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Some New Generalizations of Ostrowski Type Inequalities for $s-$Convex Functions via Fractional Integral Operators

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Abstract. Remarkably a lot of Ostrowski type inequalities involving various fractional integral operators have been investigated by many authors. Recently, Raina [34] introduced a new generalization of the Riemann-Liouville fractional integral operator involving a class of functions defined formally by $\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k$. Using this fractional integral operator, in the present note, we establish some new fractional integral inequalities of Ostrowski type whose special cases are shown to yield corresponding inequalities associated with Riemann-Liouville fractional integral operators.

1. Introduction

In 1938, A. Ostrowski [28], proved the following interesting and useful integral inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x), x \in [a,b]$.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := sup_{t\in(a,b)}|f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a),$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations [2, 14, 15, 24, 26]. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n-times differentiable mappings *etc.* appeared in a number of papers (see [3, 4, 6, 12, 13, 22, 23, 25, 29–31, 33, 43, 44, 46]). In recent years, one more dimension has been added to this studies, by introducing a number of integral inequalities involving various fractional operators like Riemann-Liouville, Erdelyi-Kober, Katugampola,

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conformable fractional integral operators *etc.* by many authors (see, e.g., [1, 8–11, 19, 20, 32, 35, 39]). Riemann-Liouville fractional integral operators are the most central between these fractional operators.

The overall structure of the study takes the form of four sections including introduction. The remaining part of the paper proceeds as follows: In Section 2, the generalized version of fractional integral operator are summarized, along with the needed definitions. In Section 3, firstly, an integral identity for generalized fractional integral operators are proved. Then, some new Ostrowski type inequalities for functions whose first derivatives in absolute value are $s-$ convex functions in the second sense utilizing this integral identity are presented and some corollary and remarks for theorems are given. Some conclusions of research are discussed in Section 4.

2. Preliminaries

In this section, we will give some previously known concepts which will be used in the proof of our main results. First of all let set of real numbers be denoted by $\mathbb{R}$. Let $[a,b]$ be an interval in $\mathbb{R}$. We follow these notations throughout the paper unless otherwise specified.

A function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [7], the class of functions which are $s-$convex in the second sense has been introduced by Breckner as the following:

Definition 2.1. A function $\varphi : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$\varphi(tx + (1-t)y) \leq t^s\varphi(x) + (1-t)^s\varphi(y).$$

for all $x, y \in [0,\infty)$, $t \in [0,1]$ and for some fixed $s \in (0,1]$. This class of s-convex functions is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, $s-$convexity reduces to ordinary convexity of functions defined on $[0,\infty)$. Also, connections between s-convexity in the first sense and s-convexity in the second sense were discussed in paper [18].

In [17] Dragomir and Fitzpatrick proved a variant of the Hermite-Hadamard inequality which holds for s-convex functions in the second sense.

Theorem 2.2. *Suppose that* $\varphi : [0,\infty) \to [0,\infty)$ *is an s-convex function in the second sense, where* $s \in (0,1]$ *and let* $a, b \in [0,\infty), a < b$. *If* $\varphi \in L[a,b]$, *then the following inequality hold:*

$$2^{s-1}\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(x)\,dx \leq \frac{\varphi(a)+\varphi(b)}{s+1} \tag{1}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1). For more study related to s-convexity in the second sense (see, e.g., [4, 5, 16]).

In [34], Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k \quad (\rho, \lambda \in \mathbb{R}^+; |x| < \infty), \tag{2}$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers. With the help of (2), Raina [34] and Agarwal *et al.* [6] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt \qquad (x > a), \tag{3}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \qquad (x < b), \tag{4}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty. \tag{5}$$

In fact, for $\varphi \in L(a,b)$, we have

$$\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)\|_1 \le \mathfrak{M}(b-a)^{\lambda}\|\varphi\|_1 \tag{6}$$

and

$$\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)\|_1 \le \mathfrak{M}(b-a)^{\lambda}\|\varphi\|_1, \tag{7}$$

where

$$\|\varphi\|_p := \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α defined by (see, e.g., [37]).

$$J^{\alpha}_{a+}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad x > a \tag{8}$$

and

$$J^{\alpha}_{b-}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \quad x < b \tag{9}$$

follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (3) and (4), and the boundedless of (8) and (9) on $L(a,b)$ is also inherited from (3) and (4), (see [6]). Here $\Gamma(\alpha)$ is the familiar Gamma function (see, e.g., [42, Section 1.1]). In the case of $\alpha = 1$, the Riemann-Liouville fractional integral reduces to the classical integral. Some recent results and properties concerning this operators can be found in [8–11, 21, 27, 36, 38, 40, 41].

