3) = -h(da_3, a_2) = w_1 \Rightarrow \lambda_X = \oint w_1. \tag{32}$$

On the other hand, it is known that $\overrightarrow{D} = a_1 \oint w_1 + a_2 \oint w_2 + a_3 \oint w_3$ and $h(D, X) = h(D, a_1)$, thus we get

$$\begin{aligned} h(D, a_1) &= \frac{1}{2}\left(D \times \overline{a}_1 + a_1 \times \overline{D}\right) \\ &= \frac{1}{2}\left[\left(a_1 \oint w_1 + a_2 \oint w_2 + a_3 \oint w_3\right) \times \overline{a}_1 + a_1 \times \left(\overline{a_1 \oint w_1 + a_2 \oint w_2 + a_3 \oint w_3}\right)\right] \\ &= \frac{1}{2}\left[(a_1 \times \overline{a}_1) \oint w_1 + (a_2 \times \overline{a}_1) \oint w_2 + (a_3 \times \overline{a}_1) \oint w_3 + (a_1 \times \overline{a}_1) \oint w_1\right. \\ &\quad \left. + (a_1 \times \overline{a}_2) \oint w_2 + (a_1 \times \overline{v}_3) \oint w_3\right] \\ &= (a_1 \times \overline{a}_1) \oint w_1 = \oint w_1. \end{aligned} \tag{33}$$

From the equation (32) and (33), we can write $\lambda_X = h(D, X)$. It is also clear that the Steiner translation vector is $L_X = \oint h(d\alpha, X) = h\left(\oint d\alpha, X\right) = h(V, X)$. □

Theorem 3.14. *Let α and $\{t, n_1, n_2\}$ be the spatial quaternionic curve and the Frenet vectors of spatial quaternionic curve α, respectively. Then the instantaneous pfaffian vector of motion is given by*

$$w = n_1 \times n_1' = rt + kn_2. \tag{34}$$

Proof. Let the instantaneous pfaffian vector of motion be

$$w = t_1 t + t_2 n_1 + t_3 n_2. \tag{35}$$

Right-multiplying both sides of (35) by t gives

$$w \times t = -t_1 - t_2 n_2 + t_3 n_1. \tag{36}$$

On the other hand, it can be written

$$w \times t = -\langle w, t\rangle + w \wedge t = -t_1 + dt = -t_1 + kn_1. \tag{37}$$

Then, $t_2 = 0$ and $t_3 = k$ are found. Similarly, right-multiplying both sides of (35) by n_1 gives

$$w \times n_1 = t_1 n_2 - t_2 - t_3 t. \tag{38}$$

On the other hand, it can be written

$$w \times n_1 = -\langle w, n_1\rangle + w \wedge n_1 = -t_2 - kt + rn_2. \tag{39}$$

From the equations (38) and (39), $t_2 = r$, $t_3 = k$ are found. If these values are used in (35), we obtain $w = rt + kn_2$. Also, it becomes

$$n_1 \times n_1' = n_1 \times (-kt + rn_2) = rt + kn_2. \tag{40}$$

And so, by considering the equation (40), the instantaneous pfaffian vector is found as $\overrightarrow{w} = n_1 \times n_1' = rt + kn_2$. $\square$

Let us choose H moving space as $H = S_p\{t, n_1, n_2\}$ in H/H' movement. In this case, the Steiner rotation and Steiner translation vectors become

$$\overrightarrow{D} = \oint \overrightarrow{w} ds = \oint (rt + kn_2) ds = t \oint rds + n_2 \oint kds$$

and

$$\overrightarrow{V} = \oint d\alpha = \oint tds,$$

respectively.

Theorem 3.15. *The angles of the pitch, pitches, and distribution parameters of the closed spatial quaternionic ruled surfaces drawn by the Frenet vectors t, n_1, n_2 are*

$$1)\ldots \begin{cases} \lambda_t = \oint rds \\ \lambda_{n_1} = 0 \\ \lambda_{n_1} = \oint kds. \end{cases} \quad 2)\ldots \begin{cases} L_t = \oint ds \\ L_{n_1} = \oint 0 \\ L_{n_2} = \oint 0. \end{cases} \quad 3)\ldots \begin{cases} P_t = 0 \\ P_{n_1} = \dfrac{r}{r^2 + k^2} \\ P_{n_2} = \dfrac{1}{r}. \end{cases}$$

Proof. 1) According to Theorem 3.13 and the equation (11), the angles of pitch of the closed spatial quaternionic ruled surfaces drawn by the motion of the Frenet vectors $\{t, n_1, n_2\}$ belonging to the spatial quaternionic curve α are as follws:

$$\begin{aligned} \lambda_t &= h(D, t) = \frac{1}{2}\Big(D \times \bar{t} + t \times \overline{D}\Big) \\ &= \frac{1}{2}\Big(- (t \oint rds + n_2 \oint kds) \times t + t \times (-t \oint rds - n_2 \oint kds)\Big) \\ &= \frac{1}{2}\Big(- (t \times t) \oint rds - \underbrace{(n_2 \times t)}_{n_1} \oint kds - (t \times t) \oint rds - \underbrace{(t \times n_2)}_{-n_1} \oint kds)\Big) = \oint rds, \end{aligned}$$

