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# Lightlike Hypersurfaces of an $(\varepsilon)$-Para Sasakian Manifold with a Semi-Symmetric Non-Metric Connection

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**Abstract.**
In the present paper, we study a lightlike hypersurface, when the ambient manifold is an $(\varepsilon)$-para Sasakian manifold endowed with a semi-symmetric non-metric connection. We obtain a condition for such a lightlike hypersurface to be totally geodesic. We define invariant and screen semi-invariant lightlike hypersurfaces of $(\varepsilon)$-para Sasakian manifolds with a semi-symmetric non-metric connection. Also, we obtain integrability conditions for the distributions $D \perp \langle\delta\rangle$ and $D' \perp \langle\delta\rangle$ of a screen semi-invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifolds with a semi-symmetric non-metric connection.

## 1. Introduction

The theory of submanifolds of semi-Riemannian manifolds is one of the most important topics in differential geometry. In case the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more difficult and is quite different from the study of non-degenerate submanifolds. The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact that in the first case the normal vector bundle has non-trivial intersection with the tangent vector bundle, and moreover in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. Lightlike submanifolds of semi-Riemannian manifolds were introduced by K. L. Duggal and A. Bejancu in [9] (see also [10]). Since then, many authors have focused to extend their ideas on this topic (for example, see [1–3, 11, 12, 16]).

The idea of semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten [13] in 1924. A linear connection $\breve{\nabla}$ on a Riemannian manifold $(M^n, g)$ is called semi-symmetric, if its torsion $\breve{T}$ satisfies

$$\breve{T}(W,Z) = \breve{\eta}(Z)\,W - \breve{\eta}(W)\,Z,$$

where $\eta$ is a non-zero 1-form associated with a vector fields $\delta$ defined by

$$\breve{\eta}(W) = \breve{g}(W,\delta)\,.$$

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In 1930, E. Bartolotti [5] gave geometrical meaning of such a connection. In 1932, H. A. Hayden [14] defined and studied semi-symmetric metric connection. In 1970, Yano [24] started the systematic study of semi-symmetric metric connection and this was further developed by various authors. In 1991, N. S. Agashe and M. R. Chafle [4] introduced a semi-symmetric connection $\breve{\nabla}$ satisfying $\breve{\nabla}g \neq 0$ and called such a connection as semi-symmetric non-metric connection. They gave the relation between the curvature tensors of the manifold with respect to the semi-symmetric non-metric connection and the Riemannian connection.

An almost paracontact structure $\left(\breve{\phi}, \delta, \breve{\eta}\right)$ satisfying $\breve{\phi}^2 = I - \breve{\eta} \otimes \delta$ and $\breve{\eta}(\delta) = 1$ on a differentiable manifold was introduced by I. Sato [17] in 1976. The structure is an analogue of the almost contact structure [7, 20]. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [22] initiated the study of almost contact manifolds equipped with an associated pseudo-Riemannian metric. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as $(\varepsilon)$-almost contact metric manifolds and $(\varepsilon)$-Sasakian manifolds [6, 8]. Also, in 1989, K. Matsumoto replaced the structure vector field $\delta$ by $-\delta$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it Lorentzian almost paracontact manifold [18]. In a Lorentzian almost paracontact manifold given by K. Matsumoto, the semi-Riemann metric has only index 1 and the structure vector field $\delta$ is always timelike. In [23], the authors introduced $(\varepsilon)$- almost paracontact structures by associating almost paracontact structure with a semi-Riemannian metric, where the structure vector field $\delta$ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$. Lightlike hypersurfaces of such an $(\varepsilon)$-para Sasakian manifolds were studied by S. Yüksel Perktaş et al. [26] (see also [21]).

In 2014, S.K. Pandey et al. [19] studied semi-symmetric non-metric connection in an indefinite para-Sasakian manifold. They obtained the relation between the semi-symmetric non-metric connection and Levi-Civita connection in an indefinite para-Sasakian manifold.

In this article, we study a lightlike hypersurface, when the ambient manifold is an $(\varepsilon)$- para Sasakian manifold with semi-symmetric non-metric connection. We obtain condition for such a lightlike hypersurface to be totally geodesic. Also, we find integrability conditions for the distributions of some special lightlike hypersurfaces. The paper is organized as follows. In Section 2, we give a brief account of lightlike hypersurfaces of a semi-Riemannian manifold, for later use. In Section 3, an $(\varepsilon)$- para Sasakian manifold with semi-symmetric non-metric connection is given. In Section 4, we investigate lightlike hypersurfaces of an $(\varepsilon)$- para Sasakian manifold with semi-symmetric non-metric connection. In Section 5, invariant lightlike hypersurfaces of such manifolds are studied. Finally, in Section 6 screen semi-invariant lightlike hypersurfaces of such manifolds are investigated and we find some necessary and sufficient conditions for integrability of distributions.

## 2. Lightlike Hypersurfaces

Let $\left(\breve{M}, \breve{g}\right)$ be an $(n+2)$-dimensional semi-Riemannian manifold of fixed index $q \in \{1, ..., n+1\}$ and $M$ a hypersurface of $\breve{M}$. Assume that the induced metric $g = \breve{g}\mid_M$ on hypersurface is degenerate on $M$. Then, there exist a vector field $\xi \neq 0$ on $M$ such that

$$g(\xi, W) = 0,$$

for all $W \in \Gamma(TM)$.

The radical space of $T_W M$, at each point $W \in M$, is defined by

$$Rad\, T_W M = \{\xi \in T_W M : g(\xi, W) = 0, W \in T_W M\}, \tag{1}$$

whose dimension is called the nullity degree of $g$ and $(M, g)$ is called a lightlike hypersurface of $\left(\breve{M}, \breve{g}\right)$. Since $g$ is degenerate and any null vector is perpendicular to itself, $T_W M^{\perp}$ is also degenerate and

$$Rad\, T_W M = T_W M \cap T_W M^{\perp}. \tag{2}$$

For a hypersurface $M$, $\dim T_W M^{\perp} = 1$ implies that

$$\begin{aligned} \dim Rad\, T_W M &= 1, \\ Rad\, T_W M &= T_W M^{\perp}. \end{aligned}$$

We call $Rad\, TM$ the radical distribution and it is spanned by the null vector field $\xi$.

Consider a complementary vector bundle $S(TM)$ of $Rad\, TM$ in $TM$. This means that

$$TM = S(TM) \perp Rad\, TM, \tag{3}$$

where $\perp$ denotes the orthogonal direct sum. The bundle $S(TM)$ is called the screen distribution on $M$. Since the screen distribution $S(TM)$ is non-degenerate, there exists a complementary orthogonal vector sub-bundle $S(TM)^{\perp}$ to $S(TM)$ in $T\breve{M}$ which is called the screen transversal bundle of dimension 2 [10].

Since $Rad\, TM$ is a lightlike vector sub-bundle of $S(TM)^{\perp}$, therefore for any local section $\xi \in \Gamma(Rad\, TM)$ there exists a unique local section $N$ of $S(TM)^{\perp}$ such that

$$g(N, N) = 0 \qquad g(\xi, N) = 1. \tag{4}$$

Hence, $N$ is not tangent to $M$ and $\{\xi, N\}$ is a local frame field of $S(TM)^{\perp}$. Moreover, we have a 1-dimensional vector sub-bundle $ltrTM$ of $T\breve{M}$, namely lightlike transversal bundle, which is locally spanned by $N$. Then we set

$$S(TM)^{\perp} = Rad\, TM \oplus ltrTM,$$

where the decomposition is not orthogonal. Thus we have the following decomposition of

$$T\breve{M} = S(TM) \perp Rad\, TM \oplus ltrTM = TM \oplus ltrTM. \tag{5}$$

From the above decomposition of a semi-Riemannian manifold $\breve{M}$ along a lightlike hypersurface $M$, we may consider the following local quasi-orthonormal field of frames of $\breve{M}$ along $M$ :

$$\{W_1, ..., W_n, \xi, N\},$$

where $\{W_1, ...W_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. According to the decomposition given by (5), we have the following the Gauss and the Weingarten formulas, respectively:

$$\breve{\nabla}_W V = \nabla_W V + B(W, V)N, \tag{6}$$

$$\breve{\nabla}_W N = -A_N W + \tau(W) N, \tag{7}$$

where $B$ is a symmetric $(0, 2)$ tensor which is called the second fundamental form and $A$ is an endomorphism of $TM$ which is called the shape operator with respect to $N$ and $\tau$ is a 1-form on $M$.

