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# Investigation of $s-$Convex Functions by Using Convex Combinations

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**Abstract.** This work is inspired by the paper [7], where the authors consider $m-$convex functions. The main aim of this paper is to establish some inequalities for $s-$convex functions without using their derivatives. Also we obtain some new results in discrete form.

## 1. Introduction

Convexity has been frequently studied since early years of the $20^{th}$ century. Since then, many convexity classes has been emerged. One of the most functional convex function class is $s-$convex function class in the second sense. In [2], Breckner introduced that a function $f : I \subseteq [0, \infty) \to \mathbb{R}$, which is in that class must ensure the following inequality:

$$f\left(tx + (1-t)\,y\right) \leq t^s f(x) + (1-t)^s f(y) \tag{1}$$

for every $x, y \in I$, $t \in [0, 1]$ and $s \in (0, 1]$. The class of $s-$convex functions in the second sense is denoted by $K_s^2$.

Many hierarchies between convexity classes have been established by different authors. By the way, $s-$convex functions in the second sense can place as follows:

**Proposition 1.1.** *If a function $f : I \subseteq [0, \infty) \to \mathbb{R}$ is convex, then it is $s-$convex in the second sense. Also if a function $f : [0, \infty) \to \mathbb{R}$ is $s-$convex in the second sense, then it is $P-$function.*

*Proof.* For $t \in [0, 1]$ and $s \in (0, 1]$, it is easy to see that $t \leq t^s$ and also $t^s \leq 1$. As we choose $f$ is convex, for every $x, y \in I$ we have

$$\begin{aligned} f\left(tx + (1-t)\,y\right) &\leq t f(x) + (1-t) f(y) \\ &\leq t^s f(x) + (1-t)^s f(y) \\ &\leq f(x) + f(y) \end{aligned}$$

which completes the proof. $\square$

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That means, a function can satisfy the $s-$convexity condition even if it does not satisfy the convexity condition.

To distinguish a convex function class from another, researchers sometimes put forward some other details instead of writing just an inequality. For example, considering the interior domains of some convex functions, they are continuos while some of them are not. If we are to mention $s$-convex functions in the second sense, these functions can be discontinuous in the interior of their domain. Such an example is given below:

**Example 1.2.** A function $f : [0,2] \to \mathbb{R}$ described as

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 3x - 1 & 1 \leq x \leq 2 \end{cases}$$

is $s-$convex in the second sense for every $s \in (0, 7/10]$ and there is discontinuity at the point $x = 1$.

Also in [4], it is proven that an $s-$convex function in the second sense is always nonnegative on $[0, \infty)$ and $f(0) = 0$.

Many results have been found on $s-$convex functions since its definition at 1978. For recent studies about $s-$convexity, you can read the references [5, 7]. Another theorem is proven by Avcı Ardıç in [1] as follows:

**Theorem 1.3.** *Let $f$ be a $\varphi_s-$convex function and $(s, t_i) \in (0,1)^2$ for $i = 1, 2, ..., n$ with $\sum\limits_{i=1}^{n} t_i = T_n = 1$. Then the following inequality is valid:*

$$f\left(\sum_{i=1}^{n} t_i \varphi(x_i)\right) \leq \sum_{i=1}^{n} t_i^s f(\varphi(x_i)).$$

## 2. Some Features of $s$-Convex Functions

In this section we handle some new results about $s-$convex functions in the second sense defined on an interval including zero without using their derivatives.

**Theorem 2.1.** *Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ be a function, $x, z \in I$ with $x < z$ and $s \in (0,1]$. If so, the following expressions are equivalent:*

(i) *The following condition holds for the function $f$*

$$f(tx + (1-t)z) \leq t^s f(x) + (1-t)^s f(z) \tag{2}$$

*for every $t \in [0,1]$ real number.*

(ii) *$f$ ensures the following inequality*

$$f(y) \leq \left(\frac{z-y}{z-x}\right)^s f(x) + \left(\frac{y-x}{z-x}\right)^s f(z) \tag{3}$$