We recall the Beta function $B(\alpha, \beta)$ defined by

$$B(\alpha,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-), \end{cases} \tag{10}$$

and the incomplete Beta function $B_x(\alpha, \beta)$ defined by

$$B(x;\alpha,\beta) := \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\Re(\alpha) > 0), \tag{11}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ are the sets of complex numbers and non-positive integers, respectively, (see, e.g., [42, Section 1.1]). Throughout this paper, the α, β in $B(\alpha, \beta)$ and $B_x(\alpha, \beta)$ are assumed to be real numbers.

Motivated by the recent results given in [6, 34, 45], in the present note, we obtain here new Ostrowski type inequalities for $s-$convex functions in the second sense via generalized fractional integral operators. An interesting feature of our results is that they would provide generalizations of those given in earlier works and new estimates on these types of inequalities.

3. Main Results

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $f \in L[a,b]$, then we have the following identity involving generalized fractional integral operators:*

$$\begin{aligned}
&\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\\
&-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a) + (\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\\
=\ &\int_0^1 \mu(t)f'(ta+(1-t)b)dt
\end{aligned} \tag{12}$$

for each $t \in [0,1]$, where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and

$$\mu(t) = \begin{cases} -t^{\lambda}\mathcal{F}^{\sigma}_{\rho+\lambda+1}[w(b-a)^{\rho}t^{\rho}] & , \quad t \in [0, \frac{b-x}{b-a}) \\ (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho+\lambda+1}[w(b-a)^{\rho}(1-t)^{\rho}] & , \quad t \in [\frac{b-x}{b-a}, 1]. \end{cases}$$

for all $x \in [a,b]$.

Proof. Integrating by parts, we get

$$\begin{aligned}
\xi &= \int_0^1 \mu(t)f'(ta+(1-t)b)dt\\
&= \int_0^{\frac{b-x}{b-a}} (-t^{\lambda})\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]f'(ta+(1-t)b)dt\\
&\quad + \int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}(1-t)^{\rho}]f'(ta+(1-t)b)dt\\
&= (-t^{\lambda})\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\frac{f(ta+(1-t)b)}{a-b}\Bigg|_0^{\frac{b-x}{b-a}}\\
&\quad + \int_0^{\frac{b-x}{b-a}} t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}]\frac{f(ta+(1-t)b)}{a-b}dt\\
&\quad + t(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}(1-t)^{\rho}]\frac{f(ta+(1-t)b)}{a-b}\Bigg|_{\frac{b-x}{b-a}}^1\\
&\quad - \int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}(1-t)^{\rho}]\frac{f(ta+(1-t)b)}{a-b}dt\\
&= +\left(\frac{b-x}{b-a}\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]\frac{f(x)}{b-a}\\
&\quad - \frac{1}{b-a}\int_0^{\frac{b-x}{b-a}} t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}t^{\rho}]f(ta+(1-t)b)dt\\
&\quad + \left(\frac{x-a}{b-a}\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\frac{f(x)}{b-a}\\
&\quad + \frac{1}{b-a}\int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^{\rho}(1-t)^{\rho}]f(ta+(1-t)b)dt.
\end{aligned}$$

Using the change of the variable $u = ta + (1-t)b$ for $t \in [0,1]$, we have

$$\begin{aligned}
\xi &= \frac{(b-x)^{\lambda}}{(b-a)^{\lambda+1}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]f(x) \\
&\quad -\frac{1}{(b-a)^{\lambda+1}}\int_x^b (b-u)^{\lambda-1})\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-u)^{\rho}]f(u)du \\
&\quad +\frac{(x-a)^{\lambda}}{(b-a)^{\lambda+1}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]f(x) \\
&\quad -\frac{1}{(b-a)^{\lambda+1}}\int_a^x (u-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(u-a)^{\rho}]f(u)du \\
&= \left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x) \\
&\quad -\frac{1}{(b-a)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f\right)(a) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f\right)(b)\right].
\end{aligned}$$

So, the proof is completed. □

Remark 3.2. In Lemma 3.1, let $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 1$. Then Lemma 3.1 reduces to Lemma 1.2 in [45].