For each $W \in \Gamma(TM)$, we can write

$$W = SW + \alpha(W)\xi, \tag{8}$$

where $S$ is projection of $TM$ on $S(TM)$ and $\alpha$ is a 1-form given by

$$\alpha(W) = \breve{g}(W, N). \tag{9}$$

From (7), for all $W, V, U \in \Gamma(TM)$, we get

$$(\nabla_W g)(V, U) = B(W, V)\alpha(U) + B(W, U)\alpha(V),$$

which implies that the induced connection $\nabla$ is a non-metric connection on $M$.

From (3), we have

$$\nabla_W S = \nabla^*_W S + C(W, S)\xi \tag{10}$$

$$\nabla_W \xi = -A^*_\xi W - \tau(W)\xi \tag{11}$$

for all $W \in \Gamma(TM)$, $S \in \Gamma(S(TM))$, where $C$, $A^*_\xi$ and $\nabla^*$ are the local second fundamental form, the local shape operator and the induced connection on $S(TM)$, respectively. Note that $\nabla^*_W S$ and $A^*_\xi W$ belong to $\Gamma(S(TM))$. Also, we have the following

$$g\left(A^*_\xi W, V\right) = B(W,V),\ g\left(A^*_\xi W, N\right) = 0,\ B(W,\xi) = 0,\ g\left(A_N W, N\right) = 0. \tag{12}$$

Moreover, from the first and third equations of (12) we have [9]

$$A^*_\xi \xi = 0. \tag{13}$$

## 3. ($\varepsilon$)-para Sasakian Manifolds with a Semi-Symmetric Non-Metric Connection

Let $\breve{M}$ be an almost paracontact manifold equipped with an almost paracontact structure $\left(\breve{\phi}, \delta, \breve{\eta}\right)$ consisting of a tensor field $\breve{\phi}$ of type $(1,1)$, a vector field $\delta$ and 1-form $\breve{\eta}$ satisfying

$$\breve{\phi}^2 = I - \breve{\eta} \otimes \delta, \tag{14}$$
$$\breve{\eta}(\delta) = 1, \tag{15}$$
$$\breve{\phi}(\delta) = 0, \tag{16}$$
$$\breve{\eta} \circ \breve{\phi} = 0. \tag{17}$$

Let $\breve{M}$ be an $n$-dimensional almost paracontact manifold and $\breve{g}$ be a semi-Riemannian metric with $index(\breve{g}) = v$, such that

$$\breve{g}\left(\breve{\phi}W, \breve{\phi}V\right) = \breve{g}(W,V) - \varepsilon \breve{\eta}(W)\breve{\eta}(V), \tag{18}$$

where $\varepsilon = \pm 1$. In this case, $\breve{M}$ is called an ($\varepsilon$)-almost paracontact metric manifold equipped with an ($\varepsilon$)-almost paracontact structure $\left(\breve{\phi}, \delta, \breve{\eta}, \breve{g}\right)$ [23].

In view of equations (15),(16) and (18), we have

$$\breve{g}\left(\breve{\phi}W, V\right) = \breve{g}\left(W, \breve{\phi}V\right) \tag{19}$$

and

$$\breve{g}(W,\delta) = \varepsilon \breve{\eta}(W), \tag{20}$$

for all $W, V \in \Gamma(T\breve{M})$. From equation (20), it follows that

$$\breve{g}(\delta,\delta) = \varepsilon, \tag{21}$$

i.e. the structure vector field $\delta$ is never lightlike. An ($\varepsilon$) −almost paracontact metric manifold $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ is said to be spacelike ($\varepsilon$)-almost paracontact metric manifold, if $\varepsilon = 1$ and $\breve{M}$ is said to be a $\breve{M}$ timelike ($\varepsilon$)-almost paracontact metric manifold if $\varepsilon = -1$.

An ($\varepsilon$)-almost paracontact metric structure is called an ($\varepsilon$)-para Sasakian structure [23] if

$$\left(\breve{\nabla}_W \breve{\phi}\right)(V) = -\breve{g}\left(\breve{\phi}W, \breve{\phi}V\right)\delta - \delta\breve{\eta}(V)\breve{\phi}^2 W, \quad \forall W, V \in \Gamma(T\breve{M}), \tag{22}$$

where $\breve{\nabla}$ the Levi-Civita connection. A manifold $\breve{M}$ endowed with an ($\varepsilon$)-para Sasakian structure is called an ($\varepsilon$)-para Sasakian manifold.

In an ($\varepsilon$)-para Sasakian manifold, we have

$$\breve{\nabla}_W \delta = \varepsilon \breve{\phi}, \tag{23}$$

$$\Omega(W,V) = \varepsilon \breve{g}\left(\breve{\phi}W,V\right) = \left(\breve{\nabla}_W \breve{\eta}\right)V, \tag{24}$$

for all $W, V \in \Gamma(T\breve{M})$, where $\Omega$ is the fundamental 2-form.

The $\tilde{\tilde{\nabla}}$ on a semi-Riemannian manifold $\left(\breve{M}, \breve{g}\right)$ is called semi-symmetric connection, if its torsion tensor $\tilde{\tilde{T}}$ satisfies

$$\begin{aligned} \tilde{\tilde{T}}(W,V) &= \breve{\eta}(V)W - \breve{\eta}(W)V, & (25)\\ \breve{\eta}(W) &= \breve{g}(W,\delta). & (26) \end{aligned}$$

Let $\tilde{\tilde{\nabla}}$ be a linear connection and $\breve{\nabla}$ be a Levi-Civita connection of an $(\varepsilon)$-para Sasakian manifold $\breve{M}$ such

$$\tilde{\tilde{\nabla}}_W V = \breve{\nabla}_W V + F(W,V), \tag{27}$$

where $F$ is a tensor of type $(1,2)$.

For a semi-symmetric non-metric connection $\tilde{\tilde{\nabla}}$ in $\breve{M}$ , we have

$$F(W,V) = \frac{1}{2}\left[\tilde{\tilde{T}}(W,V) + \tilde{\tilde{T}}^{*}(W,V) + \tilde{\tilde{T}}^{*}(V,W)\right] + \breve{g}(W,V)\delta, \tag{28}$$

where

$$\tilde{\tilde{T}}^{*}(W,V) = \breve{\eta}(V)W - \breve{g}(W,V)\delta. \tag{29}$$

Using (25) and (29) in equation (28), we get

$$F(W,V) = \breve{\eta}(V)W. \tag{30}$$

Hence in view of equations (27) and (30), a semi-symmetric connection on an $(\varepsilon)$-para Sasakian manifold $\breve{M}$ is given by

$$\tilde{\tilde{\nabla}}_W V = \breve{\nabla}_W V + \breve{\eta}(V)W. \tag{31}$$

Also, we have

$$\left(\tilde{\tilde{\nabla}}_W \breve{g}\right)(V,Z) = -\breve{\eta}(V)\breve{g}(W,Z) - \breve{\eta}(Z)\breve{g}(W,V). \tag{32}$$

In a lightlike hypersurface, we have

$$\begin{aligned} \left(\tilde{\tilde{\nabla}}_W \breve{g}\right)(V,Z) &= B(W,V)g(N,Z) + B(W,Z)g(V,N) \\ &\quad -\breve{\eta}(V)g(W,Z) - \breve{\eta}(Z)g(V,W). \end{aligned} \tag{33}$$

In view of equations (25) and (32), we conclude that the connection $\tilde{\tilde{\nabla}}$ is a semi-symmetric non-metric connection. Thus equation (31) gives the relation between the Levi-Civita connection $\breve{\nabla}$ and semi-symmetric connection $\tilde{\tilde{\nabla}}$ on an $(\varepsilon)$-para Sasakian manifold $\breve{M}$.