*for every $x$, $y$ and $z$ such that $y \in conv\{x, z\}$.*

(iii) *The next inequality is provided by $f$*

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ (z-x)^s & (z-y)^s & 0 \\ 0 & \left(\frac{y-x}{z-x}\right)^s & 1 \end{vmatrix} \geq 0 \tag{4}$$

*for every $x$, $y$ and $z$ with $y \in conv\{x, z\}$.*

*Proof.* By implementing the inequality (2) to the points $z$ and $x$ such that $x \neq z$ with coefficient $t = (z-y)/(z-x)$, (i) implies (ii). In this case, we use the convex combination of points $x$ and $z$ as,

$$y = \frac{z-y}{z-x}x + \frac{y-x}{z-x}z.$$

Inequality (3) can be lined up into the following one

$$(z-y)^s f(x) - (z-x)^s f(y) + (y-x)^s f(z) \geq 0,$$

so (ii) implies (iii). From (iii) follows (i) by calculating the determinant and dividing by $(z-x)^s$ of both sides, then change of variables $y = tx + (1-t)z$. So, the proof is done. □

Respecting the definition of functions belonging to $K_s^2$, each of statements in Theorem 2.1 is utilizable.

**Corollary 2.2.** *Let $I \subseteq \mathbb{R}$ and $s \in (0,1]$ be a real number. Then each $f : I \to \mathbb{R}$, $f \in K_s^2$ satisfies the inequality*

$$\frac{f(y) - f(x)}{y^s + x^s} \leq \frac{f(z) + f(x)}{z^s - x^s} \tag{5}$$

*for every $x, y, z \in I$ where $x < y < z$.*

*Proof.* Inequality (5) can be obtained by using (3) and the inequality

$$b^s - a^s \leq (b-a)^s \leq b^s + a^s \tag{6}$$

for each $0 < a < b$ and every $s \in (0,1]$. □

**Corollary 2.3.** *Let $I \subseteq \mathbb{R}$ be an interval including the point zero, and let $s \in (0,1]$ be a real number. So, each $f : I \to \mathbb{R}$, $f \in K_s^2$ satisfies the following two inequalities for $f(0) = 0$:*

(i) *If $x \leq y < 0$, then*

$$\left(\frac{y}{x}\right)^s f(x) \geq f(y). \tag{7}$$

(ii) *If $0 < x \leq y$, then the inequality given in inequality (7) is reversed.*

(iii) *If $x < 0 < y$, then*

$$\frac{y^s f(x) + (-x)^s f(y)}{(y-x)^s} \geq 0. \tag{8}$$

*Proof.* By reorganizing inequality (3) in varied orders of points in question, inequalities (7) and (8) can come up. □

Now let us keep in view functions that belong to $K_s^2$ which are defined on $[a,b]$ such that $a < 0 < b$. These functions are bounded as we explained in the next lemma.

**Lemma 2.4.** *Let $a < 0 < b$ and $s \in (0,1]$ be real numbers. Then each $f : [a,b] \to \mathbb{R}$, $f \in K_s^2$ satisfies next two inequalities:*

(i) *If $a \leq x < 0$, then*

$$\frac{(b-x)^s f(0) - (-x)^s f(b)}{b^s} \leq f(x) \leq \frac{(x-a)^s f(0) + (-x)^s f(a)}{(-a)^s} \tag{9}$$

(ii) *If* $0 < x \le b$, *then*

$$\frac{(x-a)^s f(0) - x^s f(a)}{(-a)^s} \le f(x) \le \frac{x^s f(b) + (b-x)^s f(0)}{b^s} \tag{10}$$

*Proof.* Inequality (9) can be obtained by using ordered triplets $x < 0 < b$ and $a \le x < 0$ with $s-$convexity in the second sense of $f$. The latter inequality (10) can be obtained by using ordered triplets $a < 0 < x$ and $0 < x \le b$ with $s-$convexity of $f$. □

**Theorem 2.5.** *Let* $a < 0 < b$ *and* $s \in (0,1]$ *be real numbers. For each* $f : [a,b] \to \mathbb{R}$, $f \in K_s^2$ *satisfying* $f(0) = 0$ *the following inequality holds*

$$-\frac{b^{2s+1} f(a) + (-a)^{2s+1} f(b)}{(s+1)(-a)^s b^s} \le \int_a^b f(x)\,dx \le \frac{bf(b) - af(a)}{s+1}. \tag{11}$$

*Proof.* By integrating the inequalities in inequalities (9) and (10), then summing them side by side, we get the desired result. □

## 3. Main Results

In this section, we prove the main theorem for functions belonging to $K_s^2$. Additionally we give some corollaries for inequalities obtained by using Riemann integrals and also for discrete inequalities.