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|$ is s-convex function in the second sense on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right. \\
&\left. -\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a) + (\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right| \\
\le &\left[\frac{(b-x)^{\lambda+s+1}}{(b-a)^{\lambda+s+1}}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-x)^{\rho}] + \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\right]|f'(a)| \\
+ &\left[\frac{(x-a)^{\lambda+s+1}}{(b-a)^{\lambda+s+1}}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(x-a)^{\rho}] + \mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\right]|f'(b)|
\end{aligned}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$, $k = 0, 1, 2, \ldots$, $B(x; a, b)$ is incompleted beta function and

$$\sigma_1(k) := \sigma(k)\frac{1}{\lambda + \rho k + s + 1},$$

$$\sigma_2(k) := \sigma(k)B\left(\frac{x-a}{b-a}; \lambda + \rho k + 1, s + 1\right),$$

$$\sigma_3(k) := \sigma(k)B\left(\frac{b-x}{b-a}; \lambda + \rho k + 1, s + 1\right).$$

Proof. From Lemma 3.1 and by using the properties of modulus, we have

$$
\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right.\\
&\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(J^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(J^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right|\\
\leq\ &\int_0^{\frac{b-x}{b-a}}|-t|^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}]|f'(ta+(1-t)b)|dt\\
&+\int_{\frac{b-x}{b-a}}^{1}|1-t|^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}]|f'(ta+(1-t)b)|dt.
\end{aligned}
$$

Since $|f'|$ is s-convex in the second sense on $[a,b]$, we get

$$
\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right.\\
&\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(J^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(J^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right|\\
\leq\ &\int_0^{\frac{b-x}{b-a}}t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}][t^{s}|f'(a)|+(1-t)^{s}|f'(b)|]dt\\
&+\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}][t^{s}|f'(a)|+(1-t)^{s}|f'(b)|]dt\\
=\ &|f'(a)|\int_0^{\frac{b-x}{b-a}}t^{\lambda+s}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}]dt\\
&+|f'(b)|\int_0^{\frac{b-x}{b-a}}t^{\lambda}(1-t)^{s}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}]dt\\
&+|f'(a)|\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\lambda}t^{s}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}]dt\\
&+|f'(b)|\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\lambda+s}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}]dt\\
=\ &|f'(a)|\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_0^{\frac{b-x}{b-a}}t^{\lambda+\rho k+s}dt\\
&+|f'(b)|\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_0^{\frac{b-x}{b-a}}t^{\lambda+\rho k}(1-t)^{s}dt\\
&+|f'(a)|\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_{\frac{b-x}{b-a}}^{1}t^{s}(1-t)^{\lambda+\rho k}dt\\
&+|f'(b)|\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\lambda+\rho k+s}dt
\end{aligned}
$$

$$= \ |f'(a)|\Big[\frac{(b-x)^{\lambda+s+1}}{(b-a)^{\lambda+s+1}}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\Big]$$
$$+ \ |f'(b)|\Big[\frac{(x-a)^{\lambda+s+1}}{(b-a)^{\lambda+s+1}}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\Big]$$

where used the facts that

$$\int_{\frac{b-x}{b-a}}^{1} t^{s}(1-t)^{\lambda+\rho k}dt = B\left(\frac{x-a}{b-a};\lambda+\rho k+1,s+1\right)$$
$$\int_{0}^{\frac{b-x}{b-a}} t^{\lambda+\rho k+s}dt = \frac{(b-x)^{\lambda+\rho k+s+1}}{(b-a)^{\lambda+\rho k+s+1}}\frac{1}{\lambda+\rho k+s+1}$$
$$\int_{\frac{b-x}{b-a}}^{1} (1-t)^{\lambda+\rho k+s}dt = \frac{(x-a)^{\lambda+\rho k+s+1}}{(b-a)^{\lambda+\rho k+s+1}}\frac{1}{\lambda+\rho k+s+1}$$
$$\int_{0}^{\frac{b-x}{b-a}} t^{\lambda+\rho k}(1-t)^{s}dt = B\left(\frac{b-x}{b-a};\lambda+\rho k+1,s+1\right).$$