In view of equation (31), we have

$$\left(\tilde{\tilde{\nabla}}_W \breve{\phi}\right)(V) = \tilde{\tilde{\nabla}}_W \breve{\phi}(V) - \breve{\phi}\left(\tilde{\tilde{\nabla}}_W V\right),$$

i.e.,

$$\left(\tilde{\tilde{\nabla}}_W \breve{\phi}\right)(V) = \left(\breve{\nabla}_W \breve{\phi}\right)(V) - \breve{\eta}(V)\breve{\phi}(W). \tag{34}$$

Replacing $W$ and $V$ by $\breve{\phi}W$ and $\breve{\phi}V$ and using equation (17), we find

$$\left(\tilde{\tilde{\nabla}}_{\breve{\phi}W}\breve{\phi}\right)\left(\breve{\phi}V\right) = \left(\breve{\nabla}_{\breve{\phi}W}\breve{\phi}\right)\left(\breve{\phi}V\right) = -\breve{g}\left(\breve{\phi}^2W, \breve{\phi}^2V\right)\delta, \tag{35}$$

for all $W, V \in T\breve{M}$ [19].

**Example 3.1.** Let us assume the manifold $R_q^{2m+1}$ with

$$\begin{aligned}
\breve{\eta} &= \frac{1}{2}\left(dz - \sum_{i=1}^{m} y^i dx^i\right),\\
\delta &= 2\partial z,\\
\breve{g} &= \breve{\eta}\otimes\breve{\eta} + \frac{1}{4}\left(-\sum_{i=1}^{\frac{q}{2}} dx^i\otimes dx^i + dy^i\otimes dy^i + \sum_{i=\frac{q}{2}+1}^{m} dx^i\otimes dx^i + dy^i\otimes dy^i\right),\\
\breve{\phi}\left(\sum^{m}(X_i\partial x_i + Y_i\partial y_i) + Z\partial z\right) &= \sum_{i=1}^{m}(Y_i\partial x_i + X_i\partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,
\end{aligned}$$

where $\left(x^i, y^i, z\right)$ are the cartesian coordinates on $R_q^{2m+1}$. Then $\left(R_q^{2m+1}, \breve{g}, \breve{\phi}, \breve{\eta}, \delta\right)$ is a usual para-Sasakian manifold [21].

**Example 3.2.** Let $R^3$ be the 3-dimensional real number space with a coordinate system $(x, y, z)$. We define

$$\begin{aligned}
\breve{\eta} &= dz,\\
\delta &= \frac{\partial}{\partial z},\\
\breve{\phi}(\frac{\partial}{\partial x}) &= \frac{\partial}{\partial x}, \breve{\phi}(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y}, \breve{\phi}(\frac{\partial}{\partial z}) = 0\\
\breve{g} &= e^{-2z}(dx)^2 + e^{2z}(dy)^2 - (dz)^2.
\end{aligned}$$

Then $\left(\breve{\phi}, \breve{g}, \breve{\eta}, \delta\right)$ is an $(\varepsilon)$-para Sasakian structure. Let $\breve{\nabla}$ and $\tilde{\tilde{\nabla}}$ denote the Levi-Civita connection and a linear connection on $R^3$, respectively. Then we have

$$\begin{aligned}
\breve{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= -e^{-2z}\frac{\partial}{\partial z}, \breve{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \breve{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = -\frac{\partial}{\partial x},\\
\breve{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} &= 0, \breve{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = e^{2z}\frac{\partial}{\partial z}, \breve{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = \frac{\partial}{\partial y},\\
\breve{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial x} &= -\frac{\partial}{\partial x}, \breve{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}, \breve{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = 0.
\end{aligned} \tag{36}$$

If we define

$$\begin{aligned}
\tilde{\tilde{\nabla}}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= -e^{2z}\frac{\partial}{\partial z}, \tilde{\tilde{\nabla}}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \tilde{\tilde{\nabla}}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = 0,\\
\tilde{\tilde{\nabla}}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} &= 0, \tilde{\tilde{\nabla}}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = e^{2z}\frac{\partial}{\partial z}, \tilde{\tilde{\nabla}}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = 2\frac{\partial}{\partial y},\\
\tilde{\tilde{\nabla}}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial x} &= -\frac{\partial}{\partial x}, \tilde{\tilde{\nabla}}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}, \tilde{\tilde{\nabla}}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.
\end{aligned} \tag{37}$$

then by using (36) and (37) we see that

$$\tilde{\tilde{T}}(W, V) = \breve{\eta}(V)W - \breve{\eta}(W)V,$$

which implies that $\tilde{\tilde{\nabla}}$ is a semi-symmetric non-metric connection.

## 4. Lightlike Hypersurfaces of an ($\varepsilon$)- para Sasakian Manifold with a Semi-Symmetric Non-Metric Connection

Let $M$ be a lightlike hypersurface of an ($\varepsilon$)- para Sasakian manifold with a semi-symmetric non-metric connection. In this case, if we take into account the fact that $\breve{\nabla}$ is a Levi-Civita connection, we can write the Gauss and Weingarten formulas as given by (6) and (7), respectively, where $\nabla$ denotes the induced connection on $M$ from Levi-Civita connection $\breve{\nabla}$.

Assume that $\tilde{\tilde{\nabla}}$ is a semi-symmetric connection on $\breve{M}$. If we denote the induced connection from $\tilde{\tilde{\nabla}}$ on $TM$ by $\mathring{\nabla}$, we can write

$$\begin{aligned} \tilde{\tilde{\nabla}}_W V &= \mathring{\nabla}_W V + m(W,V)N, && (38)\\ \tilde{\tilde{\nabla}}_W N &= -\mathring{A}_N W + w(W)N. && (39) \end{aligned}$$

Therefore, from (31) and above equations, we find

$$\begin{aligned} \mathring{\nabla}_W V &= \nabla_W V + \breve{\eta}(V)W, && (40)\\ m(W,V) &= B(W,V), && (41)\\ w(W) &= \tau(W). && (42) \end{aligned}$$

Since $\nabla$ is not a metric connection, then from (40), we obtain

$$\begin{aligned} \left(\mathring{\nabla}_W g\right)(V,Z) &= B(W,V)\theta(Z) + B(W,Z)\theta(V) - && (43)\\ &\quad \breve{\eta}(V)g(W,Z) - \breve{\eta}(Z)g(V,W), \end{aligned}$$

which implies that $\mathring{\nabla}$ is a non-metric connection. Also, we have

$$\mathring{T}(W,V) = \breve{\eta}(V)W - \breve{\eta}(W)V. \quad (44)$$

As an adaptation of [25], we have:

**Proposition 4.1.** *Let $M$ be a lightlike hypersurface of an ($\varepsilon$)- para Sasakian manifold $\breve{M}$ with a semi-symmetric non-metric connection. Then $M$ have a semi-symmetric non metric connection. Hence,*

$$\mathring{T}(W,V) = \breve{\eta}(V)W - \breve{\eta}(W)V,$$

$$\mathring{\nabla}_W V = \nabla_W V + \breve{\eta}(V)W,$$

$$\begin{aligned} \left(\mathring{\nabla}_W g\right)(V,Z) &= B(W,V)\theta(Z) + B(W,Z)\theta(V)\\ &\quad -\breve{\eta}(V)g(W,Z) - \breve{\eta}(Z)g(V,W). \end{aligned}$$