$$\begin{aligned} \lambda_{n_1} &= h(D, n_1) = \frac{1}{2}\Big(D \times \overline{n_1} + n_1 \times \overline{D}\Big) \\ &= \frac{1}{2}\Big(- (t \oint rds + n_2 \oint kds) \times n_1 + n_1 \times (-t \oint rds - n_2 \oint kds)\Big) = 0, \end{aligned}$$

$$\begin{aligned} \lambda_{n_2} &= h(D, n_2) = \frac{1}{2}\Big(D \times \overline{n_2} + n_2 \times \overline{D}\Big) = \frac{1}{2}\Big(- D \times n_2 + n_2 \times \overline{D}\Big) \\ &= \frac{1}{2}\Big(- (t \oint rds + n_2 \oint kds) \times n_2 + n_2 \times (-t \oint rds - n_2 \oint kds)\Big) = \oint kds. \end{aligned}$$

2) According to Theorem 3.13 and the equation (11), the pitches of the closed spatial quaternionic ruled surfaces drawn by the motion of the Frenet vectors $\{t, n_1, n_2\}$ belonging to the spatial quaternionic curve α are as follows:

$$L_t = h(\oint d\alpha, t) = h(\oint t ds, t) = \frac{1}{2}\Big(t \oint ds \times \bar{t} + t \times \overline{t \oint ds}\Big) = \oint ds,$$

$$L_{n_1} = h(\oint d\alpha, n_1) = h(\oint t ds, n_1) = \frac{1}{2}\Big(t \oint ds \times \overline{n_1} + n_1 \times \overline{t \oint ds}\Big) = 0,$$

$$L_{n_2} = h(\oint d\alpha, n_2) = h(\oint t ds, n_2) = \frac{1}{2}\Big(t \oint ds \times \overline{n_2} + n_2 \times \overline{t \oint ds}\Big) = 0.$$

3) Using Lemma 3.7, the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the tangent vectors t belonging to the spatial quaternionic curve α is

$$P_t = \frac{h(t \times t', \alpha')}{\mathbf{N}(t'(s))^2}. \tag{41}$$

Considering the equation (6), we can write

$$\begin{aligned} h(t \times t', t) &= \frac{1}{2}\Big((t \times t') \times \bar{t} + t \times (\overline{t \times t'})\Big) = \frac{1}{2}\Big(-(t \times t') \times t + t \times (\overline{t'} \times \bar{t})\Big) \\ &= \frac{1}{2}\Big(-(t \times k n_1) \times t + t \times (k n_1 \times t)\Big) = \frac{1}{2}\Big(-k(n_2 \times t) - k(t \times n_2)\Big) = 0 \end{aligned}$$

Thus, $P_t = 0$ is found.

Taking consideration Lemma 3.7, the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the vector n_1 belonging to the spatial quaternionic curve α is

$$P_{n_1} = \frac{h(n_1 \times n_1', \alpha')}{\mathbf{N}(n_1'(s))^2}. \tag{42}$$

Using the equation (6), we obtain

$$\begin{aligned} h(n_1 \times n_1', t) &= \frac{1}{2}\Big((n_1 \times n_1') \times \bar{t} + t \times (\overline{n_1 \times n_1'})\Big) = \frac{1}{2}\Big(-(n_1 \times (-kt + r n_2)) \times t + t \times (\overline{(-kt + r n_2)} \times \overline{n_1})\Big) \\ &= \frac{1}{2}\Big((k(n_1 \times t) - r(n_1 \times n_2)) \times t - t \times ((kt - r n_2) \times n_1)\Big) = r \end{aligned}$$

and

$$\mathbf{N}(n_1'(s))^2 = h(n_1', n_1') = \frac{1}{2}\Big(n_1' \times \overline{n_1'} + n_1' \times \overline{n_1'}\Big) = n_1' \times \overline{n_1'} = k^2 + r^2.$$

If these values are substituted in equation (42), $P_{n_1} = \dfrac{r}{k^2 + r^2}$ is found.

From Lemma 3.7, the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the vector n_2 belonging to the spatial quaternionic curve α is

$$P_{n_2} = \frac{h(n_2 \times n_2', \alpha')}{\mathbf{N}(n_2'(s))^2}. \tag{43}$$

Considering the equations (6), we get

$$\begin{aligned} h(n_2 \times n_2', t) &= \frac{1}{2}\Big((n_2 \times n_2') \times \bar{t} + t \times (\overline{n_2 \times n_2'})\Big) = \frac{1}{2}\Big(-(n_2 \times n_2') \times t + t \times (\overline{n_2'} \times \overline{n_2})\Big) \\ &= \frac{1}{2}\Big(-(n_2 \times (-r n_1)) \times t + t \times (\overline{n_2'} \times \overline{n_2})\Big) = \frac{1}{2}\Big(r - rt \times (n_1 \times n_2)\Big) = r \end{aligned}$$

and

$$\mathbf{N}(n_2'(s))^2 = h(n_2', n_2') = n_2' \times \overline{n_2'} = -rn_1 \times rn_1 = \quad r^2.$$

If these values are substituted in equation (43), $P_{n_2} = \dfrac{1}{r}$ is found. $\square$

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