So, the proof is completed. □

Corollary 3.4. *If we choose $s=1$ in Theorem 3.3, we obtain*

$$\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right.$$
$$\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right|$$
$$\leq \ \left[\frac{(b-x)^{\lambda+2}}{(b-a)^{\lambda+2}}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\right]|f'(a)| \tag{13}$$
$$+ \ \left[\frac{(x-a)^{\lambda+2}}{(b-a)^{\lambda+2}}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(b-a)^{\rho}]\right]|f'(b)|$$

where

$$\sigma_1(k):=\sigma(k)\frac{1}{\lambda+\rho k+2},$$

$$\sigma_2(k):=\sigma(k)B\left(\frac{x-a}{b-a};\lambda+\rho k+1,2\right)$$

and

$$\sigma_3(k):=\sigma(k)B\left(\frac{b-x}{b-a};\lambda+\rho k+1,2\right).$$

Remark 3.5. If we choose $\sigma(0)=1$, $w=0$ in Corollary 3.3, the inequality (13) reduces to inequality (2.1) of Theorem 2.1 in [45].

Theorem 3.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^{q}$ is s-convex function in the second sense on $[a,b]$, for some fixed $s\in(0,1]$, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following* *inequality for generalized fractional integral operators holds:*

$$\begin{aligned}
&\Bigg|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\\
&-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\Bigg|\\
\leq\ &\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\left(\frac{b-x}{b-a}\right)^{\lambda+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\\
&+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\left(\frac{x-a}{b-a}\right)^{\lambda+1}\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}$$

where $x\in[a,b]$, $\lambda,\rho>0$, $w\in\mathbb{R}$, *and*

$$\sigma_2(k):=\sigma(k)\left(\frac{1}{[\lambda+\rho k]p+1}\right)^{\frac{1}{p}}.$$

Proof. From Lemma 3.1 and using the Hölder inequality, we get

$$\begin{aligned}
&\Bigg|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\\
&-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\Bigg|\\
\leq\ &\int_0^{\frac{b-x}{b-a}}t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}]|f'(ta+(1-t)b)|dt\\
&+\int_{\frac{b-x}{b-a}}^1(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}]|f'(ta+(1-t)b)|dt\\
=\ &\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\left[\int_0^{\frac{b-x}{b-a}}t^{\lambda+\rho k}|f'(ta+(1-t)b)|dt\right.\\
&\left.+\int_{\frac{b-x}{b-a}}^1(1-t)^{\lambda+\rho k}|f'(ta+(1-t)b)|dt\right]\\
\leq\ &\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\left[\left(\int_0^{\frac{b-x}{b-a}}t^{[\lambda+\rho k]p}\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{b-x}{b-a}}^1(1-t)^{[\lambda+\rho k]p}\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f'|$ is s-convex in the second sense on $[a,b]$, by the inequality (1) we have

$$\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\leq\left(\frac{b-x}{b-a}\right)\left[\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right]$$

and

$$\int_{\frac{b-x}{b-a}}^1|f'(ta+(1-t)b)|^q dt\leq\left(\frac{x-a}{b-a}\right)\left[\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right].$$

Also, by simple compulation, we obtain

$$\int_0^{\frac{b-x}{b-a}} t^{[\lambda+\rho k]p} dt = \frac{1}{[\lambda+\rho k]p+1}\left(\frac{b-x}{b-a}\right)^{[\lambda+\rho k]p+1}$$

and

$$\int_{\frac{b-x}{b-a}}^1 (1-t)^{[\lambda+\rho k]p} dt = \frac{1}{[\lambda+\rho k]p+1}\left(\frac{x-a}{b-a}\right)^{[\lambda+\rho k]p+1}.$$

We, therefore, get

$$\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right.\\
&\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right|\\
\leq\ & \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\\
&\times\left[\left[\frac{1}{[\lambda+\rho k]p+1}\right]^{\frac{1}{p}}\left(\frac{b-x}{b-a}\right)^{\lambda+\rho k+\frac{1}{p}}\left(\frac{b-x}{b-a}\right)^{\frac{1}{q}}\left[\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\frac{1}{[\lambda+\rho k]p+1}\right]^{\frac{1}{p}}\left(\frac{x-a}{b-a}\right)^{\lambda+\rho k+\frac{1}{p}}\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right]^{\frac{1}{q}}\right]\\
=\ & \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\left(\frac{b-x}{b-a}\right)^{\lambda+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\\
&+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\left(\frac{x-a}{b-a}\right)^{\lambda+1}\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is completed. □

Corollary 3.7. *If we choose $s = 1$ in Theorem 3.6, we obtain*

$$\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right.\\
&\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right| \qquad (14)\\
\leq\ & \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\left(\frac{b-x}{b-a}\right)^{\lambda+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\\
&+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\left(\frac{x-a}{b-a}\right)^{\lambda+1}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and

$$\sigma_2(k) := \sigma(k)\left(\frac{1}{[\lambda+\rho k]p+1}\right)^{\frac{1}{p}}.$$

Remark 3.8. If we choose $\sigma(0) = 1$, $w = 0$ in Corollary 3.7, the inequality (14) reduces to inequality (2.3) of Theorem 2.2 in [45].