Now, replacing the Levi-Civita connection $\breve{\nabla}$ by semi-symmetric non-metric connection $\tilde{\tilde{\nabla}}$ in (22), the equation is reformed to

$$\left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)(V) = \left(\breve{\nabla}_W\breve{\phi}\right)(V) - \breve{\eta}(V)\breve{\phi}(W), \quad (45)$$

$$\begin{aligned} \left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)(V) &= -\breve{g}\left(\breve{\phi}W,\breve{\phi}V\right)\delta - \varepsilon\breve{\eta}(V)W && (46)\\ &\quad +\varepsilon\breve{\eta}(V)\breve{\eta}(W)\delta - \breve{\eta}(V)\breve{\phi}(W). \end{aligned}$$

Replacing $V$ by $\delta$ in (46) and using (16), $\breve{\eta}\left(\breve{\nabla}_W\delta\right) = 0$, we find

$$\tilde{\tilde{\nabla}}_W\delta = W + \varepsilon\breve{\phi}(W). \tag{47}$$

Let $(M, g)$ be a lightlike hypersurface of $\left(\breve{M}, \breve{g}\right)$. For local sections $\xi$ and $N$ of $Rad\,TM$ and $ltrTM$, respectively, in view of (26) and (14),we have

$$\breve{\eta}(\xi) = 0, \breve{\eta}(N) = 0, \tag{48}$$

$$\breve{\phi}^2\xi = 0, \breve{\phi}^2 N = 0. \tag{49}$$

For $W \in \Gamma(TM)$, we can write

$$\breve{\phi}W = \phi W + h(W)N, \tag{50}$$

where $\phi W \in \Gamma(TM)$ and

$$h(W) = g(\breve{\phi}W, \xi) = g(W, \breve{\phi}\xi). \tag{51}$$

**Proposition 4.2.** *Let $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ be an $(\varepsilon)$- para Sasakian manifold with a semi-symmetric non-metric connection and $M$ be a lightlike hypersurface of $\breve{M}$, such that structure vector field $\delta$ is tangent to $M$. Then we have*

$$g(\breve{\phi}\xi, \xi) = 0, \tag{52}$$

$$g\left(\breve{\phi}\xi, N\right) = g\left(\xi, \breve{\phi}N\right) = \varepsilon g(\delta, A_N\xi), \tag{53}$$

*where $\xi$ is a local section of $Rad\,TM$ and $N$ is a local section of $ltrTM$.*

*Proof.* From (47) and (13), we get

$$\begin{aligned} g(\breve{\phi}\xi, \xi) &= \varepsilon g\left(\tilde{\tilde{\nabla}}_\xi\delta - \xi, \xi\right) \\ &= -\varepsilon g(\delta, \nabla_\xi\xi) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} g\left(\breve{\phi}\xi, N\right) &= \varepsilon g\left(\tilde{\tilde{\nabla}}_\xi\delta - \xi, N\right) \\ &= -\varepsilon g\left(\delta, \tilde{\tilde{\nabla}}_\xi N\right) \\ &= \varepsilon g(\delta, A_N\xi) = g\left(\xi, \breve{\phi}N\right). \end{aligned}$$

Also, we find

$$g\left(\breve{\phi}\xi, \breve{\phi}N\right) = 1.$$

This completes the proof. □

From the proposition above, we can say that there is no component of $\breve{\phi}\xi$ in $ltrTM$, thus $\breve{\phi}\xi \in \Gamma(TM)$. Moreover, there may be a component of $\breve{\phi}\xi$ in $Rad\,TM$.

Therefore, for any lightlike hypersurface $M$ of $(\varepsilon)$- para Sasakian manifolds with a semi-symmetric non-metric connection $\breve{M}$, from the decomposition

$$D = S(TM) \perp Rad\,TM \perp \breve{\phi}(Rad\,TM)$$

and

$$D' = \breve{\phi}\,(ltrTM)\,,$$

we have

$$TM = D \oplus D'. \tag{54}$$

Consider two null vector field $H$ and $K$ and their 1-forms $h$ and $k$, such that

$$H = \breve{\phi}N, \quad h\,(W) = g\,(W,K)\,, \tag{55}$$
$$K = \breve{\phi}\xi, \quad k\,(W) = g\,(W,H)\,. \tag{56}$$

Denote the projection morphism of $TM$ on $D$ by $S$. Any vector field $W$ on $TM$ is expressed by

$$W = SW + h\,(W)\,H. \tag{57}$$

Applying $\breve{\phi}$ to the both sides of the last equation, we have

$$\begin{aligned} \breve{\phi}W &= \breve{\phi}SW + h\,(W)\,\breve{\phi}H, \\ \breve{\phi}W &= \phi W + h\,(W)\,N, \end{aligned} \tag{58}$$

where $\phi$ is a tensor field of type $(1,1)$ globally defined on $M$ by $\phi W = \breve{\phi}SW$.
If we apply $\breve{\phi}$ to (58) and using (14)–(17) with (55) and (56), we get

$$\begin{aligned} \breve{\phi}^2W &= \breve{\phi}\phi W + h\,(W)\,\breve{\phi}N, \\ W - \breve{\eta}\,(W)\,\delta &= \phi^2W + h\,(W)\,H, \end{aligned}$$

which imply

$$\phi^2W = W - \breve{\eta}\,(W)\,\delta - h\,(W)\,H + h\left(\phi W\right)N. \tag{59}$$

Using (32), (25), (19) and (58), we obtain

$$\begin{aligned} \left(\mathring{\nabla}_W g\right)(V,Z) &= B(W,V)g(N,Z) + B(W,Z)g(V,N) \\ &\quad -\breve{\eta}\,(V)\,g(W,Z) - \breve{\eta}\,(Z)\,g(V,W). \end{aligned} \tag{60}$$

Also, we have

$$\mathring{T}\,(W,V) = \breve{\eta}\,(V)\,W - \breve{\eta}\,(W)\,V, \tag{61}$$

for $W, V \in \Gamma\,(TM)$.

**Proposition 4.3.** *Let $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ be an $(\varepsilon)$- para Sasakian manifold with a semi-symmetric non-metric connection and $M$ be a lightlike hypersurface of $\breve{M}$, such that structure vector field $\delta$ is tangent to $M$. Then we have*

$$B(W,V) - B(V,W) = -\varepsilon\,(\breve{\eta}\otimes h)\,(W,V)\,g\,(\delta, A_N\xi)\,, \tag{62}$$

$$B(W,V) = g\left(A^*_\xi W, V\right) + \breve{\eta}\,(V)\,h(W), \tag{63}$$

$$C(W,PV) = g(PV, A_N W) + \breve{\eta}\,(PV)\,k\,(W)\,, \tag{64}$$

$$g\,(A_N W, \delta) = -\,(1+\varepsilon)\,k\,(W)\,. \tag{65}$$

*Proof.* For all $W, V, \delta \in \Gamma(TM)$, using (38) and (57), we obtain

$$\begin{aligned} \breve{\eta}(V)\, g(W,\xi) - \breve{\eta}(W)\, g(V,\xi) &= B(W,V) - B(V,W), \\ \varepsilon\left[\breve{\eta}(V)\, h(W) - \breve{\eta}(W)\, h(V)\right] g(\delta, A_N\xi) &= B(W,V) - B(V,W), \end{aligned}$$

which imply

$$B(W,V) - B(V,W) = -\varepsilon\,(\breve{\eta}\otimes h)(W,V)\, g(\delta, A_N\xi).$$

Also using (38) and (41), we find that the local second fundamental forms are related to their shape operators by

$$\begin{aligned} B(W,V) &= g\left(\tilde{\tilde{\nabla}}_W V, \xi\right) \\ &= -g(V, \nabla_W\xi) + \breve{\eta}(V)\, h(W) \\ &= g\left(A^*_\xi W, V\right) + \breve{\eta}(V)\, h(W). \end{aligned}$$