Let us have a look at an upper bound of $f \in K_s^2$, by taking into account a point $c \in [a,b]$.

**Theorem 3.1.** *Let* $I$ *be an interval in* $\mathbb{R}$ *and* $a, b, c \in I$ *with condition* $a \le c \le b$. *Let* $s \in (0,1]$ *be a real number. Then each* $f : I \to \mathbb{R}$, $f \in K_s^2$ *ensures the following inequality*

$$\int_a^b f(x)\,dx \le \frac{(c-a)[f(a)+f(c)] + (b-c)[f(c)+f(b)]}{s+1}. \tag{12}$$

*Proof.* The inequality (12) holds if $a = b$, clearly. If so, let $a < b$.

Assuming $a \le x \le c$ with $a < c$, exerting the $s-$convexity in the second sense of the function $f$ and using the convex combination shown as

$$x = \frac{c-x}{c-a}a + \frac{x-a}{c-a}c,$$

we get

$$f(x) \le \left(\frac{c-x}{c-a}\right)^s f(a) + \left(\frac{x-a}{c-a}\right)^s f(c). \tag{13}$$

By integrating both sides of the above inequality from $a$ to $c$, it follows the following integral inequality

$$\begin{aligned} \int_a^c f(x)\,dx &\le f(a)\int_a^c \left(\frac{c-x}{c-a}\right)^s dx + f(c)\int_a^c \left(\frac{x-a}{c-a}\right)^s dx \\ &= \frac{c-a}{s+1}[f(a)+f(c)] \end{aligned} \tag{14}$$

which is valid for $c = a$ also. On the other hand, by integrating (13) from $c$ to $b$, we obtain

$$\int_c^b f(x)\,dx \le \frac{b-c}{s+1}[f(c)+f(b)]. \tag{15}$$

Exerting the inequalities (14) and (15) to the related terms of the right-hand side of the seperation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

and rearranging, we achieve the inequality (12). □

By choosing $c = a$ or $c = b$ in (12) we handle the following remark which is the right hand side of an inequality proved by Dragomir and Fitzpatrick in [3].

**Remark 3.2.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, $f \in K_s^2$ and $s \in (0,1]$ be a real number. Then we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{16}$$

**Corollary 3.3.** *Let $[a,b] \subseteq \mathbb{R}$ and $a = a_0 < a_1 < a_2 < ... < a_n = b$. For each $s \in (0,1]$, the function $f : [a,b] \to \mathbb{R}$, $f \in K_s^2$ satisfies the inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \sum_{i=1}^{n}\frac{(a_i - a_{i-1})\left[f(a_i)+f(a_{i-1})\right]}{s+1}. \tag{17}$$

*Proof.* The proof can be made by using inequality (12) and the points $a_i$ $(i = 0,1,2,...,n)$. □

In Theorem 1.3, by choosing $\varphi(x) = x$, we get the next remark. On the other hand, we use mathematical induction to prove it in a different way.

**Remark 3.4.** Let $x_i \in I \subseteq \mathbb{R}$ $(i = 1,2,...,n)$ be the points which makes a convex combination $\sum\limits_{i=1}^{n} t_i x_i$ with coefficients $t_i \in [0,1]$. Let $s \in (0,1]$ be a real number. If so, each $f : I \to \mathbb{R}$, $f \in K_s^2$ providing the condition $f(0) = 0$ ensures the following inequality

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i^s f(x_i). \tag{18}$$

*Proof.* With the help of mathematical induction on the number of points $x_i$, the proof can be accomplished.