Theorem 3.9. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is s-convex function in the second sense on $[a,b]$, for some fixed $s \in (0,1]$, $q \geq 1$ and $x \in [a,b]$ then the following inequality for generalized fractional integral operators hold:*

$$\begin{aligned}
&\Bigg| \left[\frac{(b-x)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}} \right] f(x) \\
&- \frac{1}{(b-a)^{\lambda+1}} \left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w} f)(a) + (\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w} f)(b) \right] \Bigg| \\
\leq\ & \left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(b-x)^{\rho}] \right)^{1-\frac{1}{q}} \left(\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[|w|(b-x)^{\rho}] |f'(a)|^q + \mathcal{F}^{\sigma_5}_{\rho,\lambda+1}[|w|(b-x)^{\rho}] |f'(b)|^q \right)^{\frac{1}{q}} \\
&+ \left(\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[|w|(x-a)^{\rho}] \right)^{1-\frac{1}{q}} \left(\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}[|w|(x-a)^{\rho}] |f'(a)|^q + \mathcal{F}^{\sigma_8}_{\rho,\lambda+1}[|w|(x-a)^{\rho}] |f'(b)|^q \right)^{\frac{1}{q}},
\end{aligned}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$, $k = 0,1,2,\ldots$, $B(x;a,b)$ is incomplete beta function and

$$\begin{aligned}
\sigma_3(k) &:= \sigma(k) \left(\frac{b-x}{b-a} \right) \frac{1}{\rho k + 1}, & \sigma_4(k) &:= \sigma(k) \left(\frac{b-x}{b-a} \right)^{\lambda q + s + 1} \frac{1}{\lambda q + s + 1}, \\
\sigma_5(k) &:= \sigma(k) B\left(\frac{b-x}{b-a}; \lambda q + \rho k + 1, s+1 \right), & \sigma_6(k) &:= \sigma(k) \left(\frac{x-a}{b-a} \right) \frac{1}{\rho k + 1}, \\
\sigma_7(k) &:= \sigma(k) B\left(\frac{x-a}{b-a}; \lambda q + \rho k + 1, s+1 \right), & \sigma_8(k) &:= \sigma(k) \left(\frac{x-a}{b-a} \right)^{\lambda q + s + 1} \frac{1}{\lambda q + s + 1}.
\end{aligned}$$

Proof. From Lemma 3.1, using $|f'|^q$ is s-convex in the second sense and the well-known power mean inequality, we get

$$\begin{aligned}
&\Bigg| \left[\frac{(b-x)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}} \right] f(x) \\
&- \frac{1}{(b-a)^{\lambda+1}} \left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w} f)(a) + (\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w} f)(b) \right] \Bigg| \\
\leq\ & \int_0^{\frac{b-x}{b-a}} t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho} t^{\rho}] |f'(ta + (1-t)b)| dt \\
&+ \int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}] |f'(ta + (1-t)b)| dt
\end{aligned}$$