For a projection morphism $P$ to $S(TM)$ from $M$, we get

$$\begin{aligned} C(W,PV) &= g\left(\tilde{\tilde{\nabla}}_W PV, N\right) \\ &= -g\left(PV, \breve{\nabla}_W N\right) + \breve{\eta}(PV)\, k(W) \\ &= g(PV, A_N W) + \breve{\eta}(PV)\, k(W). \end{aligned}$$

Applying $\tilde{\tilde{\nabla}}_W$ to $g(\delta, N) = 0$ and using (60), (47), (56) and (39), we have

$$\begin{aligned} g\left(W + \varepsilon\breve{\phi}W, N\right) &= g(\delta, -A_N W + \tau(W)N), \\ k(W) + \varepsilon k(W) &= -g(A_N W, \delta), \\ g(A_N W, \delta) &= -(1+\varepsilon)\, k(W). \end{aligned}$$

This completes the proof. □

Now, applying $\tilde{\tilde{\nabla}}_W$ to (50), we obtain

$$\tilde{\tilde{\nabla}}_W \breve{\phi} V = \tilde{\tilde{\nabla}}_W \phi V + \left(\tilde{\tilde{\nabla}}_W h\right)(V)\, N + h(V)\, \tilde{\tilde{\nabla}}_W N,$$

$$\begin{pmatrix} -g(W,V)\,\delta + 2\varepsilon\breve{\eta}(W)\,\breve{\eta}(V)\,\delta - \varepsilon\breve{\eta}(V)\, W \\ -\breve{\eta}(V)\,\phi W - \breve{\eta}(V)\, h(W)\, N + h(\nabla_W V)\, N \\ +B(W,V)H \end{pmatrix} = \begin{pmatrix} \left(\nabla_W\phi\right)V + B(W,\phi V)N + \breve{\eta}\left(\phi V\right)W \\ \left(\tilde{\tilde{\nabla}}_W h\right)(V)\, N - h(V)A_N W - h(V)\tau(W)\, N \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \left(\nabla_W\phi\right)V &= -g(W,V)\,\delta + 2\varepsilon\breve{\eta}(W)\,\breve{\eta}(V)\,\delta \\ &\quad -\varepsilon\breve{\eta}(V)\, W - \breve{\eta}(V)\,\phi W + \breve{\eta}\left(\phi V\right)W \\ &\quad +h(V)A_N W + B(W,V)H \end{aligned} \tag{66}$$

$$\left(\tilde{\tilde{\nabla}}_W h\right)(V) = h(V)\tau(W) - \breve{\eta}(V)\, h(W) + h(\nabla_W V) - B(W,\phi V). \tag{67}$$

From (33), we have

$$\left(\tilde{\tilde{\nabla}}_W h\right)(V) = \varepsilon B(W,V)\, g(\delta, A_N\xi) - \breve{\eta}(V)\, h(W). \tag{68}$$

If we use (68) in (67), we arrive at

$$h(\nabla_W V) = h(V)\tau(W) - B(W,\phi V) - \varepsilon B(W,V) g(\delta, A_N\xi). \tag{69}$$

**Theorem 4.4.** *A lightlike hypersurface M of an* ($\varepsilon$)*- para Sasakian manifold with a semi-symmetric non-metric connection is totally geodesic if and only if*

$$\begin{aligned}\left(\nabla_W\phi\right)V &= -g(W,V)\,\delta+2\varepsilon\breve{\eta}(W)\,\breve{\eta}(V)\,\delta \\ &\quad -\varepsilon\breve{\eta}(V)\,W-\breve{\eta}(V)\,\phi W+\breve{\eta}\left(\phi V\right)W,\end{aligned} \tag{70}$$

$$A_N W=\left(\nabla_W\phi\right)H+g(W,H)\,\delta, \tag{71}$$

*where* $V\in\Gamma(D)$.

*Proof.* For any $V\in\Gamma(D)$, we have $h(V)=0$. Then, (66) is reduced to

$$\begin{aligned}\left(\nabla_W\phi\right)V &= -g(W,V)\,\delta+2\varepsilon\breve{\eta}(W)\,\breve{\eta}(V)\,\delta \\ &\quad -\varepsilon\breve{\eta}(V)\,W-\breve{\eta}(V)\,\phi W \\ &\quad +\breve{\eta}\left(\phi V\right)W-B(W,V)H.\end{aligned}$$

On the other hand, replacing $V$ by $H$ in (66), we also obtain

$$\begin{aligned}\left(\nabla_W\phi\right)H &= -g(W,H)\,\delta+2\varepsilon\breve{\eta}(W)\,\breve{\eta}(H)\,\delta \\ &\quad -\varepsilon\breve{\eta}(H)\,W-\breve{\eta}(H)\,\phi W+\breve{\eta}\left(\phi H\right)W \\ &\quad +h(H)A_N W-B(W,H)H,\end{aligned} \tag{72}$$

where

$$\breve{\eta}(H) = 0, \tag{73}$$
$$h(H) = 1, \tag{74}$$
$$\breve{\eta}\left(\phi H\right) = 0. \tag{75}$$

If taking into account (73)-(75) with (72), we find

$$\left(\nabla_W\phi\right)H=-g(W,H)\,\delta-B(W,H)H+A_N W,$$

which yields

$$A_N W=\left(\nabla_W\phi\right)H+g(W,H)\,\delta+B(W,H)H. \tag{76}$$

As a result, if we assume that $M$ is totally geodesic, then (76) is reduced (71). The converse is clear. Thus, we complete the proof. □

**Proposition 4.5.** *Let M be a lightlike hypersurface of an* ($\varepsilon$)*- para Sasakian manifold* $\breve{M}$ *with a semi-symmetric non-metric connection . Then, for any* $W\in\Gamma(TM)$,

*i) if the vector field H is parallel, then we have*

$$A_N W=\breve{\eta}(A_N W)\,\delta+h(A_N W)\,H,$$

*ii) if the vector field K is parallel, then we have*

$$\begin{aligned}A^*_\xi W-\breve{\eta}\left(A^*_\xi W\right)\delta &= 0, \\ \tau(W) &= 0, \\ h\left(\phi A^*_\xi W\right)+h\left(A^*_\xi W\right) &= 0.\end{aligned}$$

*Proof.* *i*) Applying $\phi$ to (76) and using (59), we find

$$\begin{aligned}\phi A_N W &= \phi\left(\left(\nabla_W\phi\right)H\right) + g(W,H)\phi\delta + B(W,H)\phi H\\ &= \phi\left[\nabla_W\phi H - \phi(\nabla_W H)\right] + g(W,H)\left[\breve{\phi}\delta - h(\delta)N\right]\\ &\quad + B(W,H)\left[\breve{\phi}H - h(H)N\right]\\ &= -\phi^2(\nabla_W H)\\ &= -\nabla_W H + \breve{\eta}(\nabla_W H)\delta + h(\nabla_W H)H + h\left(\phi\nabla_W H\right)N,\end{aligned}$$

for all $W \in \Gamma(TM)$. If $H$ is parallel, i.e. $\nabla_W H = 0$, then this equation reduced to

$$\phi A_N W = 0.$$

From this equation and (58), we get

$$\breve{\phi}(A_N W) = h(A_N W)N.$$

Applying $\breve{\phi}$ to this equation and using (14), we obtain

$$A_N W = \breve{\eta}(A_N W)\delta + h(A_N W)H.$$

*ii*) Suppose that the vector field $K$ is parallel. Replacing $V$ by $\xi$ in (66) and using (12), we have

$$\left(\nabla_W\phi\right)\xi = 0.$$

Hence, we find

$$\begin{aligned}\left(\nabla_W\phi\right)\xi &= \nabla_W\phi\xi - \phi(\nabla_W\xi)\\ 0 &= -\nabla_W K + \phi\left(A^*_\xi W\right) + \tau(W)K\end{aligned}$$

$$\phi\left(A^*_\xi W\right) = -\tau(W)K.$$

Applying $\phi$ to this equation and using (59), we get

$$\phi^2\left(A^*_\xi W\right) = -\tau(W)\phi K$$

$$h\left(\phi A^*_\xi W\right)N + A^*_\xi W - \breve{\eta}\left(A^*_\xi W\right)\delta - h\left(A^*_\xi W\right)H = -\tau(W)\phi K,$$

which completes the proof. □

**Theorem 4.6.** *Let $M$ be a lightlike hypersurface of an ($\varepsilon$)-para Sasakian manifold $\breve{M}$ with a semi-symmetric non-metric connection. Then, the screen distribution of $M$ is integrable if and only if*

$$\begin{aligned}C(W,\delta) &= C(\delta,W),\\ C(W,\delta) &= \varepsilon g\left(\phi W,N\right),\\ g\left(\phi W,N\right) &= g\left(W,\phi N\right).\end{aligned}$$