For $n = 1$, the inequality (18) provides as

$$\begin{aligned} f(t_1x_1) &= f(t_1x_1 + (1-t_1).0) \\ &\leq t_1^s f(x_1) + (1-t_1)^s f(0) \\ &= t_1^s f(x_1). \end{aligned}$$

Assume that inequality (18) holds for all convex combinations for $i = 2,3,...,n-1$ members. Supposing that $t_1 < 1$, and using the inductive hypothesis to the point

$$y_1 = \sum_{i=2}^{n}\frac{t_i}{1-t_1}x_i, \tag{19}$$

where the sum $\sum\limits_{i=2}^{n}(t_i/(1-t_1))\,x_i$ is convex combination belonging to $I$, we get

$$f(y_1) \leq \sum_{i=2}^{n}\left(\frac{t_i}{1-t_1}\right)^s f(x_i). \tag{20}$$

Using the inequalities (1) and (20), it is easy to handle

$$\begin{aligned} f\left(\sum_{i=1}^{n} t_i x_i\right) &= f(t_1x_1 + (1-t_1)\,y_1) \\ &\leq t_1^s f(x_1) + (1-t_1)^s f(y_1) \\ &\leq t_1^s f(x_1) + (1-t_1)^s \sum_{i=2}^{n}\left(\frac{t_i}{1-t_1}\right)^s f(x_i) \\ &= \sum_{i=1}^{n} t_i^s f(x_i), \end{aligned}$$

achieving the inequality (18). □

Abiding by Remark 3.4 we can obtain the following corollary.

**Corollary 3.5.** *Let $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a function and $s$ is a real number belongs to $(0,1]$. Then each $f : [a,b] \to \mathbb{R}$, $f \in K_s^2$ satisfies the following inequality*

$$f\left(\sum_{i=1}^{n} \frac{|I_{ni}|}{|I|} g(x_{ni})\right) \leq \frac{1}{|I|^s} \sum_{i=1}^{n} |I_{ni}|^s f(g(x_{ni})). \tag{21}$$

*Proof.* Considering then $I = [a,b]$, $n \in \mathbb{Z}^+$ and let disjoint subintervals $I_{ni}$ composes $I = \cup_{i=1}^{n} I_{ni}$. Therefore, each disjoint subinterval $I_{ni}$ contracts to the point as $n \to \infty$. Let us take a point $x_{ni}$ from each $I_{ni}$ and constitute convex combination

$$\sum_{i=1}^{n} \frac{|I_{ni}|}{|I|} g(x_{ni})$$

of points $y_{ni} = g(x_{ni})$ with coefficients $t_{ni} = |I_{ni}| / |I|$ where $\|$ denotes the length. Applying the inequality (18) to the above convex combination, we produce the inequality (21). □

**Corollary 3.6.** *Let $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a function and let $h : [a,b] \to \mathbb{R}$ be a positive function. Then each $f : [a,b] \to \mathbb{R}$, $f \in K_s^2$ satisfies the inequality with a real number $s \in (0,1]$*

$$f\left(\frac{\sum_{i=1}^{n} |I_{ni}| g(x_{ni}) h(x_{ni})}{\sum_{i=1}^{n} |I_{ni}| h(x_{ni})}\right) \leq \frac{\sum_{i=1}^{n} |I_{ni}|^s f(g(x_{ni})) h(x_{ni})}{\sum_{i=1}^{n} |I_{ni}|^s h(x_{ni})}. \tag{22}$$

*Proof.* By using the convex combination

$$\sum_{i=1}^{n} \frac{|I_{ni}| h(x_{ni})}{\sum_{i=1}^{n} |I_{ni}| h(x_{ni})} g(x_{ni}) = \frac{\sum_{i=1}^{n} |I_{ni}| g(x_{ni}) h(x_{ni})}{\sum_{i=1}^{n} |I_{ni}| h(x_{ni})}$$

of points $y_{ni} = g(x_{ni})$ with coefficients $t_{ni} = |I_{ni}| h(x_{ni}) / \sum_{i=1}^{n} |I_{ni}| h(x_{ni})$ and the method used in the proof of Corollary 3.5 we have the desired result. □

Similar results related to $m-$convex functions have been found in [6]. Interested readers can handle applications of our theorems and corollaries.

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