$$
\begin{aligned}
\leq & \left(\int_0^{\frac{b-x}{b-a}} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}]dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^{\frac{b-x}{b-a}} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}t^{\rho}]t^{\lambda q}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{b-x}{b-a}}^1 \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}]dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_{\frac{b-x}{b-a}}^1 \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}](1-t)^{\lambda q}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
\leq & \left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_0^{\frac{b-x}{b-a}} t^{\rho k}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_0^{\frac{b-x}{b-a}} t^{\lambda q+\rho k}[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q]dt\right)^{\frac{1}{q}} \\
& +\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_{\frac{b-x}{b-a}}^1 (1-t)^{\rho k}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda q+\rho k}[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q]dt\right)^{\frac{1}{q}} \\
= & \left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\frac{(b-x)^{\rho k+1}}{(b-a)^{\rho k+1}(\rho k+1)}\right)^{1-\frac{1}{q}} \\
& \times\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\left[|f'(a)|^q\int_0^{\frac{b-x}{b-a}} t^{\lambda q+\rho k+s}dt+|f'(b)|^q\int_0^{\frac{b-x}{b-a}} t^{\lambda q+\rho k}(1-t)^s dt\right]\right)^{\frac{1}{q}} \\
& +\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\frac{(x-a)^{\rho k+1}}{(b-a)^{\rho k+1}(\rho k+1)}\right)^{1-\frac{1}{q}} \\
& \times\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}\left[|f'(a)|^q\int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda q+\rho k}t^s dt+|f'(b)|^q\int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda q+\rho k+s}dt\right]\right)^{\frac{1}{q}} \\
= & \left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\right)^{1-\frac{1}{q}}\left(\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_5}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]|f'(b)|^q\right)^{\frac{1}{q}} \\
& +\left(\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\right)^{1-\frac{1}{q}}\left(\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_8}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where it is easily seen that

$$
\begin{aligned}
\int_0^{\frac{b-x}{b-a}} t^{\rho k}dt &= \left(\frac{b-x}{b-a}\right)^{\rho k+1}\frac{1}{\rho k+1}, \\
\int_0^{\frac{b-x}{b-a}} t^{\lambda q+\rho k+s}dt &= \left(\frac{b-x}{b-a}\right)^{\lambda q+\rho k+s+1}\frac{1}{\lambda q+\rho k+s+1},
\end{aligned}
$$

$$\begin{aligned}
\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\rho k}dt &= \left(\frac{x-a}{b-a}\right)^{\rho k+1}\frac{1}{\rho k+1},\\
\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\lambda q+\rho k+s}dt &= \left(\frac{x-a}{b-a}\right)^{\lambda q+\rho k+s+1}\frac{1}{\lambda q+\rho k+s+1},\\
\int_{0}^{\frac{b-x}{b-a}}t^{\lambda+\rho k}(1-t)^{s}dt &= B\left(\frac{b-x}{b-a};\lambda+\rho k+1,s+1\right),\\
\int_{\frac{b-x}{b-a}}^{1}t^{s}(1-t)^{\lambda+\rho k}dt &= B\left(\frac{x-a}{b-a};\lambda+\rho k+1,s+1\right).
\end{aligned}$$

Hence the proof is completed. □

Corollary 3.10. *If we choose $s = 1$ in Theorem 3.9, we obtain*

$$\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right. \qquad (15)\\
&\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right|\\
\leq\ &\left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\right)^{1-\frac{1}{q}}\left(\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]|f'(a)|^{q}+\mathcal{F}^{\sigma_5}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]|f'(b)|^{q}\right)^{\frac{1}{q}}\\
&+\left(\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\right)^{1-\frac{1}{q}}\left(\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]|f'(a)|^{q}+\mathcal{F}^{\sigma_8}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]|f'(b)|^{q}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
\sigma_3(k) &:= \sigma(k)\left(\frac{b-x}{b-a}\right)\frac{1}{\rho k+1}, & \sigma_4(k) &:= \sigma(k)\left(\frac{b-x}{b-a}\right)^{\lambda q+2}\frac{1}{\lambda q+2},\\
\sigma_5(k) &:= \sigma(k)B\left(\frac{b-x}{b-a};\lambda q+\rho k+1,2\right), & \sigma_6(k) &:= \sigma(k)\left(\frac{x-a}{b-a}\right)\frac{1}{\rho k+1},\\
\sigma_7(k) &:= \sigma(k)B\left(\frac{x-a}{b-a};\lambda q+\rho k+1,2\right), & \sigma_8(k) &:= \sigma(k)\left(\frac{x-a}{b-a}\right)^{\lambda q+2}\frac{1}{\lambda q+2}.
\end{aligned}$$

Remark 3.11. If we choose $\sigma(0) = 1$, $w = 0$ in Corollary 3.10, the inequality (15) reduces to inequality (2.4) of Theorem 2.3 in [45].