*Proof.* For all $W,V \in \Gamma(S(TM))$, $N \in \Gamma(ltrTM)$ the screen distribution is integrable if and only if

$$\begin{aligned}g([W,V],N) &= 0\\ g(W,N) + \varepsilon g\left(W,\breve{\phi}N\right) - g(\nabla_\delta W,N) &= 0\\ C(W,\delta) &= \varepsilon g\left(W,\breve{\phi}N\right)\\ C(W,\delta) &= \varepsilon g\left(\phi W,N\right).\end{aligned}$$

Also we can write screen distribution is integrable if and only if

$$\begin{aligned} g([W,V],N) &= 0 \\ g\left(\tilde{\tilde{\nabla}}_W\delta - \tilde{\tilde{\nabla}}_\delta W - \breve{\eta}(\delta)W - \breve{\eta}(W)\delta, N\right) &= 0 \\ g(\nabla_W\delta, N) - g(\nabla_\delta W, N) &= 0 \\ C(W,\delta) - C(\delta, W) &= 0 \\ C(W,\delta) &= C(\delta, W) \\ g\left(\phi W, N\right) &= g\left(W, \phi N\right). \end{aligned}$$

This completes the proof. □

## 5. Invariant Lightlike Hypersurfaces of an ($\varepsilon$)-para Sasakian Manifold with a Semi-Symmetric Non-Metric Connection

**Definition 5.1.** Let $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ be an $(n+2)$-dimensional ($\varepsilon$)-almost paracontact metric manifold endowed a semi-symmetric non-metric connection and $M$ be a lightlike hypersurface of $\breve{M}$. If

$$\breve{\phi}(S(TM)) = S(TM)$$

then, $M$ is called an invariant lightlike hypersurface of $\breve{M}$ [26].

**Theorem 5.2.** *Let $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ be an $(n+2)$-dimensional ($\varepsilon$)-almost paracontact metric manifold endowed semi-symmetric non-metric connection. Then $M$ is an invariant lightlike hyprsurface of $\breve{M}$ if and only if*

$$\begin{aligned} \breve{\phi}(Rad\,TM) &= Rad\,TM, \\ \breve{\phi}(ltrTM) &= ltrTM. \end{aligned}$$

*Proof.* Let $M$ be an invariant lightlike hyprsurface of $\breve{M}$. From $\breve{\phi}E = \phi E = P\phi E + \theta(\phi E)E$, for any $W \in \Gamma(TM)$, we get

$$\breve{g}\left(\breve{\phi}E, W\right) = \breve{g}\left(E, \phi W + h(w)N\right) = h(w), \tag{77}$$

$$\breve{g}\left(\breve{\phi}E, W\right) = \breve{g}\left(\breve{\phi}E, PW + h(w)H\right) = \breve{g}\left(\breve{\phi}E, PW\right) + h(w). \tag{78}$$

From (77) and (78), we find

$$\breve{g}\left(\breve{\phi}E, PW\right) = 0,$$

namely, there is not any component of $\breve{\phi}E$ in $S(TM)$ and $\breve{\phi}(Rad\,TM) = Rad\,TM$. For any local section $N$ of $ltrTM$, we can write

$$\breve{\phi}N = P\phi N + \breve{g}\left(\breve{\phi}N, N\right)E + \breve{g}\left(\breve{\phi}N, E\right)N.$$

Then, for any $W \in \Gamma(TM)$, we have

$$\begin{aligned} \breve{g}\left(\breve{\phi}N, W\right) &= \breve{g}\left(\breve{\phi}N, PW + h(w)H\right) \\ &= \breve{g}\left(\breve{\phi}N, PW\right) \\ &= \breve{g}\left(N, \breve{\phi}PW\right), \end{aligned}$$

where $PW \in S(TM)$. Since $M$ is an invariant lightlike hypersurface, $\breve{\phi}PW \in S(TM)$, then we get

$$\breve{g}\left(\breve{\phi}N, W\right) = \breve{g}\left(N, \breve{\phi}PW\right) = 0.$$

Hence, there is no component of $\breve{\phi}N$ in $S(TM)$.

Also if we apply $\breve{\phi}$ to $\breve{\phi}N = P\phi N + \breve{g}\left(\breve{\phi}N, N\right)E + \breve{g}\left(\breve{\phi}N, E\right)N$, then we find that $P\phi N = 0$. Therefore we have

$$\breve{\phi}N = \breve{g}\left(\breve{\phi}N, N\right)E + \breve{g}\left(\breve{\phi}N, E\right)N.$$

which implies

$$\breve{g}\left(\breve{\phi}N, N\right) = \breve{g}\left(\breve{\phi}N, E\right) = 0.$$

Since $\ker\breve{\phi} = Span\{\delta\}$,we find $\breve{g}\left(\breve{\phi}N, N\right) = 0$. Thus $\breve{\phi}N = \breve{g}\left(\breve{\phi}N, E\right)N$, that is $\breve{\phi}(ltrTM) = ltrTM$.

Conversely, let $\breve{\phi}(Rad\,TM) = Rad\,TM$ and $\breve{\phi}(ltrTM) = ltrTM$. For any $W \in S(TM)$, we have

$$\breve{g}\left(E, \breve{\phi}W\right) = \breve{g}\left(\breve{\phi}E, W\right) = 0.$$

Thus there is no component of $\breve{\phi}W$ in $ltrTM$. Similarly, we get

$$\breve{g}\left(\breve{\phi}N, W\right) = \breve{g}\left(N, \breve{\phi}W\right) = 0,$$

which implies that there is no component of $\breve{\phi}W$ in $Rad\,TM$. This completes the proof. □

**Example 5.3.** Let $\left(R^5_2, \breve{g}, \breve{\phi}, \breve{\eta}, \delta\right)$ be an $(\varepsilon)$-para Sasakian manifold given in Example 3.1., where $\breve{g}$ is of signature $(-,+,-,++\ )$with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial y_1, \partial y_2, \partial z\}$ . Suppose $M$ is a hypersurface of $R^5_2$ given by

$$\begin{aligned} -x^1 &= y^1 = u_1, \\ x^2 &= u_2, \\ y^2 &= u_{3,} \\ z &= u_{4.} \end{aligned}$$

Then $Rad\,TM = span\left\{E = -2\partial x_1 - 2\partial x_2 + 2\partial y_1 + 2\partial y_2 - \left(y^1 + y^2\right)\partial z\right\}$ and $ltr(TM)$ is spanned by

$$N = \frac{1}{2}\left(\partial x_1 - \partial x_2 - \partial y_1 + \partial y_2 + \left(y^1 - y^2\right)\partial z\right).$$

It can be easily checked that $\breve{\phi}E = -E$, $\breve{\phi}N = -N$.Thus $M$ is an invariant lightlike hypersurface of $R^5_2$.