Theorem 3.12. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $|f'|^q$ is s-convex function in the second sense on $[a, b]$, for some fixed $s \in (0, 1]$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for generalized fractional integral operators hold:*

$$\begin{aligned}
&\left|\left[\frac{(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]+(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)\right.\\
&\left.-\frac{1}{(b-a)^{\lambda+1}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w}f)(a)+(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w}f)(b)\right]\right|\\
\leq\ &\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\left(\frac{b-x}{b-a}\right)^{\lambda+1}\left|f'\left(\frac{b+x}{2}\right)\right| \qquad (16)\\
&+\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\left(\frac{x-a}{b-a}\right)^{\lambda+1}\left|f'\left(\frac{a+x}{2}\right)\right|
\end{aligned}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ *and*

$$\sigma_1(k) := \sigma(k) 2^{(s-1)\frac{1}{q}} \left(\frac{1}{[\lambda + \rho k]p + 1} \right)^{\frac{1}{p}}.$$

Proof. From Lemma 3.1 and using the Hölder inequality, we get

$$\begin{aligned}
& \left| \left[\frac{(b-x)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}} \right] f(x) \right. \\
& \left. - \frac{1}{(b-a)^{\lambda+1}} \left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w} f)(a) + (\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w} f)(b) \right] \right| \\
\leq \; & \int_0^{\frac{b-x}{b-a}} t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho} t^{\rho}] |f'(ta + (1-t)b)| dt \\
& + \int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[|w|(b-a)^{\rho}(1-t)^{\rho}] |f'(ta + (1-t)b)| dt \\
= \; & \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\lambda + \rho k + 1)} \left[\int_0^{\frac{b-x}{b-a}} t^{\lambda + \rho k} |f'(ta + (1-t)b)| dt \right. \\
& \left. + \int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda + \rho k} |f'(ta + (1-t)b)| dt \right] \\
\leq \; & \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\lambda + \rho k + 1)} \left[\left(\int_0^{\frac{b-x}{b-a}} t^{[\lambda + \rho k]p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{[\lambda + \rho k]p} dt \right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\lambda + \rho k} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Since $|f'|$ is s-convex in the second sense on $[a, b]$, by the inequality (1) we have

$$\int_0^{\frac{b-x}{b-a}} |f'(ta + (1-t)b)|^q dt \leq 2^{s-1} \left(\frac{b-x}{b-a} \right) \left| f' \left(\frac{b+x}{2} \right) \right|^q$$

and

$$\int_{\frac{b-x}{b-a}}^1 |f'(ta + (1-t)b)|^q dt \leq 2^{s-1} \left(\frac{x-a}{b-a} \right) \left| f' \left(\frac{a+x}{2} \right) \right|^q.$$

Therefore

$$\begin{aligned}
& \left| \left[\frac{(b-x)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}] + (x-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(b-a)^{\lambda+1}} \right] f(x) \right. \\
& \left. - \frac{1}{(b-a)^{\lambda+1}} \left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;w} f)(a) + (\mathcal{J}^{\sigma}_{\rho,\lambda,x+;w} f)(b) \right] \right|
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\lambda+\rho k+1)}$$
$$\times\Bigg[\left[\frac{1}{[\lambda+\rho k]p+1}\right]^{\frac{1}{p}}\left(\frac{b-x}{b-a}\right)^{\lambda+\rho k+\frac{1}{p}}\left(\frac{b-x}{b-a}\right)^{\frac{1}{q}}2^{(s-1)\frac{1}{q}}\left|f'\left(\frac{b+x}{2}\right)\right|$$
$$+\left[\frac{1}{[\lambda+\rho k]p+1}\right]^{\frac{1}{p}}\left(\frac{x-a}{b-a}\right)^{\lambda+\rho k+\frac{1}{p}}\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}}2^{(s-1)\frac{1}{q}}\left|f'\left(\frac{a+x}{2}\right)\right|\Bigg]$$
$$= \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-x)^{\rho}]\left(\frac{b-x}{b-a}\right)^{\lambda+1}\left|f'\left(\frac{b+x}{2}\right)\right|$$
$$+\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(x-a)^{\rho}]\left(\frac{x-a}{b-a}\right)^{\lambda+1}\left|f'\left(\frac{a+x}{2}\right)\right|.$$

So, the proof is completed. □

Remark 3.13. If we choose $\sigma(0) = 1$, $w = 0$ and $s = 1$ in Theorem 3.12, the inequality (16) reduces to inequality (2.8) of Theorem 2.4 in [45].

4. Conclusion

In this paper, we established the Ostrowski type inequalities for mappings whose first derivatives in absolute value are s-convex in the second sense involving generalised fractional integral operator. The results presented in this paper would provide generalizations of those given in earlier works.

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