**Theorem 5.4.** *Let $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ be an $(\varepsilon)$-almost paracontact metric manifold endowed semi-symmetric non-metric connection and $M$ be an invariant lightlike hypersurface of $\breve{M}$. Then $(M, \phi, \delta, \breve{\eta}, g, \varepsilon)$ is an $(\varepsilon)$-almost paracontact metric manifold with a semi-symmetric non-metric connection.*

*Proof.* Let $M$ be an invariant lightlike hypersurface of $\breve{M}$ and $W, V \in \Gamma(TM)$. From (58) and $\phi W = \breve{\phi}SW$, where $S$ denotes the projection morphism of $TM$ on $D$, we have

$$\breve{\phi}W = \phi W = \breve{\phi}SW. \tag{79}$$

If we apply $\breve{\phi}$ to (79), we can write

$$\phi^2 W = W - \eta(W)\delta. \tag{80}$$

Also from (79), it follows that

$$\breve{\phi}\delta = \phi\delta = 0. \tag{81}$$

In view of (80) and (81), we can see that

$$\breve{\eta} \circ \breve{\phi} = \breve{\eta} \circ \phi \tag{82}$$
$$\breve{\eta}(\delta) = 1. \tag{83}$$

Moreover, from (19), we find

$$g\left(\phi W, V\right) = g(W, \phi V), \tag{84}$$

and from (18), we obtain

$$g\left(\phi W, \phi V\right) = g(W, V) - \varepsilon\breve{\eta}(W)\,\breve{\eta}(V). \tag{85}$$

Therefore from (80)-(85), we completes proof. □

**Proposition 5.5.** *Let $M$ be an invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifold $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ endowed with a semi-symmetric non-metric connection. Then we have*

$$g(\delta, A_N PW) = \theta(W)(1+\varepsilon),$$

*for $W \in \Gamma(TM)$.*

*Proof.* Since $\breve{g}(\delta, N) = 0$ and using (47), we write

$$\breve{g}(W, N) + \varepsilon\breve{g}\left(\breve{\phi}W, N\right) = \breve{g}(\delta, A_N W). \tag{86}$$

For any $W \in \Gamma(TM)$, from (86), we find

$$\begin{aligned} \breve{g}(PW + \theta(W), N) + \varepsilon\breve{g}\left(W, \breve{\phi}N\right) &= \breve{g}(\delta, A_N PW) \\ \theta(W)(1+\varepsilon) &= g(\delta, A_N PW). \end{aligned}$$

□

**Corollary 5.6.** *Let $M$ be an invariant lightlike hypersurface of a timelike (resp., spacelike) $(\varepsilon)$-para Sasakian manifold $\left(\breve{M}, \breve{\phi}, \delta, \breve{\eta}, \breve{g}, \varepsilon\right)$ endowed with a semi-symmetric non-metric connection. Then we have $g(\delta, A_N PW) = 0$ (resp., $g(\delta, A_N PW) = 2\theta(W)$).*

**Theorem 5.7.** *An invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifold with semi-symmetric non metric connection is also an $(\varepsilon)$-para Sasakian manifold endowed with a semi-symmetric non-metric connection. Furthermore, we have*

$$B(W, \phi V)N - B(W, V)\phi N = 0, \tag{87}$$

$$\phi(A_N W) = A_{\phi N} W - \theta(W)\,\delta, \tag{88}$$

*for any $W, V \in \Gamma(TM)$.*

*Proof.* From (38) and (41) , we find

$$\begin{aligned} \left(\tilde{\tilde{\nabla}}_W \breve{\phi}\right)(V) &= \mathring{\nabla}_W \breve{\phi} V + B(W, \breve{\phi} V) \\ &\quad - \breve{\phi}\mathring{\nabla}_W V - B(W, V)\breve{\phi} N. \end{aligned}$$

From the definition of an invariant lightlike hypersurface, we have

$$\left(\tilde{\tilde{\nabla}}_W \breve{\phi}\right)(V) = \left(\mathring{\nabla}_W \breve{\phi}\right) V + B(W, \phi V)N - B(W, V)\breve{\phi} N.$$

Using (46), we get

$$\begin{pmatrix} -\breve{g}\left(\phi W,\phi V\right)\delta-\varepsilon\breve{\eta}\left(V\right)W \\ +\varepsilon\breve{\eta}\left(V\right)\breve{\eta}\left(W\right)\delta-\breve{\eta}\left(V\right)\phi\left(W\right) \end{pmatrix}=\begin{pmatrix} \left(\mathring{\nabla}_W\breve{\phi}\right)V+B(W,\phi V)N \\ -B(W,V)\breve{\phi}N \end{pmatrix}.$$

Equating tangential parts of above equation provides

$$\left(\mathring{\nabla}_W\breve{\phi}\right)V=-\breve{g}\left(\phi W,\phi V\right)\delta-\varepsilon\breve{\eta}\left(V\right)W+\varepsilon\breve{\eta}\left(V\right)\breve{\eta}\left(W\right)\delta-\breve{\eta}\left(V\right)\phi\left(W\right).$$

which implies that $M$ is an $(\varepsilon)$-para Sasakian manifold with semi-symmetric non metric connection via Theorem 5.1. Also, equating transversal parts of above equation gives equation (87).

Next using (46) and (39) with (34), we obtain

$$\left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)N=\tilde{\tilde{\nabla}}_W\breve{\phi}N-\breve{\phi}\left(\tilde{\tilde{\nabla}}_W N\right),$$

which implies (88) and $\tau\left(W\right)=0$. This completes the proof. □

## 6. Screen Semi-Invariant Lightlike Hypersurfaces of an $(\varepsilon)$-para Sasakian Manifold with a Semi-Symmetric Non-Metric Connection

**Definition 6.1.** Let $\left(\breve{M},\breve{\phi},\delta,\breve{\eta},\breve{g},\varepsilon\right)$ be an $(n+2)$-dimensional $(\varepsilon)$-almost paracontact metric manifold endowed with a semi-symmetric non-metric connection and $M$ be a lightlike hypersurface of $\breve{M}$. If

$$\begin{aligned} \breve{\phi}(Rad\,TM) &\subset S(TM), \\ \breve{\phi}(ltrTM) &\subset S(TM), \end{aligned}$$

then $M$ will be called a screen semi-invariant lightlike hypersurface of $\breve{M}$. (see also

**Example 6.2.** Let $\left(R_2^5,\breve{g},\breve{\phi},\breve{\eta},\delta\right)$ be an $(\varepsilon)$-para Sasakian manifold given in Example 3.1., where $\breve{g}$ is of signature $(-,+,-,++)$with respect to the canonical basis $\{\partial x_1,\partial x_2,\partial y_1,\partial y_2,\partial z\}$. Suppose $M$ is a hypersurface of $R_2^5$ given by

$$\begin{aligned} x^2 &= y^2=u_2, \\ x^1 &= u_1, \\ y^1 &= u_{3,} \\ z &= u_{4.} \end{aligned}$$

Then $Rad\,TM=span\left\{2\partial x_1+\sqrt{2}\partial x_2-2\partial y_1+\sqrt{2}\partial y_2+\left(2+2y^1+\sqrt{2}y^2\right)\partial z\right\}$ and $ltr(TM)$ is spanned by $N=\sqrt{2}\partial x_1+\sqrt{2}\partial y_1+\left(2+\sqrt{2}y^1\right)\partial z$. We easily check that

$$\begin{aligned} \breve{\phi}E &= -2\partial x_1+\sqrt{2}\partial x_2+2\partial y_1+\sqrt{2}\partial y_2+\left(2y^1+\sqrt{2}y^2\right)\partial z\in\Gamma(S(TM)), \\ \breve{\phi}N &= \sqrt{2}\partial x_1+\sqrt{2}\partial y_1+\sqrt{2}y^1\partial z\in\Gamma(S(TM)), \end{aligned}$$

thus $M$ is a screen semi invariant lightlike hypersurface of $R_2^5$.

**Proposition 6.3.** *A screen semi-invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifold with semi-symmetric non-metric connection is $(\varepsilon)$-para Sasakian manifold, if*

$$\left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)(V)=-\breve{g}\left(\phi W,\phi V\right)\delta-\breve{\eta}\left(V\right)\phi^2W-\breve{\eta}\left(V\right)\phi W.$$

In view of (55) and (56), we can find

$$\breve{g}(H,K) = 1. \tag{89}$$

Therefore $\langle H \rangle \oplus \langle K \rangle$ is a non-degenerate vector bundle of $S(TM)$ with rank2. Since $\delta$ belong to $S(TM)$ and $\breve{g}(H,\delta) = \breve{g}(K,\delta) = 0$. Hence, there exists a non-degenerate distribution $D_\circ$ of rank $n-3$ on $M$ such that

$$S(TM) = D_\circ \perp \{\langle H \rangle \oplus \langle K \rangle\} \perp \langle \delta \rangle ,$$

we note that $D_\circ$ is invariant distribution with $\breve{\phi}$, that is $\breve{\phi} D_\circ = D_\circ$. Denoting

$$D = D_\circ \perp Rad\, TM \perp \langle K \rangle$$

and

$$D' = \langle H \rangle$$

then, we have

$$TM = D \oplus D' \perp \langle \delta \rangle .$$

Thus, every $W \in \Gamma(TM)$ can be expressed as

$$W = RW + QW + \breve{\eta}(W)\,\delta,$$

where $R$ and $Q$ are projections of $TM$ into $D$ and $D'$, respectively. Hence, we may write

$$\phi W = \breve{\phi} RW, \tag{90}$$

$W \in \Gamma(TM)$. If we use (15), (50) and (51), we obtain

$$\breve{\phi}^2 W = \phi^2 W + h\left(\phi W\right) N + h(W)\, H. \tag{91}$$

By comparing the tangential and transversal parts above equation, we find

$$\phi^2 = I - \breve{\eta} \otimes \delta - h \otimes H, \tag{92}$$
$$h \otimes \phi = 0, \tag{93}$$
$$\phi\delta = 0, \tag{94}$$
$$h(\delta) = 0, \tag{95}$$

as well as

$$\breve{\eta}(H) = 0, \breve{\eta}(\delta) = 1 \tag{96}$$
$$\breve{\eta} \circ \phi = 0. \tag{97}$$

Therefore we have

**Proposition 6.4.** *Let M be a screen semi-invariant lightlike hyprsurface of an* ($\varepsilon$)*-almost paracontact metric manifold with semi-symmetric non-metric connection. Then M possesses a para* $\left(\phi, \delta, \breve{\eta}, H, h\right)$*-structure, namely, equations (92)-(97) are provided.*

Now, using equation (45), we write

$$\left(\tilde{\tilde{\nabla}}_W \breve{\phi}\right)(V) = \left(\breve{\nabla}_W \breve{\phi}\right)(V) - \breve{\eta}(V)\, \breve{\phi}(W).$$

Then, if we use (90) and (91), we have

**Proposition 6.5.** *A screen semi-invariant lightlike hypersurface of an (ε)-para Sasakian manifold with semi-symmetric non-metric connection is an (ε)-para Sasakian manifold, if*

$$\begin{aligned}\left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)(V) &= -\breve{g}\left(\phi W,\phi V\right)\delta + h(V)\breve{g}\left(\phi W,N\right)\delta \\ &+h(W)\breve{g}\left(\phi V,N\right)\delta - \varepsilon\breve{\eta}\left(V\right)\phi^2 W \\ &-\varepsilon\breve{\eta}\left(V\right)h(\phi W)N - \varepsilon\breve{\eta}\left(V\right)h(W)H \\ &-\breve{\eta}\left(V\right)\phi W - \breve{\eta}\left(V\right)h(V)N.\end{aligned}$$

Also, we have

**Theorem 6.6.** *Let M be a screen semi-invariant lightlike hyprsurface of an (ε)-para Sasakian manifold with a semi-symmetric non-metric connection $\left(\breve{M},\breve{\phi},\delta,\breve{\eta},\breve{g},\varepsilon\right)$. Then, we have*

$$\mathring{\nabla}_W K + \phi\left(A_N W\right) - \tau\left(W\right)K = 0,$$

$$B(W,K) = -h(A_N W).$$

*Proof.* From (46), we have $\left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)(N) = 0$. Further, from the Gauss and Weingarten formulas and (58), we find

$$\left(\tilde{\tilde{\nabla}}_W\breve{\phi}\right)(N) = \begin{pmatrix} \mathring{\nabla}_W K + B(W,K)N + \phi\left(A_N W\right) \\ +h\left(A_N W\right)N - \tau\left(W\right)K \end{pmatrix} = 0.$$

which completes the proof. □

### 6.1. *Integrability of* $D \perp \langle\delta\rangle$

**Theorem 6.7.** *Let M be a screen semi-invariant lightlike hyprsurface of an (ε)-para Sasakian manifold with a semi-symmetric non-metric connection. Then, the distribution $D \perp \langle\delta\rangle$ is integrable if and only if*

$$B(\phi W,V) = B(W,\phi V),$$

*for all $W,V \in \Gamma(D)$.*

*Proof.* We note that $W \in \Gamma\left(D \perp \langle\delta\rangle\right)$ if and only if $h\left(W\right) = g\left(W,K\right) = 0$. Now from (52), (65) and (69), we have

$$\begin{aligned} h\left[W,V\right] &= h(V)\tau\left(W\right) - h(W)\tau\left(V\right) \\ &+h\left(W\right)\breve{\eta}\left(V\right) - h\left(V\right)\breve{\eta}\left(W\right) \\ &-B(W,\phi V) + B(V,\phi W).\end{aligned}$$

In view of $h\left(W\right) = h(V) = 0$, we obtain

$$h\left[W,V\right] = B(\phi W,V) - B(W,\phi V),$$

for all $W,V \in \Gamma\left(D \perp \langle\delta\rangle\right)$ .Hence, we complete the proof. □

### 6.2. *Integrability of* $D' \perp \langle\delta\rangle$

**Theorem 6.8.** *Let M be a screen semi-invariant lightlike hyprsurface of an (ε)-para Sasakian manifold with a semi-symmetric non-metric connection. Then the distribution $D' \perp \langle\delta\rangle$ is integrable if and only if*

$$A_N\delta + \varepsilon H = 0.$$

*Proof.* $W \in D' \perp \langle\delta\rangle$ if and only if $\phi W = 0$. For all $W, V \in \Gamma(TM)$ and in view of (66), we have

$$\begin{aligned}\left(\nabla_W \phi\right) V &= -g(W,V)\delta + 2\varepsilon\breve{\eta}(W)\breve{\eta}(V)\delta \\ &\quad -\varepsilon\breve{\eta}(V)W - \breve{\eta}(V)\phi W + \breve{\eta}\left(\phi V\right)W \\ &\quad +h(V)A_N W + B(W,V)H.\end{aligned}$$

Then, we can write

$$\begin{aligned}\phi[W,V] &= \phi\nabla_W V - \phi\nabla_V W - \varepsilon\breve{\eta}(V)W \\ &\quad +\varepsilon\breve{\eta}(W)V + \breve{\eta}\left(\phi V\right)W - \breve{\eta}\left(\phi W\right)V \\ &\quad +h(V)A_N W + h(W)A_N V \\ &\quad +B(W,V)H + B(V,W)H.\end{aligned}$$

In particular from $\phi W = \phi V = 0$, for $W, V \in D' \perp \langle\delta\rangle$, we have

$$\phi[W,V] = -\varepsilon\breve{\eta}(V)W + \varepsilon\breve{\eta}(W)V + h(V)A_N W + h(W)A_N V.$$

Hence $D' \perp \langle\delta\rangle$ integrable if and only if

$$\phi[H,\delta] = 0,$$

namely

$$A_N\delta + \varepsilon H = 0.$$

This completes the proof. □

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